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A GEOMETRIC CRITERION FOR THE NONUNIFORM HYPERBOLICITY OF THE KONTSEVICH–ZORICH COCYCLE

GIOVANNI FORNI
(Communicated by Anton Zorich)

With an appendix
OTHER RELEVANT EXAMPLES
by CARLOS MATHEUS

ABSTRACT. We establish a geometric criterion on a SL(2,ℝ)-invariant ergodic probability measure on the moduli space of holomorphic abelian differentials on Riemann surfaces for the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle on the real Hodge bundle. Applications include measures supported on the SL(2,ℝ)-orbits of all algebraically primitive Veech surfaces (see also [7]) and of all Prym eigenforms discovered in [34], as well as all canonical absolutely continuous measures on connected components of strata of the moduli space of abelian differentials (see also [4, 17]). The argument simplifies and generalizes our proof for the case of canonical measures [17]. In the Appendix, Carlos Matheus discusses several relevant examples which further illustrate the power and the limitations of our criterion.

1. INTRODUCTION

The nonuniform hyperbolicity of the Kontsevich–Zorich cocycle with respect to the canonical absolutely continuous SL(2,ℝ)-invariant probability measures on the connected components of the moduli space of abelian differentials was originally conjectured by Zorich and Kontsevich. The conjecture was proved by the author in [17]. In this paper we develop the ideas of [17] and prove a geometric criterion for the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle with respect to a general SL(2,ℝ)-invariant probability ergodic measure. This criterion yields a streamlined proof of the original argument in [17] for the case of the canonical measures. In fact, this paper is an outgrowth of an unpublished note written to clarify and rectify parts of the original proof of Theorem 8.5 in [17] (see also [29]) in response to questions of the participants of the *Groupe de Travail in Geometry and Dynamics* of the Université de Paris VI

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The nonuniform hyperbolicity of the Kontsevich–Zorich cocycle is important in several applications to the dynamics of translation flows and interval-exchange transformations, such as the fine behavior of ergodic averages and their deviations [8, 9, 17, 27, 47–51]. The nonvanishing of the second exponent Kontsevich–Zorich exponent is crucial in the approach of A. Avila and the author to weak mixing for translation flows and interval-exchange transformations [3]. While the Lebesgue generic case is now well-understood, there are only a few scattered results on the Kontsevich–Zorich spectrum for generic translation flows with respect to $SL(2,\mathbb{R})$-invariant probability ergodic measures other than the canonical ones. Most of these results, for instance [7] and [20] (as well as work in progress by C. Matheus, M. Möller and J.-C. Yoccoz) are limited to $SL(2,\mathbb{R})$-invariant probability measures associated to Veech surfaces and leave open natural questions such as the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle for strata of quadratic differentials. Finally, we stress that, as far as we can see, our methods cannot yield the stronger property of *simplicity* of the Kontsevich–Zorich spectrum, which was proved by A. Avila and M. Viana [4] for the canonical absolutely continuous measures on the moduli space of abelian differentials. It would be interesting, in our opinion, to formulate geometric criteria for the simplicity of the spectrum based on Avila's and Viana's approach. However, since there exist nonuniformly hyperbolic $SL(2,\mathbb{R})$-invariant ergodic measures for which the spectrum is not simple, the criterion we propose in this paper has a somewhat different scope (while of course yielding a weaker property). For instance, for a particular sequence of square-tiled cyclic covers, the so-called "stairs" square-tiled cyclic covers (double covers of square-tiled "stairs" surfaces), all the Kontsevich–Zorich exponents of the associated $SL(2,\mathbb{R})$-invariant measure are nonzero and all but the top and bottom exponents are double (explicit values for all the exponents are computed in [11, Appendix B.1]). In fact, the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle for all "stairs" square-tiled surfaces follows easily from our criterion (see [11, Appendix C] for the verification of the main condition of our criterion, the *Lagrangian* property of the horizontal/vertical foliation introduced in [17, Definition 4.3], and recalled below in Definition 1.4). Other examples of measures on strata of abelian differentials on surfaces of genus 3 with multiple Kontsevich–Zorich exponents to which our criterion applies have been found by C. Matheus. He communicated to the author that he can prove the following result: (a) all measures on $\mathscr{H}(1,1,1,1)$ coming from a regular (unbranched) double cover construction over a square-tiled surface in the stratum $\mathscr{H}(1,1)$;

(b) the measure on $\mathscr{H}(2,2)$ coming from a certain regular double cover construction over an L-shaped square-tiled surface with three squares (in the stratum $\mathscr{H}(2)$) have a double second exponent. The details and the complete argument for case (b) are given in Appendix A.1.

In genus 2, the simplicity of the Kontsevich–Zorich spectrum for the canonical absolutely continuous $\mathrm{SL}(2,\mathbb{R})$-invariant probability measures on the space of abelian differentials was proved in [17]. In unpublished work the author was able to extend this result to all $\mathrm{SL}(2,\mathbb{R})$-invariant probability ergodic measures. Later, M. Bainbridge [5] was able to establish the exact numerical values of the second exponent (the only nontrivial one) for all such measures. Bainbridge result, based on C. McMullen's classification [33] theorem for $\mathrm{SL}(2,\mathbb{R})$-invariant probability measures in genus 2, implies the simplicity of the Kontsevich–Zorich spectrum.

In order to state our geometric criterion we recall a few fundamental properties of probability measures invariant under the Teichmüller flow and introduce a couple of definitions. Let $\mathscr{H}_g$ be the moduli space of abelian differentials of unit total area on Riemann surfaces of genus $g \geq 2$. The space $\mathscr{H}_g$ has a natural stratification defined as follows. For any $\kappa := (k_1, \dots, k_\sigma) \in \mathbb{Z}_+^\sigma$ such that $\sum k_i = 2g-2$, the subset $\mathscr{H}(\kappa) \subset \mathscr{H}_g$ of all abelian differentials which have exactly $\sigma \in \mathbb{Z}_+$ zeroes $p_1, \dots, p_\sigma$ with multiplicities $k_1, \dots, k_\sigma$ is nonempty. Each stratum $\mathscr{H}(\kappa)$ has the structure of an affine orbifold locally parametrized by the period map into the relative complex cohomology $H^1(S,\Sigma;\mathbb{C})$ of the surface S relative to zero set $\Sigma := \{p_1, \dots, p_\sigma\}$. The strata are in general not connected and can have up to three connected components [28].

Every abelian differential $\omega \in \mathscr{H}_g$ induces a pair $(\mathscr{F}^h_\omega, \mathscr{F}^v_\omega)$ of transverse orientable measured foliations on the topological surface S of genus $g \geq 2$, called respectively the *horizontal* and the *vertical* foliation of the abelian differential. Such foliations are respectively defined as follows:

$$\mathscr{F}^h_\omega := \{\mathrm{Im}\,\omega = 0\} \qquad \text{and} \qquad \mathscr{F}^v_\omega := \{\mathrm{Re}\,\omega = 0\}.$$

The transverse measure for the horizontal foliation $\mathscr{F}^h_\omega$ is defined by integration along transverse arcs of the density $|\mathrm{Re}\,\omega|$; the transverse measure for the vertical foliation $\mathscr{F}^v_\omega$ is defined by integration along transverse arcs of the density $|\mathrm{Im}\,\omega|$.

We recall that the canonical action of the group $\mathrm{SL}(2,\mathbb{R})$ on $\mathscr{H}_g$ can be defined as follows. For any $A \in \mathrm{SL}(2,\mathbb{R})$ and for any holomorphic differential ω on a Riemann surface S_ω, there exists a unique Riemann surface $S_{A\omega}$ and a unique abelian differential $A\omega$ holomorphic on $S_{A\omega}$ such that

$$A\begin{pmatrix}\mathrm{Re}\,\omega\\ \mathrm{Im}\,\omega\end{pmatrix} = \begin{pmatrix}\mathrm{Re}\,A\omega\\ \mathrm{Im}\,A\omega\end{pmatrix}.$$

The *Teichmüller geodesic flow* is defined as the action of the diagonal subgroup $\{g_t\} = \{\mathrm{diag}(e^t, e^{-t})\} < \mathrm{SL}(2,\mathbb{R})$. It is immediate by the definition that strata are invariant under the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathscr{H}_g$, hence under the Teichmüller flow. In addition, it is possible to define a natural smooth $\mathrm{SL}(2,\mathbb{R})$-invariant measure on every stratum. A fundamental result by H. Masur [31] and W. Veech [42, 43] states that every such measure has finite mass and is ergodic for the Teichmüller flow.

The Teichmüller flow admits two invariant foliations, $\mathscr{W}^+$ and $\mathscr{W}^-$ on $\mathscr{H}_g$, which are (locally) defined as follows. The leaves $\mathscr{W}^\pm(\omega)$ through any abelian differential $\omega \in \mathscr{H}_g$, are given by the formulas

$$\mathscr{W}^+(\omega) := \{\tilde\omega \in \mathscr{H}_g \mid \mathrm{Im}\,\tilde\omega \in \mathbb{R}^+ \cdot \mathrm{Im}\,\omega\} = \{\tilde\omega \in \mathscr{H}_g \mid [\mathscr{F}^h_{\tilde\omega}] = [\mathscr{F}^h_\omega]\},$$
$$\mathscr{W}^-(\omega) := \{\tilde\omega \in \mathscr{H}_g \mid \mathrm{Re}\,\tilde\omega \in \mathbb{R}^+ \cdot \mathrm{Re}\,\omega\} = \{\tilde\omega \in \mathscr{H}_g \mid [\mathscr{F}^v_{\tilde\omega}] = [\mathscr{F}^v_\omega]\}.$$

Let $\mathscr{W}^\pm_\kappa$ denote the intersections of the foliations $\mathscr{W}^\pm$ with a stratum $\mathscr{H}(\kappa)$ of the moduli space. For any $\omega \in \mathscr{H}(\kappa)$, the intersection $\mathscr{W}^+_\kappa(\omega) \cap \mathscr{W}^-_\kappa(\omega)$ coincides with the Teichmüller orbit $\{g_t\omega \mid t \in \mathbb{R}\} \subset \mathscr{H}(\kappa)$. For any pair ω^+, $\omega^- \in \mathscr{H}(\kappa)$, the intersection $\mathscr{W}^+_\kappa(\omega^+) \cap \mathscr{W}^-_\kappa(\omega^-)$ is nonempty if and only if the horizontal foliation $\mathscr{F}^h_{\omega^+}$ of ω^+ and the vertical foliation $\mathscr{F}^v_{\omega^-}$ of ω^- are transverse (in the sense of measured foliations). In fact, a pair $(\mathscr{F}^h, \mathscr{F}^v)$ of measured foliations is transverse if and only there exists a holomorphic abelian differential $\omega \in \mathscr{H}_g$ such that $\mathscr{F}^h(\omega) \in \mathbb{R}^+ \cdot \mathscr{F}^h$ and $\mathscr{F}^v(\omega) \in \mathbb{R}^+ \cdot \mathscr{F}^v$.

It is an immediate consequence of a result of [17] (see Theorem 1.1 below) that every probability Teichmüller-invariant ergodic measure is nonuniformly hyperbolic, in the sense that the tangent cocycle of the Teichmüller flow restricted to the relevant stratum has nonzero Lyapunov exponents in all directions except for the flow direction. In fact, this result is a consequence of a stronger property: there exists on every stratum $\mathscr{H}(\kappa) \subset \mathscr{H}_g$ a Hodge Riemannian metric whose restrictions to the leaves of the foliations $\mathscr{W}^+_\kappa$ and $\mathscr{W}^-_\kappa$ is contracted by the backward or forward action, respectively, of the Teichmüller flow (in all directions except for the flow direction), *uniformly* on every compact subset of the (noncompact) space $\mathscr{H}(\kappa)$, see [2, 17]. The Hodge Riemannian metrics have been explicitly constructed and studied in depth in [1].

We recall the definition of the Kontsevich–Zorich cocycle over the Teichmüller flow [17, 27], a continuous-time version of the Rauzy–Veech–Zorich cocyle [39, 42, 48] over the Rauzy–Veech–Zorich map.

Let $\hat{\mathscr{H}}_g$ be the Teichmüller space of holomorphic abelian differentials of unit total area on Riemann surfaces of genus $g \geq 2$. It can be defined as the moduli space of abelian differentials on *marked* Riemann surfaces. Let S denote the underlying smooth surface of genus $g \geq 2$. Points of the Teichmüller space $\hat{\mathscr{H}}_g$ are equivalence classes of holomorphic abelian differentials on the surface S, endowed with some holomorphic structure, with respect to the equivalence relation given by the natural action of the group $\mathrm{Diff}^+_0(S)$ of orientation-preserving diffeomorphisms isotopic to the identity. The moduli space $\mathscr{H}_g$ can be defined as the quotient $\hat{\mathscr{H}}_g/\Gamma_g$ of the Teichmüller space of holomorphic abelian differentials of unit total area with respect to the action of the *mapping-class group* $\Gamma_g := \mathrm{Diff}^+(S)/\mathrm{Diff}^+_0(S)$ on $\hat{\mathscr{H}}_g$. The Teichmüller flow $\{g_t\}$ on the moduli space $\mathscr{H}_g$ lifts to a flow $\{\hat g_t\}$ on the Teichmüller space $\hat{\mathscr{H}}_g$. Let $\{\hat\rho_t\}$ be the trivial cocycle over the flow $\{\hat g_t\}$ on the trivial cohomology bundle

$\hat{\mathcal{H}}_g \times H^1(S,\mathbb{R})$ defined as follows:

$$\hat{\rho}_t := \hat{g}_t \times \mathrm{Id}\colon \hat{\mathcal{H}}_g \times H^1(S,\mathbb{R}) \to \hat{\mathcal{H}}_g \times H^1(S,\mathbb{R}).$$

The mapping-class group Γ_g acts on the trivial bundle $\hat{\mathcal{H}}_g \times H^1(S,\mathbb{R})$ by pullback on each coordinate. The quotient bundle

$$H^1_g := \big(\hat{\mathcal{H}}_g \times H^1(S,\mathbb{R})\big)/\Gamma_g$$

is an orbifold vector bundle over the moduli space $\mathcal{H}_g$ of holomorphic abelian differentials of unit total area called the (real) *Hodge bundle*. The *Kontsevich–Zorich cocycle* can be defined as the projection $\{\rho_t\}$ to the Hodge bundle H^1_g of the trivial cocycle $\{\hat{\rho}_t\}$. By definition, it is a cocycle over the Teichmüller geodesic flow $\{g_t\}$ on the moduli space $\mathcal{H}_g$ of holomorphic abelian differentials of unit total area.

It is an immediate consequence of the definition that the top exponent of the Kontsevich–Zorich cocycle is equal to 1. In addition, since the action of the group $\mathrm{Diff}^+(S)$ on $H^1(S,\mathbb{R})$ is symplectic with respect to the standard symplectic structure given by the intersection form, the cocyle is *symplectic*, hence for any probability measure μ on the moduli space $\mathcal{H}_g$, invariant under the Teichmüller flow and ergodic, its *Lyapunov spectrum* has g nonnegative and g nonpositive exponents (counting multiplicities) and it is symmetric with respect to the origin, that is, it has the form

$$\lambda_1^\mu = 1 \ge \lambda_2^\mu \ge \cdots \ge \lambda_g^\mu \ge -\lambda_g^\mu \ge \cdots \ge -\lambda_2^\mu \ge -\lambda_1^\mu = -1.$$

There is a simple well-known relation between the Lyapunov spectrum of the Kontsevich–Zorich cocycle and the Lyapunov spectrum of the Teichmüller flow (that is, the Lyapunov spectrum of the tangent cocycle) restricted to any stratum of the moduli space [17, 18, 27, 48, 51]. The Lyapunov spectrum of the Teichmüller flow with respect to any invariant, ergodic probability measure μ supported on a stratum $\mathcal{H}(\kappa) \subset \mathcal{H}_g$ of the moduli space can be written as follows in terms of the Lyapunov spectrum of the Kontsevich–Zorich cocycle:

$$\begin{aligned} 2 \ge (1+\lambda_2^\mu) \ge \cdots \ge (1+\lambda_g^\mu) \ge \overbrace{1 = \cdots = 1}^{\sigma-1} \ge (1-\lambda_g^\mu) \\ \ge \cdots \ge (1-\lambda_2^\mu) \ge 0 \ge -(1-\lambda_2^\mu) \ge \cdots \ge -(1-\lambda_g^\mu) \\ \ge \underbrace{-1 = \cdots = -1}_{\sigma-1} \ge -(1+\lambda_g^\mu) \ge \cdots \ge -(1+\lambda_2^\mu) \ge -2. \end{aligned}$$

It is immediate from the above formula that the nonuniform hyperbolicity of the Teichmüller flow with respect to any ergodic probability measure μ on $\mathcal{H}(\kappa)$ is equivalent to a *spectral gap* property for the Kontsevich–Zorich cocycle, *i.e.,* the strict inequality $\lambda_2^\mu < \lambda_1^\mu = 1$. For a class of measures satisfying a certain integrability condition (class which includes all the canonical absolutely continuous measures), the nonuniform hyperbolicity of the Teichmüller flow was proved by W. Veech in [43]. In [17], Corollary 2.2 (see also [18, Theorem 5.1]), the author proved the following generalization of Veech's result.

THEOREM 1.1. *For any Teichmüller-invariant ergodic probability measure μ on $\mathcal{H}_g$, the Kontsevich–Zorich cocycle has a spectral gap, that is,*

$$\lambda_2^\mu < \lambda_1^\mu = 1.$$

Equivalently, any Teichmüller-invariant ergodic probability measure is nonuniformly hyperbolic for the Teichmüller flow.

The above theorem is a rather straightforward application of the variational formulas for the Hodge norm on the Hodge bundle (see [17, §2]).

The nonvanishing of the Kontsevich–Zorich exponents is much harder to establish. For the canonical absolutely continuous $SL(2,\mathbb{R})$-invariant measures on strata of the moduli space, Kontsevich and Zorich conjectured that the spectrum of the Rauzy–Veech–Zorich cocycle (or, equivalently, of the Kontsevich–Zorich cocycle) is *simple*, hence in particular the cocycle is *nonuniformly hyperbolic* (that is, all the Lyapunov exponents are nonzero). We recall that the nonuniform hyperbolicity was proved by the author in [17] and the full conjecture was later proved by A. Avila and M. Viana [4] by a different approach. The nonuniformly hyperbolicity of the Kontsevich–Zorich cocycle fails in general even for $SL(2,\mathbb{R})$-invariant probability measures. The first example of an $SL(2,\mathbb{R})$-invariant ergodic probability measure with zero Lyapunov exponents was found by the author in 2002 (later published in [18, §7]). It is the unique $SL(2,\mathbb{R})$-invariant probability measure supported on the closed $SL(2,\mathbb{R})$-orbit of an "exceptionally symmetric" translation surface of genus 3, often called the *Eierlegende Wollmilchsau* surface [22]. For such a measure, the Kontsevich–Zorich spectrum is maximally degenerate, that is, all nontrivial exponents are zero [18, 20]. Another maximally degenerate example of genus 4 was later discovered by C. Matheus and the author [19]. Both these examples belong to the class of *square-tiled cyclic covers* which includes many examples of partially degenerate Lyapunov spectrum [11, 20]. Outside of this class, it seems that there are no explicit examples in the literature of $SL(2,\mathbb{R})$-invariant probability measures with some zero exponents. However, the latter should not be hard to construct. In fact, as pointed out by C. Matheus, examples should be contained implicitly in the work of C. McMullen [35], although Kontsevich–Zorich exponents are never mentioned there.

For the maximally degenerate examples, the action on homology of the affine group is given by finite symmetry groups [32]. According to a theorem of Möller [37], there are no other maximally degenerate examples coming from Veech surfaces (except possibly in genus 5). For square-tiled cyclic cover, this result can be derived from the Kontsevich–Zorich formula (see [20]). Recently A. Avila and M. Möller have announced work that confirms the conjectured negative answer to the question on the existence of $SL(2,\mathbb{R})$-invariant measures with completely degenerate Kontsevich–Zorich spectrum outside of the class of Veech surfaces.

Our criterion for the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle applies to certain $SL(2,\mathbb{R})$-invariant probability ergodic measures which have a local product structure in the sense defined below.

For every open subset $\mathscr{U} \subset \mathscr{H}(\kappa)$, the local invariant foliations $\mathscr{W}^{\pm}_{\mathscr{U}}$ are defined as follows: the leaf $\mathscr{W}^{\pm}_{\mathscr{U}}(\omega)$ is the unique connected component of the intersection $\mathscr{W}^{\pm}_{\kappa}(\omega) \cap \mathscr{U}$ which contains the abelian differential $\omega \in \mathscr{U}$.

DEFINITION 1.2. An open set $\mathscr{U} \subset \mathscr{H}(\kappa)$ is said to be of *product type* if, for any pair of abelian differentials $(\omega^+,\omega^-) \in \mathscr{U} \times \mathscr{U}$, there exist an abelian differential $\omega \in \mathscr{U}$ and an open interval $(a,b) \subset \mathbb{R}$ such that

$$\mathscr{W}^{+}_{\mathscr{U}}(\omega^+) \cap \mathscr{W}^{-}_{\mathscr{U}}(\omega^-) = \{g_t\omega \mid t \in (a,b)\}.$$

Since every stratum $\mathscr{H}(\kappa)$ of the moduli space of abelian differentials has an affine structure with local charts given by the relative period map, by writing the invariant foliations $\mathscr{F}^{\pm}_{\kappa}$ in coordinates, it can be verified that the topology $\mathscr{H}(\kappa)$ has a (countable) basis of open sets of product type.

Let $\mathscr{U} \subset \mathscr{H}(\kappa)$ be an open set of product type. For every set $\Omega \subset \mathscr{U}$, let

$$\mathscr{W}^{\pm}_{\mathscr{U}}(\Omega) := \bigcup_{\omega\in\Omega} \mathscr{W}^{\pm}_{\mathscr{U}}(\omega).$$

DEFINITION 1.3. A Teichmüller-invariant measure μ supported on $\mathscr{H}(\kappa)$ is said to have a *product structure* on an open subset $\mathscr{U} \subset \mathscr{H}(\kappa)$ of product type if for any pair of Borel subsets Ω^+, $\Omega^- \subset \mathscr{U}$,

$$\mu(\Omega^+) \neq 0 \text{ and } \mu(\Omega^-) \neq 0 \quad \Longrightarrow \quad \mu\big(\mathscr{W}^{+}_{\mathscr{U}}(\Omega^+) \cap \mathscr{W}^{-}_{\mathscr{U}}(\Omega^-)\big) \neq 0.$$

A Teichmüller-invariant measure μ on $\mathscr{H}(\kappa)$ is said to have a *local product structure* if every abelian differential $\omega \in \mathscr{H}(\kappa)$ has an open neighborhood $\mathscr{U}_\omega \subset \mathscr{H}(\kappa)$ of product type on which μ has a product structure.

We remark that any $SL(2,\mathbb{R})$-invariant *affine* measure has a local product structure. A measure on a stratum of the moduli space of abelian differentials is called affine if it is locally equal (up to normalization) to the the restriction of the Lebesgue measure to a complex affine subspace with respect to the natural affine structure induced by the complex (relative) period map. It is part of a broader Ratner-type conjecture on the action of $SL(2,\mathbb{R})$ on the moduli space of abelian differentials that all $SL(2,\mathbb{R})$-invariant probability measures are affine. A proof of a Ratner-type conjecture, which includes the statement that every $SL(2,\mathbb{R})$-invariant probability measure is affine, has been recently announced by A. Eskin and M. Mirzakhani [13]. For the genus two case, all $SL(2,\mathbb{R})$-invariant probability measures were classified by C. McMullen [33] and K. Calta [10] and are known to be affine.

Next we recall the notion of a completely periodic *Lagrangian* measured foliation (introduced in [17, Definition 4.3]).

DEFINITION 1.4. A *completely periodic* measured foliation $\mathscr{F}$ on a compact orientable surface S of genus $g \geq 2$ is a measured foliation on S such that all its regular leaves are closed (compact) curves.

The *homological dimension* of a completely periodic measured foliation $\mathscr{F}$ on S is the dimension of the (isotropic) subspace $\mathscr{L}(\mathscr{F}) \subset H_1(S,\mathbb{R})$, generated by the homology classes of the regular leaves of $\mathscr{F}$.

A *Lagrangian* measured foliation $\mathscr{F}$ on S is a completely periodic measured foliation of maximal homological dimension (equal to the genus of the surface), that is, a measured foliation such that the subspace $\mathscr{L}(\mathscr{F})$ is a Lagrangian subspace of the space $H_1(S,\mathbb{R})$, endowed with the symplectic structure given by the intersection form.

A completely periodic measured foliation $\mathscr{F}$ is Lagrangian if and only if it has $g \geq 2$ distinct regular leaves $\gamma_1,\ldots,\gamma_g$ such that $\widehat{S} := S \setminus \cup\{\gamma_1,\ldots,\gamma_g\}$ is homeomorphic to a sphere minus $2g$ (paired) disjoint disks.

DEFINITION 1.5. A Teichmüller-invariant probability measure on a stratum is said to be *cuspidal* if it has a local product structure and its support contains a holomorphic differential with completely periodic horizontal or vertical foliation.

The *homological dimension* of a Teichmüller-invariant measure is the maximal homological dimension of a completely periodic vertical or horizontal foliation of a holomorphic differential in its support.

A Teichmüller-invariant probability measure is said to be *Lagrangian* if it has maximal homological dimension, that is, if its support contains a holomorphic differential with Lagrangian horizontal or vertical foliation.

We are grateful to J. Smillie who pointed out to us that any closed SL(2,$\mathbb{R}$)-invariant set, hence in particular the support of any SL(2,$\mathbb{R}$)-invariant measure, contains a holomorphic differential with completely periodic horizontal or vertical foliation. In fact, according to a theorem by Smillie and B. Weiss [40], any closed SL(2,$\mathbb{R}$)-invariant subset of the moduli space contains a minimal set for the Teichmüller horocycle flow and every such minimal set corresponds to a cylinder decomposition. By this result and by the Ratner's type result announced by A. Eskin and M. Mirzakhani [13] that every SL(2,$\mathbb{R}$)-invariant probability measure is affine, it follows that every SL(2,$\mathbb{R}$)-invariant probability measure is cuspidal.

Our main result is the following criterion.

THEOREM 1.6. *Let μ be a* SL(2,$\mathbb{R}$)*-invariant ergodic probability measure on a stratum $\mathscr{H}(\kappa)$ of the moduli space of abelian differential. If μ is cuspidal Lagrangian, the Kontsevich–Zorich cocycle is nonuniformly hyperbolic μ-almost everywhere. In fact, the Lyapunov exponents $\lambda_1^\mu \geq \cdots \geq \lambda_{2g}^\mu$ of the Kontsevich–Zorich cocycle form a symmetric subset of the real line and the following inequalities hold:*

$$1 = \lambda_1^\mu > \lambda_2^\mu \geq \cdots \geq \lambda_g^\mu > 0 > \lambda_{g+1}^\mu = -\lambda_g^\mu \geq \cdots \geq \lambda_{2g-1}^\mu = -\lambda_2^\mu > \lambda_{2g}^\mu = -\lambda_1^\mu = -1.$$

A weaker statement holds without assuming that the SL(2,$\mathbb{R}$)-invariant probability measure has a local product structure. In fact, the following holds.

THEOREM 1.7. *Let μ be a* $\mathrm{SL}(2,\mathbb{R})$*-invariant ergodic probability measure on a stratum $\mathscr{H}(\kappa) \subset \mathscr{H}_g$ of the moduli space of abelian differential on Riemann surfaces of genus $g \geq 3$. If μ is Lagrangian, the following inequalities hold for the Lyapunov spectrum of the Kontsevich–Zorich cocycle:*

$$1 = \lambda_1^\mu > \lambda_2^\mu \geq \cdots \geq \lambda_{[g+1/2]}^\mu > 0. \tag{1.1}$$

As recalled above, in genus 2 the Kontsevich–Zorich spectrum is simple for all $\mathrm{SL}(2,\mathbb{R})$-invariant ergodic probability measures [5]. This result can be derived from the Kontsevich–Zorich formula (see Corollary 2.5 below) and from a description of possible boundary points of Teichmüller disks in the Deligne–Mumford compactification of the moduli space.

For cuspidal measures, Theorem 1.6 can be generalized as follows.

THEOREM 1.8. *Let μ be a* $\mathrm{SL}(2,\mathbb{R})$*-invariant ergodic probability cuspidal measure on a stratum $\mathscr{H}(\kappa)$ of the moduli space of abelian differential. If μ has homological dimension $k \in \{1,\ldots,g\}$, there are at least k strictly positive Kontsevich–Zorich exponents, that is,*

$$1 = \lambda_1^\mu > \lambda_2^\mu \geq \cdots \geq \lambda_k^\mu > 0.$$

The proof of the above theorem can be obtained along the same lines of the proof of Theorem 1.6 and will not be explained in detail in this paper.

We remark that the lower bound given in Theorem 1.8 is optimal in general. In fact, the *maximally degenerate examples* of [18, 19] (see also [20]) are both given by $\mathrm{SL}(2,\mathbb{R})$ invariant probability measures (supported on closed $\mathrm{SL}(2,\mathbb{R})$-orbits) of *homological dimension equal to* 1 (in both cases completely periodic directions split the surface into two homologous cylinders). However, there exist cuspidal measures where the number of strictly positive Kontsevich–Zorich exponents is *greater* than the homological dimension of the measure. For instance, the family of square-tiled cyclic covers studied by C. Matheus and J.-C. Yoccoz [32, §3.1] provide examples of $\mathrm{SL}(2,\mathbb{R})$-invariant probability measures on the strata $\mathscr{H}(q-1,q-1,q-1)$ for any odd $q \geq 3$ with homological dimension equal to 1 and a number of nonzero Kontsevich–Zorich exponents equal to $1+(q-3)/2$ when $q = 3 \pmod 4$, and $1+(q-1)/2$ when $q = 1 \mod 4$. We are grateful to C. Matheus who computed the above formulas for us. His calculations appear at the end of this paper in Appendix A.2.

The case $q = 3$ in the Matheus–Yoccoz family corresponds to the example by C. Matheus and the author of a square-tiled cyclic cover of genus 4 with completely degenerate spectrum [19, 20] mentioned above.

Recently, Delecroix and Matheus found examples of cuspidal nonuniformly hyperbolic $\mathrm{SL}(2,\mathbb{R})$-invariant probability measures which are not Lagrangian: the measures supported on the $\mathrm{SL}(2,\mathbb{R})$ orbits of a couple of square-tiled surfaces, one in genus 3 and one in genus 4. Such examples were found with the aid of A. Zorich's computer program to compute the exponents to have simple Lyapunov spectrum and, as communicated to us by Carlos Matheus, it seems possible to prove simplicity by some version of Avila–Viana's criterion [4].

As we have remarked above, by a result of J. Smillie and B. Weiss [40] on minimal sets for the Teichmüller horocycle flow and by the Ratner type result announced by A. Eskin and M. Mirzakhani [13] it follows that every $SL(2,\mathbb{R})$-invariant probability measure is cuspidal. However, not all $SL(2,\mathbb{R})$-invariant probability ergodic measures are Lagrangian. In fact, as mentioned above, there are many examples among $SL(2,\mathbb{R})$-invariant measures supported on $SL(2,\mathbb{R})$-orbits of square-tiled cyclic covers which fail to be nonuniformly hyperbolic [11, 20]. By our criterion such measures are not Lagrangian. To the author's best knowledge, no examples of (cuspidal) $SL(2,\mathbb{R})$-invariant probability measures which are not Lagrangian outside of the class of measures supported on closed $SL(2,\mathbb{R})$-orbits or on strata of branched covers are known.

All the $SL(2,\mathbb{R})$-invariant probability measures supported on $SL(2,\mathbb{R})$-orbits of *algebraically primitive* Veech surfaces are cuspidal Lagrangian (Lemma 6.5). We are grateful to P. Hubert for explaining to us the proof of this basic fact. Our criterion therefore implies a corollary of formulas by I. Bouw and M. Möller [7] which states that the Kontsevich–Zorich cocycle is nonuniformly hyperbolic for all algebraically primitive Veech surfaces (Corollary 6.6). In fact, in Section 6.1 we prove nonuniform hyperbolicity for a wider class of Veech surfaces, the Veech surfaces of *maximal homological rank* (Definition 6.3). In addition to algebraically primitive Veech surfaces, this class also includes many geometrically primitive examples which are not algebraically primitive, namely all the primitive Prym eigenforms of genus 3 and 4 constructed by C. McMullen in [34], as well as many non geometrically primitive Veech surfaces, for instance all the nonprimitive Prym eigenforms (Lemma 6.7). The Kontsevich–Zorich cocycle is thus nonuniformly hyperbolic with respect to all $SL(2,\mathbb{R})$-invariant probability measures given by Prym eigenforms (Corollary 6.8). We are grateful to P. Hubert who suggested to test our criterion on Prym eigenforms, as a main example of geometrically primitive, nonalgebraically primitive Veech surfaces.

All canonical $SL(2,\mathbb{R})$-invariant absolutely continuous invariant probability measures on connected components of strata of abelian differentials are cuspidal Lagrangian (Lemma 6.9). Our criterion therefore implies the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle with respect to all canonical measures on the moduli space of abelian differentials (Corollary 6.10), a result proved in [17] (see also [4]). The argument given here for this case is in fact a simplified and streamlined version of the original argument.

There are several interesting $SL(2,\mathbb{R})$-invariant probability measures to which our criteria may be applied. One of the most interesting in our opinion is given by the algebraic (singular) measures on the moduli space of abelian differentials coming from canonical measures on strata of quadratic differentials by the standard double (orienting) cover construction. This application has been carried out recently by R. Treviño who has derived a proof that all such measures are nonuniformly hyperbolic [41].

This paper is organized as follows. In Section 2 we recall the variational formulas for the exponents proved in [17]. In Section 3, we prove a transversality result for the unstable space of the Kontsevich–Zorich cocycle with respect to integral Lagrangian subspaces in homology. This is a crucial improvement over [17]. In Section 4, we prove following [17] the key asymptotic formulas near appropriate boundary points of the moduli space. Section 5 is devoted to carrying out the proof of the the main theorem on the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle. Finally, Section 6 presents two fundamental applications. The first, in Section 6.1, is to a wide class of Veech surfaces, which includes algebraically primitive Veech surfaces already covered by Bouw–Möller results, but it also applies to many other cases. The second, in Section 6.2, is a review of the case of canonical measures on connected components of strata of abelian differential, already treated in [17]. The argument we present here is somewhat simpler and hopefully provides a template for other cases, such as measures coming from strata of quadratic differentials.

2. FORMULAS FOR THE EXPONENTS

We recall below for the convenience of the reader, the relevant formulas for all partial sums of the Kontsevich–Zorich exponents. Such formulas were first derived in [17] (see also [18, 29, 45]) as a generalization of the Kontsevich–Zorich formula for the sum of all the exponents. The exposition below follows [21].

Let S be a Riemann surface. The natural Hermitian intersection form on the complex cohomology $H^1(S,\mathbb{C})$ of the Riemann surface can be defined on closed 1-forms representing cohomology classes as

$$(\omega_1,\omega_2) := \frac{i}{2}\int_S \omega_1 \wedge \bar{\omega}_2. \tag{2.1}$$

Restricted to the subspace $H^{1,0}(S,\mathbb{C})$ of holomorphic 1-forms, this induces a positive-definite Hermitian form. Hence, by the Hodge representation theorem, it induces a positive-definite bilinear form on the cohomology $H^1(S,\mathbb{R})$. The real Hodge bundle H^1_g (over the moduli space of abelian differentials) is thus endowed with an inner product, called the *Hodge inner product* and a norm, called the *Hodge norm*.

Given a cohomology class $c \in H^1(S,\mathbb{R})$, let ω_c be the unique holomorphic 1-form such that $c = [\mathrm{Re}(\omega_c)]$. Define $*c$ to be the real cohomology class $[\mathrm{Im}(\omega_c)]$. The Hodge norm $\|c\|$ satisfies

$$\|c\|^2 = \frac{i}{2}\int_S \omega_c \wedge \bar{\omega}_c = \int_S \mathrm{Re}[\omega_c] \wedge \mathrm{Im}[\omega_c],$$

or, in other words, $\|c\|^2$ is the value of $(c \cdot *c)$ on the fundamental cycle. The operator $c \mapsto *c$ on the real cohomology $H^1(S,\mathbb{R})$ of a Riemann surface S is called the *Hodge operator*.

Associated to any holomorphic 1-form ω on the Riemann surface S, there is a naturally defined space $L^2_\omega(S)$ defined as the completion of smooth functions on S with respect to the topology generated by the inner product

$$\langle f,g\rangle_\omega = \frac{i}{2}\int_S f\,\bar g\,\omega\wedge\bar\omega \qquad \text{for any } f,g\in C^\infty(S).$$

In other words, the space $L^2_\omega(S)$ is the standard space of square-integrable functions with respect to the area form $\frac{i}{2}\omega\wedge\bar\omega$ on S.

The subspaces $\mathcal{M}_\omega \subset L^2_\omega(S)$ and $\bar{\mathcal{M}}_\omega \subset L^2_\omega(S)$ of meromorphic and antimeromorphic functions, respectively, are both finite-dimensional, with dimension equal to the genus of the surface. In fact, they can be described as follows. Let $H^{1,0}(S)$ and $H^{0,1}(S)$ be the spaces of holomorphic and antiholomorphic 1-forms on S, respectively. Both have dimension equal to the genus of the surface by Riemann–Roch Theorem. The following characterization of the subspaces $\mathcal{M}_\omega$ and, consequently, of $\bar{\mathcal{M}}_\omega$, holds (see [16, 17]):

$$\mathcal{M}_\omega = \left\{\tilde\omega/\omega \mid \tilde\omega\in H^{1,0}(S)\right\}.$$

We remark the space $\mathcal{M}_\omega$ endowed with the norm of the Hilbert space $L^2_\omega(S)$ is isometric to the space $H^{1,0}(S)$ endowed with the restriction of the hermitian intersection form (2.1). In fact,

$$\left\langle \frac{\omega_1}{\omega},\frac{\omega_2}{\omega}\right\rangle_\omega = (\omega_1,\omega_2), \qquad \text{for all } \omega_1,\omega_2\in H^{1,0}(S).$$

Let $\pi_\omega\colon L^2_\omega(S)\to\bar{\mathcal{M}}_\omega$ denote the orthogonal projection and let H_ω be the following positive-semidefinite hermitian form on $H^{1,0}(S)$:

$$H_\omega(\omega_1,\omega_2) := \left\langle \pi_\omega\Big(\frac{\omega_1}{\omega}\Big),\pi_\omega\Big(\frac{\omega_2}{\omega}\Big)\right\rangle_\omega \qquad \text{for all } \omega_1,\omega_2\in H^{1,0}(S).$$

Let B_ω be the complex bilinear form on $H^{1,0}(S)$ defined as follows:

$$B_\omega(\omega_1,\omega_2) := \left\langle \frac{\omega_1}{\omega},\frac{\bar\omega_2}{\bar\omega}\right\rangle_\omega \qquad \text{for all } \omega_1,\omega_2\in H^{1,0}(S). \tag{2.2}$$

The geometric significance of the forms B_ω and H_ω is related to notions of *second fundamental form* and *curvature* of the Hodge bundle (see [21, §1]). Their significance for the dynamics of the Kontsevich–Zorich cocycle lies in the variational formulas proved in [17, Sections 2–5], which give the variation of the Hodge norm on the (real) Hodge bundle H^1_g under the action of the Kontsevich–Zorich cocycle (see also [18, 21]).

The forms H_ω and B_ω are related as follows (see [17, §4]). Let $\{\omega_1,\dots,\omega_g\}$ be any orthonormal basis of $H^{1,0}(S)$. The restriction of the projection operator $\pi_\omega\restriction_{\mathcal{M}_\omega}$ is then given by the following formula:

$$\pi_\omega\Big(\frac{\tilde\omega}{\omega}\Big) = \sum_{i=1}^{g} B_\omega(\tilde\omega,\omega_i)\bar\omega_i \qquad \text{for all } \tilde\omega\in H^{1,0}(S).$$

It follows that the matrices of H and B of the forms H_ω and B_ω respectively, with respect to any orthonormal basis of the space $H^{1,0}(S)$ are related by the following identity:

$$H = BB^*. \tag{2.3}$$

In particular, the forms H_ω and B_ω have the same rank and their eigenvalues are related. Note that the above formula (2.3) is the correct version of the formula $H = B^*B$ that appears as formula (4.3) in [17] and as formula (44) in [18]. This mistake is of no consequence.

Let $\mathrm{EV}(H_\omega)$ and $\mathrm{EV}(B_\omega)$ denote the set of eigenvalues of the forms H_ω and B_ω, respectively. The following identity holds:

$$\mathrm{EV}(H_\omega) = \{|\lambda|^2 : \lambda \in \mathrm{EV}(B_\omega)\}.$$

By the Hodge representation theorem for Riemann surfaces (which states that any real cohomology class can be represented as the real or imaginary part of a holomorphic 1-form), the forms H_ω and B_ω on $H^{1,0}(S)$ can be interpreted as bilinear forms, denoted by $H_\omega^{\mathbb{R}}$ and $B_\omega^{\mathbb{R}}$, respectively, on the real cohomology $H^1(S,\mathbb{R})$. The forms $H_\omega^{\mathbb{R}}$ and $B_\omega^{\mathbb{R}}$ on $H^1(S,\mathbb{R})$ have the same rank, which is equal to twice the common rank of the forms H_ω and B_ω on $H^{1,0}(S)$. The form $H_\omega^{\mathbb{R}}$ is real-valued and positive-semidefinite, while the form $B_\omega^{\mathbb{R}}$ is complex-valued. For every holomorphic differential $\omega \in \mathcal{H}_g$, the eigenvalues of the positive-semidefinite form $H_\omega^{\mathbb{R}}$ on $H^1(S,\mathbb{R})$ will be denoted as follows:

$$\Lambda_1(\omega) \equiv 1 \geq \Lambda_2(\omega) \geq \cdots \geq \Lambda_g(\omega) \geq 0.$$

We remark that the above eigenvalues induce well-defined continuous nonnegative bounded functions on the moduli space of all abelian differentials.

In [17], Sections 2, 3, and 5, several variational formulas for the Hodge norm on the real cohomology bundle along trajectories of the Teichmüller flow were proved and formulas for the Lyapunov exponents of the Kontsevich–Zorich cocycle were derived. These generalize the fundamental Kontsevich–Zorich formula for the sum of all Lyapunov exponents [27]. The formulas are written in [17] with different notational conventions. In fact, as explained above, any abelian holomorphic differential ω on a Riemann surface S induces an isomorphism between the space $H^{1,0}(S)$ of all abelian holomorphic differentials on S, endowed with the Hodge norm, and the subspace $\mathcal{M}_\omega \subset L^2_\omega(S)$ of all square integrable meromorphic functions (with respect the area form of the abelian differential ω on S). In [17, 18], variational formulas are written in the language of meromorphic functions. We will adopt here the language of holomorphic abelian differentials.

For any holomorphic abelian differential ω on a Riemann surface S, we define functions on the Grassmannian $G_k(S,\mathbb{R})$ of k-dimensional isotropic subspaces of $H^1(S,\mathbb{R})$ as follows. Let $I_k \subset H^1(S,\mathbb{R})$ be any isotropic subspace (with respect to the intersection form) of dimension $k \in \{1,\ldots,g\}$. Let $\{c_1,\ldots,c_k\} \subset I_k$ be any Hodge-orthonormal basis and let

$$\{c_1,\ldots,c_k,c_{k+1},\ldots,c_g\} \subset H^1(S,\mathbb{R})$$

be any Hodge-orthonormal Lagrangian completion. Let

$$\Phi_k(\omega, I_k) := \sum_{i=1}^{g} \Lambda_i(\omega) - \sum_{i,j=k+1}^{g} \left|B_\omega^{\mathbb{R}}(c_i,c_j)\right|^2. \tag{2.4}$$

We remark that the above definition is independent of the choice of the orthonormal basis $\{c_1,\dots,c_k\}\subset I_k$ and of its Hodge-orthonormal Lagrangian completion $\{c_1,\dots,c_k,c_{k+1},\dots,c_g\}$. The function Φ_k is therefore well-defined and equivariant under the action of the mapping-class group on the Grassmannian bundle of the Hodge bundle over the Teichmüller space. It induces therefore a function on the Grassmannian bundle of the Hodge bundle over the moduli space of all abelian differentials.

REMARK 2.1. The function Φ_g is the pullback to the Grassmannian bundle of Lagrangian subspaces of the Hodge bundle H^1_g of a function on the moduli space. In other terms, it has no dependence on the Lagrangian subspace. In fact, for every $\omega\in\mathcal{H}_g$ and every Lagrangian subspace $I_g\subset H^1(S,\mathbb{R})$,

$$\Phi_g(\omega,I_g):=\sum_{i=1}^{g}\Lambda_i(\omega). \tag{2.5}$$

This fundamental fact (discovered in [27]) is crucial for the validity of the Kontsevich–Zorich formula for the sum of exponents. A version of the formula is given below.

The functions Φ_k arise in the computation of the hyperbolic Laplacian of the Hodge norm of isotropic polyvectors along Teichmüller disks.

Let $\{c_1,\dots,c_k\}\subset I_k$ be any Hodge-orthonormal basis of an isotropic subspace $I_k\subset H^1(S,\mathbb{R})$ on a Riemann surface S. The Euclidean structure defined by the Hodge scalar product on $H^1(S,\mathbb{R})$ defines the natural norm $\|c_1\wedge\cdots\wedge c_k\|_\omega$ of any polyvector which we also call the Hodge norm. Similarly to the case of the Hodge norm it is defined only by the complex structure of the underlying Riemann surface. Thus, for any (S_0,ω_0) in $\mathcal{H}_g$ the Hodge norm $\|c_1\wedge\cdots\wedge c_k\|_\omega$ defines a smooth function on the hyperbolic surface obtained as a left quotient $SO(2,\mathbb{R})\setminus SL(2,\mathbb{R})\omega_0$ of the orbit of ω_0. Variation formulas for the Hodge norm of polyvectors were established in [17, Lemmas 5.2 and 5.2'].

LEMMA 2.2. *For all $k\in\{1,\dots,g\}$, the following formula holds:*

$$\triangle\log\|c_1\wedge\cdots\wedge c_k\|_\omega=2\Phi_k(\omega,I_k).$$

From the variational formula of Lemma 2.2, it is possible by integration to derive a formula for the average growth of the spherical averages of the Hodge norm of any isotropic polyvector on any Teichmüller disk.

LEMMA 2.3. *Let ω_0 be an abelian differential on the Riemann surface S_0. For any k-dimensional isotropic subspace $I_k\subset H^1(S_0,\mathbb{R})$ and any basis $\{c_1,\dots,c_k\}\subset I_k$, let $\|c_1\wedge\cdots\wedge c_k\|_\omega$ denote the Hodge norm of the polyvector $c_1\wedge\cdots\wedge c_k$ at $(S,\omega)\in\mathcal{H}_g$ for any $\omega\in SO(2,\mathbb{R})\setminus SL(2,\mathbb{R})\omega_0$. Let $D_t(\omega_0)$ denote the disk of hyperbolic radius $t>0$ centered at the origin $SO(2,\mathbb{R})\omega_0$ of the hyperbolic surface $SO(2,\mathbb{R})\setminus SL(2,\mathbb{R})\omega_0$, let $|D_t|$ denote its hyperbolic area and let $d\mathcal{A}_P$ denote the Poincaré area element. We have that*

$$\left(\frac{1}{2\pi}\right)\frac{\partial}{\partial t}\int_0^{2\pi}\log\|c_1\wedge\cdots\wedge c_k\|_\omega\,d\theta=\frac{\tanh(t)}{|D_t|}\int_{D_t(\omega_0)}\Phi_k(\omega,I_k)\,d\mathcal{A}_P. \tag{2.6}$$

Proof. The formula was derived in [17] as formula (5.10). It follows from the variational formula of Lemma 2.2 by Green's formula or explicit integration of the Poisson equation for the hyperbolic Laplacian on the Poincaré disk (as in [17, Lemma 3.1]). □

By formula (2.6), it is possible to derive formulas for the partial sums of the Lyapunov exponents of the Kontsevich–Zorich cocyle. Here is our main result.

THEOREM 2.4. *Let μ be any $\mathrm{SL}(2,\mathbb{R})$-invariant Borel probability ergodic measure on the moduli space $\mathcal{H}_g$ of normalized abelian differentials. Assume that $\lambda_k^\mu > \lambda_{k+1}^\mu$, for $k \in \{1,\dots,g-1\}$, and let E_k^+ denote the Oseledec's subbundle carrying the subset $\{\lambda_1^\mu,\dots,\lambda_k^\mu\}$ of the Lyapunov spectrum. Then the following holds:*

$$\lambda_1^\mu + \cdots + \lambda_k^\mu = \int_{\mathcal{H}_g} \Phi_k\big(\omega, E_k^+(\omega)\big)\, d\mu(\omega). \tag{2.7}$$

Proof (see [17, Corollary 5.5]). For any given $(S,\omega) \in \mathcal{H}_g$, let σ_ω^k denote the normalized canonical (Haar) measure on the Grassmannian $G_k(S,\mathbb{R})$ of isotropic k-dimensional subspaces $I_k \subset H^1(S,\mathbb{R})$, endowed with the Euclidean structure given by the Hodge inner product. The probability measure σ_ω^k is invariant under the action of the circle group $\mathrm{SO}(2,\mathbb{R})$ on $\mathcal{H}_g$. Let $(t,\theta) \in \mathbb{R}^+ \times S^1$ denote the geodesic polar coordinates centered at the origin $\mathrm{SO}(2,\mathbb{R})\omega$ on the hyperbolic surface $\mathrm{SO}(2,\mathbb{R})\backslash \mathrm{SL}(2,\mathbb{R})\omega$, let $\mathrm{dist}_{(t,\theta)}^k$ denote the Hodge distance on the Grassmannian of isotropic k-dimensional subspaces, and let $E_k^+(t,\theta)$ denote the unstable Oseledec's subspace at the abelian differential $\omega_{(t,\theta)} \in \mathrm{SO}(2,\mathbb{R})\backslash \mathrm{SL}(2,\mathbb{R})\omega$.

By Oseledec's Theorem, for μ-almost all $(S,\omega) \in \mathcal{H}_g$, almost all $\theta \in S^1$ and σ_ω^k-almost all k-dimensional isotropic subspaces $I_k \in G_k(S,\mathbb{R})$,

$$\lim_{t\to+\infty} \mathrm{dist}_{(t,\theta)}^k\big(I_k, E_k^+(t,\theta)\big) = 0.$$

Consequently, since the function Φ_k is bounded and continuous, by Fubini's Theorem and the Dominated-Convergence Theorem, for μ-almost all $\omega \in \mathcal{H}_g$,

$$\lim_{t\to+\infty} \frac{1}{|D_t|}\left[\int_{G_k(S,\mathbb{R})}\int_{D_t(\omega)} \Big|\Phi_k\big(\omega_{(s,\theta)}, I_k\big) - \Phi_k\big(\omega_{(s,\theta)}, E_k^+(s,\theta)\big)\Big|\, d\mathcal{A}_P(s,\theta)\, d\sigma_\omega^k\right] = 0. \tag{2.8}$$

For any $(S,\omega) \in \mathcal{H}_g$ and any isotropic subspace $I_k \in G_k(S,\mathbb{R})$, the Hodge norm $\|c_1 \wedge \cdots \wedge c_k\|_\omega$ does not depend on the Hodge orthonormal basis $\{c_1,\dots,c_k\} \subset I_k$, hence it defines a function on the Grassmannian bundle of k-dimensional isotropic subspaces of the (real) Hodge bundle. By formula (2.8) and by averaging formula (2.6) over $G_k(S,\mathbb{R})$ with respect to the measure σ_ω^k, then with respect to the $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure μ on $\mathcal{H}_g$ and by applying Fubini's Theorem and the Dominated-Convergence Theorem, we find that, since E_k^+ is an invariant subbundle of the Kontsevich–Zorich cocycle, the following formula holds:

$$\lim_{t\to+\infty}\left[\frac{\partial}{\partial t}\int_{\mathcal{H}_g}\int_{G_k(S,\mathbb{R})}\log\|c_1\wedge\cdots\wedge c_k\|_{\omega_{(t,\theta)}}\,d\sigma_\omega^k d\mu(\omega)\right.$$
$$\left.-\tanh(t)\int_{\mathcal{H}_g}\Phi_k\big(\omega,E_k^+(\omega)\big)\,d\mu(\omega)\right]=0.$$

Finally, by averaging over $[0,T]$ with respect to the radial coordinate $t>0$ (Teichmüller time) and by Oseledec's Theorem, we obtain formula (2.7). □

In the case $k=g$, since Φ_g does not depend on the Lagrangian subspace of the cohomology, Theorem 2.4 reduces to a version of the Kontsevich–Zorich formula for the sum of all nonnegative Lyapunov exponents (see [27] and [17, Corollary 5.3]).

COROLLARY 2.5. *Let μ be any* $\mathrm{SL}(2,\mathbb{R})$*-invariant Borel probability ergodic measure on the moduli space $\mathcal{H}_g$ of normalized abelian differentials. Then,*

$$\lambda_1^\mu+\cdots+\lambda_g^\mu=\int_{\mathcal{H}_g}(\Lambda_1+\cdots+\Lambda_g)\,d\mu.$$

3. LAGRANGIAN ABELIAN DIFFERENTIALS

The following crucial transversality result holds.

LEMMA 3.1. *The unstable bundle E^+ of the Kontsevich–Zorich cocycle is μ-almost everywhere transverse to all integral Lagrangian subspaces, with respect to any* $\mathrm{SL}(2,\mathbb{R})$*-invariant ergodic probability measure μ on a stratum $\mathcal{H}(\kappa)$, that is, for μ-almost all $(S,\omega)\in\mathcal{H}(\kappa)$ and for any integral Lagrangian subspace $\Lambda\subset H^1(S,\mathbb{R})$,*

$$E^+(\omega)\cap\Lambda=\{0\}.$$

Proof. Let μ be any $\mathrm{SL}(2,\mathbb{R})$-invariant Borel probability measure μ on $\mathcal{H}_g$. By the Oseledec and Fubini Theorems, for μ-almost all $(S,\omega)\in\mathcal{H}_g$ the Oseledec's stable/unstable spaces $E^\pm(e^{i\theta/2}\omega)\subset H^1(S,\mathbb{R})$ are well-defined for (Lebesgue) almost all $\theta\in S^1$ and the following identity holds:

$$E^+(e^{i\theta/2}\omega)=E^-(e^{i(\theta+\pi)/2}\omega)\qquad\text{for almost all }\theta\in S^1.$$

It follows that the unstable bundle E^+ is μ-almost everywhere transverse to all integral Lagrangian subspaces if and only if the stable subbundle E^- is.

Let $(S,\omega)\in\mathcal{H}_g$ and let $(t,\theta)\in\mathbb{R}^+\times S^1$ denote the geodesic polar coordinate centered at the origin $\mathrm{SO}(2,\mathbb{R})\omega$ on the hyperbolic surface $\mathrm{SO}(2,\mathbb{R})\backslash\mathrm{SL}(2,\mathbb{R})\omega$. Let $E^-(t,\theta)$ denote the stable Oseledec's space at $\omega_{(t,\theta)}\in\mathrm{SO}(2,\mathbb{R})\backslash\mathrm{SL}(2,\mathbb{R})\omega$. Let $\Lambda\subset H^1(S,\mathbb{R})$ be any Lagrangian subspace and let $\{c_1,\dots,c_g\}\subset\Lambda$ be any basis. By Oseledec's Theorem, for μ-almost all $\omega\in\mathcal{H}_g$ and for almost all $\theta\in S^1$, if $\Lambda\cap E^-(0,\theta)\neq\{0\}$, then

$$\lim_{t\to+\infty}\frac{1}{t}\log\|c_1\wedge\cdots\wedge c_g\|_{\omega_{(t,\theta)}}<\lambda_1^\mu+\cdots+\lambda_g^\mu. \tag{3.1}$$

Thus, for all μ-almost all $\omega \in H_g$, if the set

$$\mathcal{N}_\Lambda^-(\omega) := \{\theta \in S^1 \mid \Lambda \cap E^-(0,\theta) \neq \{0\}\} \subset S^1 \tag{3.2}$$

has positive Lebesgue measure then, by averaging the limit in formula (3.1) over the circle and applying the Lebesgue Dominated-Convergence Theorem, we conclude that

$$\lim_{t\to+\infty} \frac{1}{2\pi t} \int_0^{2\pi} \log \| c_1 \wedge \cdots \wedge c_g \|_{\omega(t,\theta)} \, d\theta < \lambda_1^\mu + \cdots + \lambda_g^\mu. \tag{3.3}$$

However, since the function $\Phi_g \colon \mathrm{SO}(2,\mathbb{R})\backslash\mathcal{H}_g \to \mathbb{R}^+$ defined in formulas (2.4) and (2.5) is bounded, by the Pointwise Ergodic Theorem for $\mathrm{SL}(2,\mathbb{R})$-balls of A. Nevo and E. M. Stein [38] and by the Kontsevich–Zorich formula (see Corollary 2.5), it follows that, for μ-almost all $(S,\omega) \in \mathcal{H}_g$,

$$\lim_{t\to+\infty} \frac{1}{|D_t|} \int_{D_t(\omega)} \Phi_g \, d\mathcal{A}_P = \int_{\mathcal{H}_g} \Phi_g \, d\mu = \lambda_1^\mu + \cdots + \lambda_g^\mu. \tag{3.4}$$

By averaging formula (2.6) over $[0,t]$ with respect to the radial coordinate and by formula (3.4), we derive that for μ-almost all $(S,\omega) \in \mathcal{H}_g$, for any Lagrangian subspace $\Lambda \subset H^1(S,\mathbb{R})$, and for any basis $\{c_1,\ldots,c_g\} \subset \Lambda$,

$$\lim_{t\to+\infty} \frac{1}{2\pi t} \int_0^{2\pi} \log \| c_1 \wedge \cdots \wedge c_g \|_{\omega(t,\theta)} \, d\theta = \lambda_1^\mu + \cdots + \lambda_g^\mu.$$

This implies that, for μ-almost all $(S,\omega) \in \mathcal{H}_g$ and for any Lagrangian subspace $\Lambda \in H^1(S,\mathbb{R})$, the set $\mathcal{N}_\Lambda^-(\omega)$ defined in formula (3.2) has zero Lebesgue measure, otherwise the inequality (3.3) holds. Since the set of all integral Lagrangian subspaces is countable, it follows by the Fubini Theorem that the stable subbundle E^- is μ-almost everywhere transverse to all integral Lagrangian subspaces. □

Lemma 8.4 of [17] is replaced by the following statement.

LEMMA 3.2. *For every cuspidal Lagrangian* $\mathrm{SL}(2,\mathbb{R})$*-invariant ergodic probability measure* μ *on a stratum* $\mathcal{H}(\kappa)$ *of the moduli space of abelian differentials, there exists an abelian differential* $\omega_0 \in \mathrm{supp}(\mu)$ *that satisfies the following:*

1. *The vertical foliation* $\mathcal{F}_{\omega_0}^v$ *is Lagrangian.*
2. *The differential* ω_0 *is a density point of a compact positive measure set* $\mathcal{P}_\kappa \subset \mathcal{H}(\kappa)$ *such that*
   (a) *all* $\omega \in \mathcal{P}_\kappa$ *are Oseledec's regular points for the Kontsevich–Zorich cocycle and the unstable subspace* $E^+(\omega)$ *of the cocycle depend continuously on* $\omega \in \mathcal{P}_\kappa$*, and*
   (b) *the Poincaré dual* $P(\mathcal{L}_0) \subset H^1(S,\mathbb{R})$ *of the Lagrangian subspace* $\mathcal{L}_0 := \mathcal{L}(\mathcal{F}_{\omega_0}^v)$*, generated by the regular trajectories of the vertical foliation* $\mathcal{F}_{\omega_0}^v$*, is transverse to* $E^+(\omega)$ *for all* $\omega \in \mathcal{P}_\kappa$*.*

Proof. Let $\omega^- \in \mathrm{supp}(\mu)$ be an abelian differential with Lagrangian vertical foliation $\mathcal{F}$. Let $\mathcal{U} \subset \mathcal{H}(\kappa)$ be a neighborhood of ω^- of product type on which the measure μ has a product structure. Let $\mathcal{R}_\kappa \subset \mathcal{H}(\kappa)$ be the set of Oseledec's regular points for the Kontsevich–Zorich cocycle. By Oseledec's and Luzin's Theorems and by the transversality Lemma 3.1, there exists a compact subset $\mathscr{R}^+ \subset \mathscr{R}_\kappa \cap \mathscr{U}$ of positive μ-measure such that the restriction $E^+ \upharpoonright_{\mathscr{R}^+}$ of the unstable space of the cocycle is continuous and transverse to the Poincaré dual $P(\mathscr{L}_{\mathscr{F}})$ of the Lagrangian subspace $\mathscr{L}_{\mathscr{F}} \subset H_1(S,\mathbb{R})$, generated by the homology classes of the regular leaves of the measured foliation $\mathscr{F}$.

Let $\omega^+ \in \mathscr{R}^+$ be any density point of $\mathscr{R}^+$ and let ω_0 be any abelian differential in $\mathscr{W}^+_{\mathscr{U}}(\omega^+) \cap \mathscr{W}^-_{\mathscr{U}}(\omega^-)$. By construction the vertical foliation $\mathscr{F}^v_{\omega_0}$ of the abelian differential $\omega_0 \in \mathscr{W}^-_{\mathscr{U}}(\omega^-)$ is Lagrangian. Let $\mathscr{K} \subset \mathscr{U}$ be a compact neighborhood of ω^- and let $\mathscr{P}_\kappa := \mathscr{W}^+_{\mathscr{U}}(\mathscr{R}^+) \cap \mathscr{W}^-_{\mathscr{U}}(\mathscr{K})$. Since the measure μ has a product structure on $\mathscr{U}$, it follows that for any neighborhood $\mathscr{V}$ of ω_0 in $\mathscr{U}$ the set $\mathscr{P}_\kappa \cap \mathscr{V}$ has positive μ-measure. In fact, there exist neighborhoods $\mathscr{V}^\pm \subset \mathscr{U}$ of $\omega^\pm$ such that

$$\mathscr{W}^+_\kappa(\mathscr{V}^+) \cap \mathscr{W}^-_\kappa(\mathscr{V}^-) \subset \bigcup_{t\in\mathbb{R}} g_t(\mathscr{V}).$$

By construction, $\mu(\mathscr{R}^+ \cap \mathscr{V}^+) > 0$, since ω^+ is a density point of $\mathscr{R}^+$; moreover, $\mu(\mathscr{V}^-) > 0$, since $\omega^- \in \operatorname{supp}(\mu)$. Hence,

$$\mathscr{W}^+_{\mathscr{U}}(\mathscr{R}^+ \cap \mathscr{V}^+) \cap \mathscr{W}^-_{\mathscr{U}}(\mathscr{V}^-) \subset \bigcup_{t\in\mathbb{R}} g_t(\mathscr{P}_\kappa \cap \mathscr{V})$$

has positive μ-measure, which implies that $\mathscr{P}_\kappa \cap \mathscr{V}$ has positive μ-measure by the Teichmüller invariance of the measure.

The unstable bundle E^+ of the Kontsevich–Zorich cocycle is locally constant along the leaves of the foliation $\mathscr{W}^+_\kappa$ in the following sense. Let $\mathscr{U} \subset \mathscr{H}(\kappa)$ be any open set of product type. For any Oseledec's regular point $\omega \in \mathscr{U}$ and for any $\tilde\omega \in \mathscr{W}^+_{\mathscr{U}}(\omega)$, the unstable space $E^+(\tilde\omega) = E^+(\omega)$. This property can be derived from the representation theorem (see [17, Thm. 8.3]), which states that the unstable space $E^+(\omega)$ can be identified with the space of cohomology classes of all basic currents in the dual Sobolev space $\mathscr{H}^{-1}_\omega(S)$ for the horizontal foliation $\mathscr{F}^h_\omega$. It follows that the function $E^+ \upharpoonright_{\mathscr{P}_\kappa}$ is continuous and transverse to $P(\mathscr{L}_{\mathscr{F}})$, since by construction $\mathscr{W}^+_{\mathscr{U}}(\mathscr{P}_\kappa) = \mathscr{W}^+_{\mathscr{U}}(\mathscr{R}^+)$ and since the function $E^+ \upharpoonright_{\mathscr{R}^+}$ is continuous and transverse to $P(\mathscr{L}_{\mathscr{F}})$. The argument is complete. □

4. ASYMPTOTIC FORMULAS

In this section we compute the Hodge curvature near certain boundary points of the moduli space. Such boundary points are obtained by pinching a higher-genus surface along the waist curves of cylinders of a Lagrangian foliation, so they are represented by Riemann surfaces with nodes made of one or several punctured Riemann spheres carrying a meromorphic differential with at least $2g$ paired poles.

Let $\{a_1,\dots,a_g,b_1,\dots,b_g\}$ be a system of curves defining a canonical basis of the (real) homology (a marking) of the topological surface S and let $\{(S_\tau,\omega_\tau) \mid \tau \in [\mathbb{C}\setminus\{0\}]^{g+s}\}$ be any smooth family of holomorphic differentials (the differential ω_τ is holomorphic on the Riemann surface S_τ) obtained by pinching the system of nonintersecting curves

$$\{a_1,\dots,a_g,a_{g+1},\dots,a_{g+s}\},$$

which converges (projectively) as $\tau\to 0$ in $\mathbb{C}^{g+s}$ to a meromorphic differential ω_0 with $2g$ paired poles on a Riemann surface with nodes S_0 which is the union of s punctured Riemann spheres.

The following results give the asymptotics of the period matrix and of its derivative in the direction of the Teichmüller flow as the pinching parameter $\tau\to 0$ along the deformation described above. Let us recall that for any Riemann surface R and for any canonical basis of the homology $H_1(R,\mathbb{Z})$, the period matrix Π is the $g\times g$ symmetric complex matrix with positive-definite imaginary part defined as follows. Let $\{\theta_1,\dots,\theta_g\}$ be the unique basis of holomorphic differentials on R, dual to the canonical homology basis $\{[a_1],\dots,[a_g],[b_1],\dots,[b_g]\}$, in the sense that

$$\int_{a_i}\theta_j=\delta_{ij}\qquad \text{for all } i,j\in\{1,\dots,g\}.$$

The period matrix of the Riemann surface R is then given by

$$\Pi_{ij}:=\int_{b_i}\theta_j\qquad \text{for all } i,j\in\{1,\dots,g\}.$$

Let us fix a canonical homology basis $\{[a_1],\dots,[a_g],[b_1],\dots,[b_g]\}$ on the topological surface S and let $\Pi(\tau)$ denote the period matrix of the Riemann surface S_τ for all pinching parameters $\tau\in[\mathbb{C}\setminus\{0\}]^{g+s}$.

Let $\tau'=(\tau_1,\dots,\tau_g)$ denote the pinching parameters relative to the system of curves $\{a_1,\dots,a_g\}$ and let $\tau''=(\tau_{g+1},\dots,\tau_{g+s})$ be the pinching parameters relative to the system of curves $\{a_{g+1},\dots,a_{g+s}\}$.

LEMMA 4.1. *The family of $g\times g$ complex matrices*

$$\Pi_{ij}(\tau)-\frac{1}{2\pi\imath}\delta_{ij}\log|\tau'_i|$$

is bounded as the pinching parameter $\tau':=(\tau_1,\dots,\tau_g)\to(0,\dots,0)$ in $\mathbb{C}^g$, uniformly with respect to $\tau''\in\mathbb{C}^s$, $|\tau''|\le 1$.

Proof. The asymptotics follows from classical formulas for the period matrix near the boundary of the Deligne–Mumford compactification of the moduli space of abelian differentials (see for instance [15, III, p.54] or [46, §3, Cor. 6]). A similar formula for the particular case $s=0$, which corresponds to boundary points given by meromorphic differentials supported on a single punctured Riemann sphere (with $2g$ paired punctures), appears in [17, formula (4.33)]. □

The asymptotics of the Lie derivative $d\Pi/d\mu$ of the period matrix Π in the direction of the Teichmüller flow is given by the following result.

Let $p_1^\pm,p_2^\pm,\dots,p_g^\pm$ denote the $2g$ paired punctures on the punctured Riemann surface S_0 corresponding to the pinching of the curves $a_1,a_2,\dots,a_g$ and let $\pm\rho_1,\dots,\pm\rho_g\in\mathbb{C}\setminus\{0\}$ denote the residues of the limit meromorphic abelian differential ω_0 at $p_1^\pm,\dots,p_g^\pm$, respectively.

LEMMA 4.2. *The family of $g \times g$ complex matrices*

$$\frac{d\Pi}{d\mu}(\tau) - \frac{1}{2\pi}\frac{\bar{\rho}_i}{\rho_i}\delta_{ij}\log|\tau'_i| \tag{4.1}$$

is bounded as the pinching parameter $\tau' := (\tau_1, \ldots, \tau_g) \to (0, \ldots, 0)$ in $\mathbb{C}^g$, uniformly with respect to $\tau'' \in \mathbb{C}^s$, $|\tau''| \le 1$.

Proof. The proof is based on Lemmas 4.2 and 4.2' of [17]. In that paper, the asymptotic was proved for the particular case $s = 0$ of boundary points given by meromorphic differentials supported on a single punctured Riemann sphere (with $2g$ paired punctures). In the more general case, considered here, the limit points in the Deligne–Mumford boundary as $\tau' \to 0 \in \mathbb{C}^g$ consists of meromorphic differentials on pinched Riemann surfaces S_0 with possibly several parts (at most $s+1$) which are punctured Riemann spheres. The asymptotics claimed above follows immediately from [17, Lemma 4.2'], as $\tau' \to 0 \in \mathbb{C}^g$ while $\tau'' \in \mathbb{C}^s$ varies within a compact subset of the set

$$D''_0 := \{\tau'' \in \mathbb{C}^s : |\tau''| \le 1\} \cap \left(\bigcap_{i=1}^{s} \{\tau'' \in \mathbb{C}^s : \tau_{g+i} \ne 0\}\right).$$

Let $J \subset \{g+1, \ldots, g+s\}$ and let $\tau''_J = (\tau_j)_{j \in J} \in \mathbb{C}^{\#J}$. By [17, Lemma 4.2], the derivative of the period matrix $d\Pi(\tau)/d\mu$ has well-defined asymptotics as $\tau' \to 0 \in \mathbb{C}^g$ and $\tau''_J \to 0 \in \mathbb{C}^{\#J}$ under the condition that $|\tau''| \le 1$. Furthermore, the asymptotics of the matrix $d\Pi(\tau)/d\mu$ as $\tau' \to 0$ do not depend, up to uniformly bounded terms, on $\tau'' \in D''_0$ and are given by formula (4.1). Thus, the general asymptotics claimed above follow. □

We remark that it is also possible to give a more direct proof of Lemma 4.2 based on Lemma 4.2 in [17], along the lines of the proof of Lemma 4.2' in [17]. The main difference lies in the fact that in the general case the limit meromorphic differentials are supported on pinched Riemann surfaces composed of several punctured Riemann spheres. This feature makes essentially no difference in the calculations carried out in Lemma 4.2'.

COROLLARY 4.3. *If the residues $\rho_1, \ldots, \rho_g$ of the limit meromorphic differential ω_0 on S_0 are all real (and nonzero), then the following asymptotics hold*

$$\operatorname{Im}\Pi(\tau)^{-1/2}\left[\frac{d\Pi}{d\mu}(\tau)\right]\operatorname{Im}\Pi(\tau)^{-1/2} \to -I_g$$

as the pinching parameter $\tau' := (\tau_1, \ldots, \tau_g) \to (0, \ldots, 0)$ in $\mathbb{C}^g$, uniformly with respect to $\tau'' \in \mathbb{C}^s$, $|\tau''| \le 1$.

Let $P(\mathscr{A}) \subset H^1(M,\mathbb{R})$ be the Poincaré dual of the Lagrangian subspace $\mathscr{A} \subset H_1(M,\mathbb{R})$ generated by the system $\{[a_1], \ldots, [a_g]\}$ and let $G_{\mathscr{A}}(S,\mathbb{R})$ denote the open subset of the Grassmannian of all Lagrangian subspaces $\Lambda \subset H^1(M,\mathbb{R})$ transverse to the Poincaré dual $P(\mathscr{A}) \subset H^1(S,\mathbb{R})$.

LEMMA 4.4. *Assume that the residues $\rho_1, \ldots, \rho_g$ of the limit meromorphic differential ω_0 on S_0 are all real (and nonzero). For any $\Lambda \in G_{\mathscr{A}}(S,\mathbb{R})$, let $\mathscr{B}^{\Lambda}_{\tau} :=$*

$\{c_1^\Lambda(\tau),\ldots,c_g^\Lambda(\tau)\}$ *be any Hodge orthonormal basis of the Lagrangian subspace* $\Lambda\subset H^1(S,\mathbb{R})$ *on the Riemann surface* S_τ*, for any* $\tau\neq 0$*. The following limit holds*

$$B^{\mathbb{R}}_{\omega_\tau}\big(c_i^\Lambda(\tau),c_j^\Lambda(\tau)\big)\to -\delta_{ij}\qquad \text{as } \tau\to 0 \tag{4.2}$$

uniformly with respect to Λ *in any given compact subsets of* $G_{\mathscr{A}}(S,\mathbb{R})$ *and to any family* $\{\mathscr{B}_\tau\}$ *of Hodge orthonormal bases for* $\Lambda\in G_{\mathscr{A}}(S,\mathbb{R})$.

Proof. Let $\{\theta_1(\tau),\ldots,\theta_g(\tau)\}$ be the dual basis of holomorphic differentials on the Riemann surface S_τ, defined by the standard condition $\theta_i(a_j)=\delta_{ij}$, for all i, $j\in\{1,\ldots,g\}$. Let $\big\{c_1^\Lambda,\ldots,c_g^\Lambda\big\}\subset H^1(M,\mathbb{R})$ be a fixed basis of the Lagrangian subspace Λ, represented by the system of harmonic differentials $\{\mathrm{Re}\,\xi_1(\tau),\ldots,\mathrm{Re}\,\xi_g(\tau)\}$ on the Riemann surface S_τ. The differentials $\xi_1(\tau),\ldots,\xi_g(\tau)$ are holomorphic and form a basis of the space of holomorphic differentials on S_τ. As a consequence, there exists a complex $g\times g$ invertible matrix $\zeta(\tau):=(\zeta_{ij}(\tau))$ such that, for all $i\in\{1,\ldots,g\}$,

$$\xi_i(\tau)=\sum_{j=1}^{g}\zeta_{ij}(\tau)\,\theta_j(\tau). \tag{4.3}$$

Let α and β be the $g\times g$ real matrices defined by

$$\alpha_{ij}:=\int_{a_j}\mathrm{Re}\,\xi_i(\tau)\qquad\text{and}\qquad \beta_{ij}:=\int_{b_j}\mathrm{Re}\,\xi_i(\tau). \tag{4.4}$$

Since the harmonic differential $\mathrm{Re}\,\xi_i(\tau)$ represents the fixed cohomology class $c_i^\Lambda\in H^1(S,\mathbb{R})$, for all $i\in\{1,\ldots,g\}$, the matrices α and β do not depend on $\tau\in\mathbb{C}^{g+s}$. In addition, since $\Lambda\cap P(\mathscr{A})=\{0\}$, the matrix α is invertible. By the definitions of the dual basis $\{\theta_1(\tau),\ldots,\theta_g(\tau)\}$ and of the period matrix $\Pi(\tau)$ of the Riemann surface S_τ with respect to the canonical homology basis $\{[a_1],\ldots,[a_g],[b_1],\ldots,[b_g]\}$, we have

$$\mathrm{Re}\,\zeta(\tau)=\alpha\qquad\text{and}\qquad \mathrm{Im}\,\zeta(\tau)=(\alpha\,\mathrm{Re}\,\Pi(\tau)-\beta)(\mathrm{Im}\,\Pi(\tau))^{-1}. \tag{4.5}$$

Let $\{c_1^\Lambda(\tau),\ldots,c_g^\Lambda(\tau)\}\subset H^1(M,\mathbb{R})$ be any orthonormal basis of the subspace Λ with respect to the Hodge inner product on S_τ, represented by a orthonormal system of harmonic differentials $\{\mathrm{Re}\,h_1(\tau),\ldots,\mathrm{Re}\,h_g(\tau)\}$ on the Riemann surface S_τ. There exist a *real* invertible matrix $R(\tau):=(R_{ij}(\tau))$ and a complex invertible matrix $Z(\tau):=(Z_{ij}(\tau))$ such that the basis $\{h_1(\tau),\ldots,h_g(\tau)\}$ can be written in terms of the bases $\{\xi_1(\tau),\ldots,\xi_g(\tau)\}$ and $\{\theta_1(\tau),\ldots,\theta_g(\tau)\}$ of holomorphic differentials by the formulas

$$h_i(\tau)=\sum_{j=1}^{g}R_{ij}(\tau)\,\xi_j(\tau)=\sum_{j=1}^{g}Z_{ij}(\tau)\,\theta_j(\tau). \tag{4.6}$$

We remark that the matrix $Z(\tau)$ is the inverse of the matrix C of change of basis which appears in [17, formulas (4.5), (4.6), and (8.32)].

By (4.3) and (4.6) it follows that $Z(\tau)=R(\tau)\zeta(\tau)$. Hence, by (4.5),

$$\begin{aligned}\mathrm{Re}\,Z(\tau)&=R(\tau)\,\alpha,\\ \mathrm{Im}\,Z(\tau)&=R(\tau)\big(\alpha\,\mathrm{Re}\,\Pi(\tau)-\beta\big)\,\mathrm{Im}\,\Pi(\tau)^{-1}.\end{aligned}$$

Since the matrix α is invertible, we can write

$$\operatorname{Im} Z(\tau) = \operatorname{Re} Z(\tau)\big(\operatorname{Re}\Pi(\tau) - \alpha^{-1}\beta\big)\operatorname{Im}\Pi(\tau)^{-1}. \tag{4.7}$$

Let $B(\tau)$ be the symmetric complex matrix defined as follows:

$$B_{ij}(\tau) := B^{\mathbb{R}}_{\omega_\tau}\big(c_i^\Lambda(\tau), c_j^\Lambda(\tau)\big) \qquad \text{for all } i, j \in \{1,\dots,g\}. \tag{4.8}$$

By definition (2.2), the matrix $B(\tau)$ depends on the holomorphic differential $\omega_\tau \in \mathcal{H}(\kappa)$ as well as on the Hodge orthonormal basis $\{c_1^\Lambda(\tau),\dots,c_g^\Lambda(\tau)\}$ of the Lagrangian subspace $\Lambda \in G_{\mathscr{A}}(S,\mathbb{R})$.

Let $d\Pi(\tau)/d\mu$ denote the derivative of the period matrix in the direction of the Teichmüller flow at the holomorphic differential $\omega_\tau \in \mathcal{H}(\kappa)$. We claim that the following formula holds (see also [17, formula (8.32)]):

$$B(\tau) = Z(\tau)\left[\frac{d\Pi}{d\mu}(\tau)\right] Z(\tau)^t. \tag{4.9}$$

In fact, by Rauch's variational formula, for all $i, j \in \{1,\dots,g\}$,

$$B_{\omega_\tau}(\theta_i(\tau),\theta_j(\tau)) = \frac{\imath}{2}\int_S \theta_i(\tau)\theta_j(\tau)\,\frac{\bar\omega_\tau}{\omega_\tau} = \frac{d\Pi_{ij}}{d\mu}(\tau).$$

Since the form B_{ω_τ} is bilinear, it then follows from the identities (4.6) that

$$\begin{aligned} B_{ij}(\tau) &= \sum_{s,t=1}^{g} Z_{is}(\tau)Z_{jt}(\tau)B_{\omega_\tau}(\theta_s(\tau),\theta_t(\tau)) \\ &= \sum_{s,t=1}^{g} Z_{is}(\tau)\,Z_{jt}(\tau)\left[\frac{d\Pi_{st}}{d\mu}(\tau)\right]. \end{aligned}$$

According to a classical formula [14, III.2.3], for any basis $\{\theta_1,\dots,\theta_g\}$ dual to a canonical homology basis $\{[a_1],\dots,[a_g],[b_1],\dots,[b_g]\}$,

$$\frac{\imath}{2}\int_M \theta_i \wedge \overline{\theta}_j = \frac{\imath}{2}\sum_{s=1}^{g}\left[\int_{a_s}\theta_i\int_{b_s}\overline{\theta}_j - \int_{b_s}\theta_i\int_{a_s}\overline{\theta}_j\right] = \operatorname{Im}\Pi_{ij}.$$

Hence, the orthonormality condition on the basis $\{h_1(\tau),\dots,h_g(\tau)\}$ yields

$$\delta_{ij} = \frac{\imath}{2}\int_S h_i(\tau)\wedge\overline{h_j(\tau)} = \sum_{s,t=1}^{g} Z_{is}(\tau)\operatorname{Im}\Pi_{st}(\tau)\overline{Z_{jt}(\tau)}.$$

Hence, $Z(\tau)\operatorname{Im}\Pi(\tau)\,Z(\tau)^* = I_g$, equivalent to [17, (4.5)]. It follows that

$$\operatorname{Re} Z(\tau)\operatorname{Im}\Pi(\tau)\operatorname{Re} Z(\tau)^t + \operatorname{Im} Z(\tau)\operatorname{Im}\Pi(\tau)\operatorname{Im} Z(\tau)^t = I_g, \tag{4.10}$$

$$\operatorname{Im} Z(\tau)\operatorname{Im}\Pi(\tau)\operatorname{Re} Z(\tau)^t - \operatorname{Re} Z(\tau)\operatorname{Im}\Pi(\tau)\operatorname{Im} Z(\tau)^t = 0. \tag{4.11}$$

A calculation shows that (4.11) is equivalent to the symmetry of the matrix $\alpha^{-1}\beta$, a property which will play no role in the argument. We can rewrite (4.10) as follows. Let

$$\mathscr{E}(\tau) := \operatorname{Im}\Pi(\tau)^{-1/2}\big(\operatorname{Re}\Pi(\tau) - \alpha^{-1}\beta\big)\operatorname{Im}\Pi(\tau)^{-1/2}.$$

By formulas (4.7) and (4.10), we have

$$\operatorname{Re} Z(\tau)\operatorname{Im}\Pi(\tau)^{1/2}\big(I_g + \mathscr{E}(\tau)\mathscr{E}(\tau)^t\big)\operatorname{Im}\Pi(\tau)^{1/2}\operatorname{Re} Z(\tau)^t = I_g.$$

It follows from Lemma 4.1 that the matrix $\mathscr{E}(\tau) \to 0$ as $\tau \to 0$. Hence, the invertible matrix $\operatorname{Re} Z(\tau) \operatorname{Im}\Pi(\tau)^{1/2}$ converges to the compact subgroup $O(g,\mathbb{R}) \subset \mathrm{GL}(g,\mathbb{R})$ of $g \times g$ real orthogonal matrices. Hence, for all $\tau \in [\mathbb{C} \setminus \{0\}]^{g+s}$ near 0, there exist a orthogonal real matrix $O(\tau) \in O(g,\mathbb{R})$ and a matrix $\mathscr{E}'(\tau)$, which converges to 0 in $\mathrm{GL}(g,\mathbb{R})$ as $\tau \to 0$, such that

$$\operatorname{Re} Z(\tau) \operatorname{Im}\Pi(\tau)^{1/2} = O(\tau)\big(I + \mathscr{E}'(\tau)\big). \tag{4.12}$$

Since $O(g,\mathbb{R})$ is compact, it follows by (4.7) and (4.12) that, as $\tau \to 0$,

$$\operatorname{Im} Z(\tau) \operatorname{Im}\Pi(\tau)^{1/2} = O(\tau)\big(I + \mathscr{E}'(\tau)\big)\mathscr{E}(\tau) \to 0. \tag{4.13}$$

By formulas (4.12) and (4.13), it follows that

$$\mathscr{O}(\tau) := Z(\tau) \operatorname{Im}\Pi(\tau)^{1/2} \to O(g,\mathbb{R}) \qquad \text{as } \tau \to 0. \tag{4.14}$$

Since, by formula (4.9), the matrix $B(\tau)$ can be written as

$$B(\tau) = \mathscr{O}(\tau) \operatorname{Im}\Pi(\tau)^{-1/2} \left[\frac{d\Pi}{d\mu}(\tau)\right] \operatorname{Im}\Pi(\tau)^{-1/2} \mathscr{O}(\tau)^t,$$

the desired asymptotic formula (4.2) follows from the definition of the matrix $B(\tau)$ in formula (4.8), from formula (4.14), and from Corollary 4.3.

The convergence in (4.2) is uniform since as the Lagrangian subspace Λ varies in a compact subset of $G_{\mathscr{A}}(S,\mathbb{R})$, the basis $\{c_1^\Lambda, \dots, c_g^\Lambda\}$ of Λ (fixed with respect to the pinching parameter $\tau \in [\mathbb{C} \setminus \{0\}]^{g+s}$) can be chosen in such a way that the matrices α^{-1} and β, defined in (4.4), are uniformly bounded. Hence, the matrices $\mathscr{E}(\tau)$ and $\mathscr{E}'(\tau)$ converge to 0 uniformly as $\tau \to 0$. □

5. NON-UNIFORM HYPERBOLICITY

The proof of the main theorem proceeds by contradiction or contraposition based on the following corollary of the formulas for the Kontsevich–Zorich exponents given in Section 2.

LEMMA 5.1. *Let μ be a $\mathrm{SL}(2,\mathbb{R})$-invariant ergodic probability measure on a stratum $\mathscr{H}(\kappa)$ of the moduli space of abelian differential. If the Kontsevich–Zorich cocycle has $g-k$ zero exponents, that is, if*

$$\lambda_{k+1}^\mu = \cdots = \lambda_g^\mu = 0,$$

then for μ-almost all $\omega \in \mathscr{H}(\kappa)$ and for any Hodge-orthonormal isotropic system $\{c_{k+1}, \dots, c_g\} \subset H^1(S_\omega,\mathbb{R})$, Hodge orthogonal and symplectic orthogonal to the k-dimensional unstable space $E_k^+(\omega) \subset H^1(S_\omega,\mathbb{R})$ of the cocycle, the following holds:

$$B_\omega^{\mathbb{R}}(c_i, c_j) = 0$$

for all $i, j \in \{k+1, \dots, g\}$.

Proof. By the hypothesis on the exponents, it follows that

$$\lambda_1^\mu + \cdots + \lambda_k^\mu = \lambda_1^\mu + \cdots + \lambda_g^\mu.$$

Hence, by the formula on partial sums of exponents (see Theorem 2.4) and by the Kontsevich–Zorich formula (see Corollary 2.5), it follows that

$$\int_{\mathscr{H}_g} \Phi_k\big(\omega, E_k^+(\omega)\big)\, d\mu(\omega) = \int_{\mathscr{H}_g} \big(\Lambda_1(\omega) + \cdots + \Lambda_g(\omega)\big)\, d\mu(\omega). \tag{5.1}$$

By the definition of the functions Φ_k in formula (2.4), it follows that

$$\Phi_k\big(\omega, E_k^+(\omega)\big) \le \Lambda_1(\omega) + \cdots + \Lambda_g(\omega).$$

Hence, by the integral identity (5.1), equality holds μ-almost everywhere. It follows that for μ-almost all $\omega \in \mathscr{H}(\kappa)$ and for any Hodge orthonormal isotropic system $\{c_{k+1}, \dots, c_g\} \subset H^1(S_\omega, \mathbb{R})$, Hodge orthogonal and symplectic orthogonal to $E_k^+(\omega) \subset H^1(S_\omega, \mathbb{R})$,

$$\sum_{i,j=k+1}^{g} \left|B_\omega^{\mathbb{R}}(c_i, c_j)\right|^2 = 0,$$

which immediately implies the desired conclusion. □

We finally prove our main result.

Proof of Theorem 1.6. The measure μ on $\mathscr{H}(\kappa)$ is cuspidal Lagrangian, so there is a holomorphic abelian differential $\omega_0 \in \mathrm{supp}(\mu)$ such that the properties (1) and (2) stated in Lemma 3.2 hold.

Let $\{\omega_t \mid t \ge 0\}$ denote the forward Teichmüller orbit of the holomorphic differential ω_0 on S_0. For each $t \ge 0$, the differential ω_t is holomorphic on a (unique) Riemann surface S_t. Since ω_0 has Lagrangian vertical foliation, the *marked* abelian differential ω_t converges projectively as $t \to +\infty$ to a meromorphic differential on a union of punctured Riemann sphere with poles at all (paired) punctures, obtained by pinching the waist curves $\{a_1, \dots, a_{g+s}\}$ of all cylinders of the vertical foliation $\mathscr{F}^v_{\omega_0}$ on S_0 (see [30, Theorem 3]). Since only vertical waist curves are pinched, the limit differential has real nonzero residues at all punctures. Hence, Lemma 4.4 applies.

Let $\mathscr{A} := \mathscr{L}(\mathscr{F}^v_{\omega_0}) \subset H_1(S_0, \mathbb{R})$ be the Lagrangian subspace generated by the system $\{[a_1], \dots, [a_{g+s}]\} \subset H_1(S_0, \mathbb{R})$. By Lemma 3.2, the unstable manifold $E^+(\omega_0)$ is well-defined and transverse to the Poincaré dual $P(\mathscr{A}) \subset H^1(S_0, \mathbb{R})$. Thus there exists a Lagrangian subspace $\Lambda_0 \subset H^1(S_0, \mathbb{R})$ such that $E^+(\omega_0) \subset \Lambda_0$ and $\Lambda_0 \cap P(\mathscr{A}) = \{0\}$, that is $\Lambda_0 \in G_{\mathscr{A}}(S_0, \mathbb{R})$.

By Lemma 4.4, for any compact subset $\mathscr{G}_0 \subset G_{\mathscr{A}}(S_0, \mathbb{R})$, there exists $t(\mathscr{G}_0) > 0$ such that, for all $t \ge t(\mathscr{G}_0)$, for all $\Lambda \in \mathscr{G}_0$, and for any Hodge orthonormal basis $\{c_1^\Lambda(t), \dots, c_g^\Lambda(t)\}$ of $\Lambda \subset H^1(S_0, \mathbb{R})$,

$$\left|B^{\mathbb{R}}_{\omega_t}\big(c_i^\Lambda(t), c_j^\Lambda(t)\big) + \delta_{ij}\right| \le 1/4.$$

Let $\mathscr{G}_0 \subset G_{\mathscr{A}}(S_0, \mathbb{R})$ be any given compact neighborhood of the Lagrangian subspace $\Lambda_0 \subset H^1(S_0, \mathbb{R})$. Fix any $t > t(\mathscr{G}_0)$. By continuity of the Hodge product and of the form B_ω with respect to the abelian differential $\omega \in \mathscr{H}(\kappa)$, there exists a neighborhood $\mathscr{V}_t \subset \mathscr{H}(\kappa)$ such that, for all $\Lambda \in \mathscr{G}_0$ and for any Hodge orthonormal basis $\{c_1^\Lambda(\omega), \dots, c_g^\Lambda(\omega)\}$ of $\Lambda \subset H^1(S_0, \mathbb{R})$,

$$\left|B_\omega^{\mathbb{R}}\big(c_i^\Lambda(\omega), c_j^\Lambda(\omega)\big) + \delta_{ij}\right| \le 1/2.$$

Let $\mathscr{V}$ be a neighborhood of ω_0 in $\mathscr{H}(\kappa)$ such that $g_t(\mathscr{V}) \subset \mathscr{V}_t$ and such that for any $\omega \in \mathscr{V} \cap \mathscr{P}_\kappa$, the unstable subspace $E^+(\omega)$ is contained in a Lagrangian subspace $\Lambda_\omega \in \mathscr{G}_0$. Since ω_0 is a density point of $\mathscr{P}_\kappa$, by the Teichmüller invariance of the measure, the set $\mathscr{P}_\kappa(t) := g_t(\mathscr{P}_\kappa \cap \mathscr{V})$ has positive μ-measure. By construction $g_t(\mathscr{P}_\kappa \cap \mathscr{V})$ and, for all $\omega \in \mathscr{P}_\kappa \cap \mathscr{V}$, the unstable space $E^+(g_t\omega) = E^+(\omega) \subset \Lambda_\omega \in \mathscr{G}_0$. It follows that for all $\omega \in \mathscr{P}_\kappa(t)$ and for any Hodge orthonormal basis $\{c_1(\omega), \dots, c_g(\omega)\}$ of the Lagrangian subspace $\Lambda_\omega \subset H^1(S_0, \mathbb{R})$,

$$\left|B_\omega^{\mathbb{R}}\big(c_i(\omega), c_i(\omega)\big)\right| \ge 1/2 \qquad \text{for all } i \in \{1, \dots, g\}. \tag{5.2}$$

Let us assume that there exists $k < g$ such that $\lambda_k^\mu > \lambda_{k+1}^\mu = 0$. It follows from Lemma 5.1 that, for μ-almost all $\omega \in \mathscr{H}(\kappa)$ and for *any* Hodge orthonormal isotropic system $\{c_{k+1}, \dots, c_g\} \subset H^1(S_\omega, \mathbb{R})$, Hodge orthogonal and symplectic orthogonal to $E^+(\omega) \subset H^1(S_\omega, \mathbb{R})$,

$$B_\omega^{\mathbb{R}}(c_i, c_i) = 0 \qquad \text{for all } i \in \{k+1, \dots, g\}. \tag{5.3}$$

Since by construction $E^+(\omega) \subset \Lambda_\omega$ for all $\omega \in \mathscr{P}_\kappa(t)$, there exists a Hodge orthonormal basis $\{c_1(\omega), \dots, c_g(\omega)\}$ such that $\{c_1(\omega), \dots, c_k(\omega)\}$ is a basis of $E^+(\omega)$. However, since $\mathscr{P}_\kappa(t)$ has positive μ-measure, both the estimate (5.2) and the identity (5.3) hold on $\mathscr{P}_\kappa(t)$ leading to a contradiction. It follows that all the Kontsevich–Zorich exponents are all nonzero. □

For the sake of completeness we sketch below the proof of Theorem 1.7 following the argument given in [17, Cor. 5.4] and [21].

Proof of Theorem 1.7. If μ is Lagrangian, there exists $\omega \in \text{supp}(\mu)$ such that the form $B_\omega^{\mathbb{R}}$ has maximal rank. In fact, let $\omega_0 \in \text{supp}(\mu)$ be a holomorphic abelian differential with Lagrangian vertical foliation. The marked abelian differential $g_t\omega$ converges projectively to a meromorphic differential on a union of punctured Riemann spheres with poles at all (paired) punctures. It follows from Lemma 4.4 that for $t > 0$ sufficiently large, the form $B_{g_t\omega}^{\mathbb{R}}$ has maximal rank. By continuity, there exists an open set $\mathscr{U} \subset \mathscr{H}(\kappa)$ of positive μ-measure such that $B_\omega^{\mathbb{R}}$ has maximal rank for all $\omega \in \mathscr{U}$. By an elementary linear algebra argument, since $B_\omega^{\mathbb{R}}$ has maximal rank, the existence of a Hodge orthonormal Lagrangian system $\{c_1, \dots, c_g\} \subset H^1(S, \mathbb{R})$ such that

$$B_\omega^{\mathbb{R}}(c_i, c_j) = 0 \qquad \text{for all } i, j \in \{r+1, \dots, g\},$$

implies that $r \ge g/2$. Thus the result follows from Lemma 5.1. □

6. FUNDAMENTAL APPLICATIONS

6.1. **Veech surfaces.** The case of the Veech surfaces (n-gons) found by W. Veech in [44] was one of the original motivation of the construction in [17], in particular it inspired the notion of a Lagrangian measured foliation and the related focus on meromorphic abelian differentials on spheres with $2g$ paired punctures at the boundary of the moduli space. P. Hubert has recently explained to the author that the relevant properties of the Veech n-gons of [44] are in fact shared by a larger class of Veech surfaces, called *algebraically primitive* Veech surfaces. For all $SL(2,\mathbb{R})$-invariant probability measure associated to algebraically primitive Veech surfaces, the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle follows from the formulas of Bouw and Möller [7] for individual exponents. We remark that from the Bouw–Möller formulas, it seems possible to construct examples of nonuniformly hyperbolic $SL(2,\mathbb{R})$-invariant probability measures with multiple exponents.

We briefly recall below basic definitions concerning Veech surfaces.

Let ω be an abelian holomorphic differential on a Riemann surface S (or equivalently, let (S,ω) be a translation surface). The $SL(2,\mathbb{R})$-orbit of ω in the Teichmüller space of abelian differentials is called its *Teichmüller disk* (it is in fact isomorphic to the unit tangent bundle of a Poincaré disk).

The stabilizer $SL(\omega) < SL(2,\mathbb{R})$ of ω is a Fuchsian group, called the *Veech group*. A translation surface (S,ω) is called a *Veech surface* if and only if the Veech group $SL(\omega)$ is a lattice in $SL(2,\mathbb{R})$. In [44] Veech proved that if the Veech group is a lattice a dichotomy holds for the directional flows of the translation surface: a directional flow is either uniquely ergodic or it is completely periodic.

By definition, the $SL(2,\mathbb{R})$-orbit of a Veech surface in the moduli space $\mathscr{H}_g$ of abelian differentials is isomorphic to an immersed hyperbolic surface of finite volume. The (normalized) canonical hyperbolic measure supported on the $SL(2,\mathbb{R})$-orbit of a Veech surface in $\mathscr{H}_g$ is a fundamental example of a $SL(2,\mathbb{R})$-invariant probability measure on $\mathscr{H}_g$.

An important invariant of a Veech surface is the *trace field* of its Veech group, that is, the group generated by all the traces of elements of the Veech group. By a theorem of R. Kenyon and J. Smillie [26], the trace field of a translation surface of genus $g \geq 1$ has degree at most g over $\mathbb{Q}$.

DEFINITION 6.1 ([36, §2])**.** A Veech surface (S,ω) of genus $g \geq 1$ is said to be *algebraically primitive* if the trace field of its Veech group $SL(\omega)$ has exactly degree g over $\mathbb{Q}$, that is, it has maximal degree.

We recall that a Veech surface is said to be *geometrically primitive* if it is not a branched cover over another Veech surface (of lower genus). While any algebraically primitive Veech surface is geometrically primitive, the converse is not true in genus $g \geq 3$ (see [36, §2] and [34]).

We introduce below a class of Veech surfaces, which includes all algebraically primitive Veech surfaces and many examples of geometrically, but not algebraically, primitive Veech surfaces (as well as many nonprimitive surfaces) to which our criterion applies.

DEFINITION 6.2. A translation surface (S,ω) is said to be *decomposable* if the set $\mathscr{P}(\omega)\subset\mathbb{P}^1(\mathbb{R})$ of completely periodic directions has at least two distinct elements. It is called a *prelattice surface* (or a *bouillabaisse surface*, see [23]) if its Veech group $SL(\omega)$ contains two transverse parabolic elements. Any prelattice surface has two completely periodic transverse directions (and in each direction the surface split as a union of flat cylinders with commensurable moduli [44]). In particular, it is decomposable.

Let (S,ω) be a decomposable translation surface. For any pair of distinct (transverse) directions $\alpha,\beta\in\mathscr{P}(\omega)$, the *intersection matrix* $E^\omega(\alpha,\beta)$ is defined as follows (see [23, 44]). Let $\{C_i^\alpha \mid i\in\{1,\dots,r\}\}$ and $\{C_i^\beta \mid i\in\{1,\dots,s\}\}$ be the families of all flat cylinders in the direction α and $\beta\in\mathscr{P}(\omega)$, respectively. The matrix $E^\omega(\alpha,\beta)$ is the $r\times s$ nonnegative integer matrix such that $E^\omega_{ij}(\alpha,\beta)$ is equal to the number of parallelograms in the intersection $C_i^\alpha\cap C_j^\beta$. In other terms, the nonnegative integer $E^\omega_{ij}(\alpha,\beta)$ is equal to the (algebraic) intersection number of the (positively oriented) waist curves γ_i^α and γ_j^β of the cylinders C_i^α and C_j^β, respectively.

DEFINITION 6.3. The *homological rank* $r(\omega)\in\{0,1,\dots,g\}$ of a decomposable translation surface (S,ω) of genus $g\ge 1$ is the integer defined as follows:

$$r(\omega) := \max\{\operatorname{rank} E^\omega(\alpha,\beta) \mid \alpha\neq\beta\in\mathscr{P}(\omega)\}.$$

We remark that it is quite immediate to prove that the homological rank of a translation surface is at most equal to its genus. In fact, the waist curves of any cylinder decomposition generate an isotropic subspace of the homology with respect to the intersection form. Since the dimension of isotropic subspace is at most equal to the genus of the surface, our conclusion follows.

We can thus state the main application of our criterion to Veech surfaces.

THEOREM 6.4. *Let μ_ω denote the unique $SL(2,\mathbb{R})$-invariant probability measure on the moduli space $\mathscr{H}_g$ supported on the $SL(2,\mathbb{R})$-orbit of a Veech surface $(S,\omega)\in\mathscr{H}_g$. If the Veech surface has maximal homological rank, then the Kontsevich–Zorich cocycle is nonuniformly hyperbolic. In fact, the following general statement holds. Let $r := r(\omega)\in\{1,\dots,g\}$ denote the homological rank of a Veech surface (S,ω). The Kontsevich–Zorich exponents satisfy the inequalities*

$$\lambda_1^{\mu_\omega} = 1 > \lambda_2^{\mu_\omega} \ge \cdots \ge \lambda_r^{\mu_\omega} > 0.$$

Proof. It is immediate to verify that the measure μ_ω has a local product structure and it is cuspidal. By definition its homological dimension is at least equal to the homological rank of the Veech surface (S,ω). In fact, if $\alpha\in\mathscr{P}(\omega)$ is a completely periodic direction of homological dimension at most $d\in\{1,\dots,g\}$, then by definition the rank of the intersection matrix $E^\omega(\alpha,\beta)$ is at most d, for all $\beta \neq \alpha \in \mathscr{P}(\omega)$. The statement then follows from Theorem 1.6, in case the homological rank is maximal, and from Theorem 1.8 in general. □

We are grateful to P. Hubert for telling us about the following result from [23, §2.5]. We sketch the argument for the convenience of the reader.

LEMMA 6.5. *Any prelattice (bouillabaisse) algebraically primitive translation surface has maximal homological rank. In fact, for any prelattice surface the rank of the trace field is at most equal to its homological rank.*

Proof. Let $\alpha \neq \beta \in \mathscr{P}(\omega)$ be two transverse parabolic directions of a prelattice surface (S,ω). Let us denote the width and height vectors of the cylinders C_i^α and C_j^β by (w_i^α, h_i^α) and (w_j^β, h_j^β), respectively. We remark that by construction, the width vectors w_i^α, $w_j^\beta \in \mathbb{R}^2$ have directions $\alpha, \beta \in P^1(\mathbb{R})$, respectively, for all $i \in \{1,\dots,r\}$ and $j \in \{1,\dots,s\}$.

Let $E := E^\omega(\alpha,\beta)$ be the intersection matrix. Let

$$x := (|w_1^\alpha|,\dots,|w_r^\alpha|), \qquad y := (|h_1^\alpha|,\dots,|h_r^\alpha|),$$
$$\xi := (|w_1^\beta|,\dots,|w_s^\beta|), \qquad \eta := (|h_1^\beta|,\dots,|h_s^\beta|).$$

By construction, the following identities hold:

$$x = E\xi \quad \text{and} \quad \eta = E^t y. \tag{6.1}$$

Up to taking a power of the parabolic elements, one can assume that the parabolic elements $P^\alpha, P^\beta \in \mathrm{SL}(\omega)$ corresponding respectively to the parabolic directions $\alpha, \beta \in \mathscr{P}(\omega)$, are each a multiple of the Dehn twist of each cylinder C_i^α, C_j^β for all $i \in \{1,\dots,r\}$, $j \in \{1,\dots,s\}$. Under this assumption, there exist numbers $a, b \in \mathbb{R}^+$ such that, with respect to a system of coordinates with axis parallel to the directions $\{\alpha,\beta\}$,

$$P^\alpha = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad P^\beta = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$

By construction, all the ratios x_i/y_i are commensurable with a and all the ratios ξ_j/η_j are commensurable with b, that is, there are integer vectors $(m_1,\dots,m_r) \in \mathbb{Z}^r$ and $(n_1,\dots,n_s) \in \mathbb{Z}^s$ such that

$$m_i x_i = a y_i \quad \text{for all } i \in \{1,\dots,r\},$$
$$n_j \eta_j = b \xi_j \quad \text{for all } j \in \{1,\dots,s\}.$$

Let $D_m := \mathrm{diag}(m_1,\dots,m_r)$ and $D_n := \mathrm{diag}(n_1,\dots,n_s)$. The above equations can respectively be written in matrix form as

$$D_m x = a y \quad \text{and} \quad D_n \eta = b y. \tag{6.2}$$

By equations (6.1) and (6.2), it follows after some calculations that

$$E D_n (E^t) D_m x = (ab) x \quad \text{and} \quad (E^t) D_m E D_n \eta = (ab)\eta.$$

It follows that $t := ab \in \mathbb{R}$ is an eigenvalue of the matrices $ED_n(E^t)D_m$ and $(E^t)D_m E D_n$. Since the rank of both the above matrices is no higher than that of the intersection matrix E, which by definition is no higher than the homological rank $r(\omega)$ of the translation surface, it follows that $t \in \mathbb{R}$ is an algebraic number of degree at most equal to $r(\omega)$. Finally, it is proved in [23, Claim 2.1] that the trace field of the Veech group $\mathrm{SL}(\omega)$ is equal to $\mathbb{Q}[t]$, hence its degree is bounded above by the homological rank $r(\omega)$. The argument is thus completed. □

By Theorem 6.4 and Lemma 6.5 we can prove the following result, which can also be derived from the formulas of I. Bouw and M. Möller for single Kontsevich–Zorich exponents (see [7, Thm. 8.2 and Cor. 8.3]).

COROLLARY 6.6. *Let μ_ω denote the unique $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure on the moduli space $\mathcal{H}_g$ supported on the $\mathrm{SL}(2,\mathbb{R})$-orbit of a Veech surface $(S,\omega) \in \mathcal{H}_g$. If the Veech surface is algebraically primitive, then the Kontsevich–Zorich cocycle is nonuniformly hyperbolic. In fact, the following general statement holds. Let $r \in \{1,\dots,g\}$ denote the rank of the trace field of the Veech group $\mathrm{SL}(\omega)$ of a Veech surface (S,ω). The Kontsevich–Zorich exponents satisfy the inequalities:*

$$\lambda_1^{\mu_\omega} = 1 > \lambda_2^{\mu_\omega} \geq \cdots \geq \lambda_r^{\mu_\omega} > 0.$$

An important family of Veech surfaces that are geometrically primitive but not algebraically primitive, and given by *Prym eigenforms* in genus 3 and 4, was discovered by C. McMullen [34]. The Prym class contains an example discovered earlier by M. Möller [36, §2] and studied earlier in [24]. In genus 3 it appears that most geometrically primitive Veech surfaces are not algebraically primitive. In fact, conjecturally there are only finitely many algebraically primitive Veech surfaces, while there exist infinitely many geometrically primitive Veech surfaces that are not algebraically primitive (the geometrically primitive Prym eigenforms). Bainbridge and Möller [6] have recently proved a finiteness result for algebraically primitive Veech surfaces in the stratum $\mathcal{H}(3,1)$, a result that supports the aforementioned conjecture.

All Prym eigenforms have a quadratic trace field, hence they are not algebraically primitive in genus 3 and 4. However, the following result holds.

LEMMA 6.7. *All Prym eigenforms (geometrically primitive or not) in genus $g = 2$, 3, and 4 have maximal homological rank.*

Proof. The intersection matrices for the Prym eigenforms are computed in [34] (see for instance Figure 1 which gives the corresponding Coxeter graphs). The result is as follows. Let E_g^P denote intersection matrices for the Prym eigenforms in genus $g \in \{2,3,4\}$. The following formulas hold:

$$E_2^P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad E_3^P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \qquad E_4^P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is quite immediate to verify that the above matrices have maximal rank. □

COROLLARY 6.8. *The Kontsevich–Zorich cocycle is nonuniformly hyperbolic with respect to the unique* $\mathrm{SL}(2,\mathbb{R})$*-invariant probability measure* μ_ω *on the moduli space* $\mathscr{H}_g$ *supported on the* $\mathrm{SL}(2,\mathbb{R})$*-orbit of any Prym eigenform* $\omega \in \mathscr{H}_g$ *in genus* $g = 2, 3$ *or* 4*.*

6.2. **Canonical measures.** In the proof of the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle with respect to the canonical absolutely continuous $\mathrm{SL}(2,\mathbb{R})$-invariant probability measures on connected components of strata of the moduli space of abelian differentials, a key result is the following density statement (see [17, Lemma 4.4]).

LEMMA 6.9. *The subset of Lagrangian abelian differentials is dense in every stratum* $\mathscr{H}(\kappa) \subset \mathscr{H}_g$ *of the moduli space of abelian differentials.*

Proof. Let $\mathscr{F}(\kappa)$ be the set of isotopy classes of all orientable measured foliations which can be realized as horizontal (or vertical) foliation of an abelian differential in $\mathscr{H}(\kappa)$. The multiplicative group $\mathbb{R}^+$ of nonzero real numbers acts on $\mathscr{F}(\kappa)$. Let $(\mathscr{F}(\kappa)\times\mathscr{F}(\kappa))/\mathbb{R}^+$ denote the quotient of the product space $\mathscr{F}(\kappa)\times\mathscr{F}(\kappa)$ with respect to the diagonal action. The map $\mathscr{H}(\kappa) \to (\mathscr{F}(\kappa)\times\mathscr{F}(\kappa))/\mathbb{R}^+$ defined as

$$\omega \to [(\mathscr{F}^h_\omega, \mathscr{F}^v_\omega)] \in (\mathscr{F}(\kappa)\times\mathscr{F}(\kappa))/\mathbb{R}^+ \qquad \text{for all } \omega \in \mathscr{H}(\kappa)$$

is locally well-defined and open on the stratum $\mathscr{H}(\kappa)$. We claim that the set of Lagrangian foliation is dense in $\mathscr{F}(\kappa)$. The proof of the claim will conclude the argument.

Let $\mathscr{F} := \{\eta_{\mathscr{F}} = 0\} \in \mathscr{F}(\kappa)$ be an orientable measure foliation on a surface S of genus $g \geq 1$, and let $\Sigma \subset S$ be the subset of its singular points. It follows from Poincaré Recurrence Theorem that $\mathscr{F}$ is completely periodic whenever the relative cohomology class $[\eta_{\mathscr{F}}] \in \mathbb{R}\cdot H^1(S,\Sigma_{\mathscr{F}},\mathbb{Z})$, hence in particular whenever $[\eta_{\mathscr{F}}] \in \mathbb{R}\cdot H^1(S,\Sigma_{\mathscr{F}},\mathbb{Q})$. It follows that completely periodic foliations are dense in $\mathscr{F}(\kappa)$. In fact, by the Katok local classification theorem [25], the relative period map

$$\mathscr{F} \to [\eta_{\mathscr{F}}] \in H^1(S,\Sigma_{\mathscr{F}},\mathbb{R})$$

is a local homeomorphism on the space $\mathscr{F}(\kappa)$.

Let $\mathscr{F}$ be completely periodic. In this case, the surface can be decomposed, by cutting along the singular leaves, into a finite union of cylindrical components whose number is at most $3g-3$. Let $P\colon H_1(S,\mathbb{R}) \to H^1(S,\mathbb{R})$ be the (symplectic) map given by the Poincaré duality. We claim that, if $\mathscr{F}$ is completely periodic, then $P^{-1}[\eta_{\mathscr{F}}] \in \mathscr{L}(\mathscr{F})$. More precisely, if $\{a_1,\dots,a_s\}$ are the oriented waist curves of the cylinders $\{A_1,\dots,A_s\}$ of $\mathscr{F}$, which are respectively of transverse heights $\{h_1,\dots,h_s\}$, then

$$P^{-1}[\eta_{\mathscr{F}}] = \sum_{i=1}^{s} h_i [a_i] \in H_1(S,\mathbb{R}). \tag{6.3}$$

In fact, if $\gamma \subset S$ is any simple oriented closed curve, then $\gamma \cap A_i$ is homologous to $([a_i]\cap[\gamma])\cdot v_i$ relative to ∂A_i, where v_i is a positively oriented vertical segment joining the ends of A_i. Hence, formula (6.3) follows.

Let $\mathscr{L}(\mathscr{F}) \subset H_1(S,\mathbb{R})$ denote the isotropic subspace generated by the homology classes of the regular leaves of $\mathscr{F}$ and let $d(\mathscr{F}) := \dim \mathscr{L}(\mathscr{F}) \in \{1,\dots,g\}$. If $d(\mathscr{F}) = g$, then $\mathscr{F}$ is a Lagrangian measured foliation. Let us assume $d(\mathscr{F}) := d < g$ and let us construct an arbitrarily small perturbation $\mathscr{F}'$ of the foliation $\mathscr{F}$ such that $d(\mathscr{F}') > d$. Let $\{a_1,\dots,a_d\}$ be a maximal system of regular leaves of $\mathscr{F}$ such that the system of homology classes $\{[a_1],\dots,[a_d]\}$ is linearly independent in $H_1(S,\mathbb{R})$, hence it is a basis of the isotropic subspace $\mathscr{L}(\mathscr{F}) \subset H^1(S,\mathbb{R})$. Since $d < g$, there exists a smooth closed curve $\gamma \subset S$ such that $[\gamma] \notin \mathscr{L}(\mathscr{F})$,

$$\gamma \cap \Sigma_{\mathscr{F}} = \emptyset \qquad \text{and} \qquad \gamma \cap a_j = \emptyset \quad \text{for all } j \in \{1,\dots,d\}. \tag{6.4}$$

The existence of a curve $\gamma \subset S$ with the above properties can be proved as follows. Since $d < g$, there exists a closed surface S' of strictly positive genus with $2d$ distinct paired punctures $p_1^{\pm},\dots,p_d^{\pm} \in S'$ such that the open surface $S \setminus (\cup\{a_1,\dots,a_d\})$ is homeomorphic to the surface

$$S'' := S' \setminus \{p_1^+, p_1^-, \dots, p_d^+, p_d^-\}.$$

Since S' has strictly positive genus, there exists a continuous closed curve $\gamma' \subset S'' \subset S'$ such that $[\gamma] \neq 0 \in H_1(S',\mathbb{Z})$. Let $\gamma \subset S$ be any smooth curve isotopic to the image of $\gamma' \subset S'$ in $S \setminus (\cup\{a_1,\dots,a_d\})$.

Let us construct a representative of the Poincaré dual $P[\gamma] \in H^1(S,\mathbb{Z})$ supported in a compact subset of the open set

$$S \setminus (\cup\{a_1,\dots,a_d\} \cup \Sigma_{\mathscr{F}}).$$

By (6.4) there exists an open tubular neighborhood $\mathscr{U} \subset S$ of γ such that

$$\overline{\mathscr{U}} \cap \Sigma_{\mathscr{F}} = \emptyset \qquad \text{and} \qquad \overline{\mathscr{U}} \cap a_j = \emptyset \quad \text{for all } j \in \{1,\dots,d\}.$$

Let $\mathscr{V} \subset \bar{\mathscr{V}} \subset \mathscr{U}$ be an open tubular neighborhood of γ in $\mathscr{U}$. Let $\mathscr{U}^{\pm}$ be the two connected components of the open set $\mathscr{U} \setminus \gamma$ and let $\mathscr{V}^{\pm} := \mathscr{V} \cap \mathscr{U}^{\pm}$. Let $f : S \to \mathbb{R}$ be a function, smooth on $S \setminus \gamma$, with the following properties:

$$f(p) = \begin{cases} 0 & \text{for all } p \in \mathscr{U}^- \cup (S \setminus \mathscr{U}^+), \\ 1 & \text{for all } p \in \overline{\mathscr{V}^+}. \end{cases}$$

For any $r \in \mathbb{Q} \setminus \{0\}$, the 1-form $\eta_r := \eta_{\mathscr{F}} + r\,df$ is smooth and closed (but not cohomologous to $\eta_{\mathscr{F}}$) and $\eta_r \to \eta_{\mathscr{F}}$, as $r \to 0$, in the space of smooth 1-forms on S. Since $df = 0$ on $S \setminus \mathscr{U}$, $\eta_r \equiv \eta_{\mathscr{F}}$ in a neighborhood of $\Sigma_{\mathscr{F}}$. Hence, if $r \neq 0$ is sufficiently small, $\eta_r(p) = 0$ if and only if $p \in \Sigma_{\mathscr{F}}$, and the isotopy class of the orientable measured foliation $\mathscr{F}_r := \{\eta_r = 0\}$ belongs to the space $\mathscr{F}(\kappa)$. Since $r \in \mathbb{Q}$, the fundamental class $[\eta_r] \in H^1(S,\Sigma_{\mathscr{F}};\mathbb{Q})$. Hence, $\mathscr{F}_r$ is periodic. The simple closed curves $a_1,\dots,a_d$ are regular leaves of $\mathscr{F}_r$. In fact, $df = 0$ on $S \setminus \mathscr{U}$ and $\cup\{a_1,\dots,a_d\} \subset S \setminus \cup\overline{\mathscr{U}}$. Hence, $\eta_r \equiv \eta_{\mathscr{F}}$ in a neighborhood of $\cup\{a_1,\dots,a_d\}$. It follows that $a_1,\dots,a_d$ are also regular leaves of the foliation $\mathscr{F}_r$, hence the isotropic subspace $\mathscr{L}(\mathscr{F}_r) \subset H_1(S,\mathbb{R})$ generated by the homology classes of the regular leaves of the perturbed foliation $\mathscr{F}_r$ contains the isotropic subspace $\mathscr{L}(\mathscr{F})$ generated by the homology classes of the regular leaves of the foliation $\mathscr{F}$.

We claim that $\mathscr{L}(\mathscr{F}_r) \neq \mathscr{L}(\mathscr{F})$, hence $\dim \mathscr{L}(\mathscr{F}_r) > d := \dim \mathscr{L}(\mathscr{F})$. In fact, if equality holds $P^{-1}[\eta_r] \in \mathscr{L}(\mathscr{F})$ by formula (6.3). However, $P^{-1}[\eta_{\mathscr{F}}] \in \mathscr{L}(\mathscr{F})$, again by formula (6.3), and $P^{-1}[df] = [\gamma] \notin \mathscr{L}(\mathscr{F})$ by construction. Hence,

$$P^{-1}[\eta_r] = P^{-1}[\eta_{\mathscr{F}}] + rP^{-1}[df] \notin \mathscr{L}(\mathscr{F}).$$

The claim is thus proved.

By a finite iteration of the previous construction, we can show that the closure in $\mathscr{F}(\kappa)$ of the subset of all Lagrangian measured foliations contains the subset of all periodic measured foliations. Hence it coincides with the entire space $\mathscr{F}(\kappa)$. The density lemma is therefore proved. □

By the above density lemma, we can derive from our criterion the nonuniform hyperbolicity of the Kontsevich–Zorich cocycle for all canonical (absolutely continuous) $SL(2,\mathbb{R})$-invariant measures on connected components of strata of abelian differentials (see [4, 17]). The resulting proof is a simplified version of the original proof given in [17, Thm. 8.5].

COROLLARY 6.10. *All canonical measures on connected components of strata of abelian differentials are cuspidal Lagrangian, hence the Kontsevich–Zorich cocycle is nonuniformly hyperbolic with respect to all such measures.*

Proof. In the coordinates given by the relative period map, all canonical measures are (locally) equivalent to the Lebesgue measure and the invariant foliations of the Teichmüller flow are linear. It follows that canonical measures have a local product structure. By Lemma 6.9 every canonical measure is Lagrangian (on every connected component), hence by definition it is cuspidal Lagrangian. By Theorem 1.6 the Kontsevich–Zorich cocycle is nonuniformly hyperbolic with respect to all such measures. □

APPENDIX A. OTHER RELEVANT EXAMPLES BY CARLOS MATHEUS

In this appendix, we present some examples of closed $SL(2,\mathbb{R})$-orbits generated by square-tiled surfaces which provide interesting examples in the discussion on G. Forni's geometric criterion for the nonuniform hyperbolicity of the Kontsevich–Zorich cocyle.

While preparing the manuscript about his geometric criterion for the nonvanishing of Lyapunov exponents of the Kontsevich–Zorich cocycle, G. Forni asked me some natural questions originating from his paper. In particular, the following two questions arose:

- are there some examples of *cuspidal Lagrangian* $SL(2,\mathbb{R})$-invariant ergodic probability measures with *nonsimple* Lyapunov exponents on the corresponding Kontsevich–Zorich spectrum?
- are there some examples of *cuspidal* $SL(2,\mathbb{R})$-invariant ergodic probability measures whose *homological dimension* is *strictly less than* the number of positive Lyapunov exponents on the corresponding Kontsevich–Zorich spectrum?

We refer to Definition 1.4 and Definition 1.5 of the Introduction for more details on the terms marked in italic (namely, *cuspidal*, *Lagrangian*, and *homological dimension*).

The first question is related to Theorem 1.6 of the Introduction and the simplicity theorem of Avila and Viana [4]: in fact, as pointed out by G. Forni in the Introduction, while his Theorem 2 shows that any *cuspidal Lagrangian* $SL(2,\mathbb{R})$-invariant ergodic probability is nonuniformly hyperbolic (*i.e.,* 0 does not belong to the Kontsevich–Zorich spectrum of this measure), it does not provide any hints about the simplicity of the Kontsevich–Zorich cocycle (*i.e.,* the multiplicity of the Lyapunov exponents is 1). In Section A.1 of this Appendix, we present certain regular (*i.e.,* unbranched) double-covers of genus 2 square-tiled surfaces leading to cuspidal Lagrangian $SL(2,\mathbb{R})$-invariant ergodic probabilities with multiple (*i.e.,* nonsimple) Kontsevich–Zorich spectrum. These examples (together with the "stairs" square-tiled cyclic covers mentioned in the Introduction) provide some square-tiled surfaces such that the canonical $SL(2,\mathbb{R})$-invariant ergodic probability measure supported on its $SL(2,\mathbb{R})$-orbit is cuspidal Lagrangian with multiple (nonzero) Lyapunov exponents of the Kontsevich–Zorich cocycle (see Theorem A.4 below), so the first question has a positive answer.

The second question is related to Theorem 1.8 of the the Introduction: while this theorem ensures that any cuspidal $SL(2,\mathbb{R})$-invariant ergodic probability measure with homological dimension $k \in \{1,\dots,g\}$ has *at least* k strictly positive Lyapunov exponents in its Kontsevich–Zorich spectrum, and the lower bound on the number of nonvanishing Kontsevich–Zorich exponents provided by this result is the best possible in view of the *maximally degenerate examples* [18, 19] of cuspidal $SL(2,\mathbb{R})$-invariant ergodic probabilities with homological dimension 1 and *exactly* one nonvanishing Kontsevich–Zorich exponent (see also [20]), it does not give upper bounds on the number of nonvanishing Kontsevich–Zorich exponents based on the homological dimension. In Section A.2 of this Appendix, as it was suggested by G. Forni during our conversations, we show that a family of square-tiled cyclic covers indexed by odd integers $q \geq 3$ studied by J.-C. Yoccoz and myself (see [32, §3.1]) give cuspidal $SL(2,\mathbb{R})$-invariant ergodic probabilities with homological dimension 1 such that the number of nonvanishing Kontsevich–Zorich exponents equal to $1+(q-3)/2$ when $q \equiv 3 \pmod 4$ and $1+(q-1)/2$ when $q \equiv 1 \pmod 4$, so that the answer to the second question is also positive (see Theorem A.5 below).

Closing this introduction, I would like to acknowledge G. Forni for his kind invitation to contribute with this Appendix, A. Zorich for allowing me to use his excellent computer programs in order to numerically test some ideas and conjectures, and G. Forni and A. Zorich for several fruitful discussions.

A.1. **Cuspidal Lagrangian measures with multiple spectrum.** Given an abelian differential ω_S on a Riemann surface S, one can produce further examples of abelian differentials on Riemann surfaces by the following *covering* procedure: given a (possibly ramified) covering $p: R \to S$ of Riemann surfaces, we can define an abelian differential ω_R by pulling back ω_S under the covering map p, *i.e.*, $\omega_R = p^*(\omega_S)$. In this situation, we can relate the Kontsevich–Zorich spectrum $\mathscr{L}(S,\omega_S)$ of (S,ω_S) with the Kontsevich–Zorich spectrum $\mathscr{L}(R,\omega_R)$ of (R,ω_R).

LEMMA A.1. *Let $p\colon R \to S$ be a possibly ramified covering. Then, for any abelian differential ω_S on S, we have the inclusion $\mathscr{L}(S,\omega_S) \subset \mathscr{L}(R,\omega_R)$, where $\omega_R := p^*(\omega_S)$. In other words, any Kontsevich–Zorich exponent of (S,ω_S) is also a Kontsevich–Zorich exponent of $(R, p^*(\omega_S))$.*

REMARK A.2. While this elementary lemma is a common knowledge of several authors (for instance, G. Forni and A. Zorich were aware of it for quite a long time), I included a brief indication of its proof for sake of completeness.

Proof. A direct inspection of the definitions (see Section 2 of the main article) shows that the covering $p\colon R \to S$ induces a natural injective linear isometry

$$(H^1(S,\mathbb{R}), \|\cdot\|_{\omega_S}) \to (H^1(R,\mathbb{R}), \|\cdot\|_{\omega_R}),$$

where $\|\cdot\|_\omega$ stands for the *Hodge norm* on $H^1(M,\mathbb{R})$. The desired lemma follows from Lemma 2.1' of [17] (or equivalently, Lemma 4.3 of [18]). □

In the sequel, we will specialize this covering procedure to the case of *double unramified* covers $p\colon R \to S$ of a *square-tiled surface*[1] (S,ω_S) of genus 2. In this situation, $(R,\omega_R := p^*(\omega_S))$ is a square-tiled surface of genus 3. Of course, since we are dealing with unbranched coverings, if $(S,\omega_S) \in \mathscr{H}(2)$ (*i.e.*, ω_S has one double zero), then $(R,\omega_R) \in \mathscr{H}(2,2)$ (*i.e.*, ω_R has two double zeroes), and if $(S,\omega_S) \in \mathscr{H}(1,1)$ (*i.e.*, ω_S has two simple zeroes), then $(R,\omega_R) \in \mathscr{H}(1,1,1,1)$ (*i.e.*, ω_R has four simple zeroes). On the other hand, after the works of Bainbridge [5] (see also [12]), we know that the Kontsevich–Zorich spectrum $\mathscr{L}(S,\omega_S)$ in the case of a genus 2 surface S is:

$$\mathscr{L}(S,\omega_S) = \begin{cases} \{1, 1/3\} & \text{if } (S,\omega_S) \in \mathscr{H}(2), \\ \{1, 1/2\} & \text{if } (S,\omega_S) \in \mathscr{H}(1,1). \end{cases}$$

Thus, the preceding lemma implies the following fact.

COROLLARY A.3. *Let $p\colon R \to S$ be an unramified double covering of a genus two square-tilled surface (S,ω_S). Then,*

$$\mathscr{L}(R, p^*\omega_S) \supset \begin{cases} \{1, 1/3\} & \textit{if } (R, p^*\omega_S) \in \mathscr{H}(2,2), \\ \{1, 1/2\} & \textit{if } (R, p^*\omega_S) \in \mathscr{H}(1,1,1,1). \end{cases}$$

[1] Recall that, in general, (S,ω_S) is square-tiled surface when the abelian differential ω_S has *integral* periods. Alternatively (and equivalently), we say that (S,ω_S) is a square-tiled surface when its stabilizer $SL(S,\omega_S)$ (called *Veech group*) under the natural $SL(2,\mathbb{R})$ action on the moduli space of abelian differentials is commensurable to $SL(2,\mathbb{Z})$. For more details on square-tiled surfaces, see [52] and references therein.

In order to simplify the exposition, we will present our square-tiled surfaces in a combinatorial fashion, namely, we label its unit squares (tiles) using positive integers $i = 1,\dots,N$ and we consider a pair of permutations $(h,v) \in S_N \times S_N$ such that $h(i)$ (resp., $v(i)$) is the neighbor to the right (resp., on the top) of the square i. Since our Riemann surfaces are connected, we require that h and v act transitively on $\{1,\dots,N\}$. Also, since we can relabel the squares (tiles) of our surface without changing it, we will say that (h_0, v_0) is equivalent to (h_1, v_1) whenever they are *simultaneously* conjugate (*i.e.,* there exists $\phi \in S_N$ such that $h_1 = \phi^{-1} h_0 \phi$ *and* $v_1 = \phi^{-1} v_0 \phi$). Below, we will always write permutations through their cycles (and we will write even their 1-cycles to make the total number of square tiles of our surfaces more evident).

Consider the square-tiled surface associated with $h_{S_0} = (1,2)(3)$ and $v_{S_0} = (1,3)(2)$. It is an L-shaped square-tiled surface S_0 formed by 3 unit squares glued via the recipe provided by h_{S_0} and v_{S_0}. We can form double covers R_0 of S_0 by taking two copies of S_0 and changing the side identifications conveniently (using a pair of permutations h_{R_0} and v_{R_0}). For our purposes, we take $h_{R_0} = (1,2,3,4)(5,6)$ and $v_{R_0} = (1,5)(2)(3,6)(4)$.

THEOREM A.4. *The canonical (absolutely continuous)* $\mathrm{SL}(2,\mathbb{R})$*-invariant ergodic probability measure* μ_{R_0} *supported on the (closed)* $\mathrm{SL}(2,\mathbb{R})$*-orbit of the square-tiled surface* (R_0, ω_{R_0}) *associated to*

$$h_{R_0} = (1,2,3,4)(5,6) \qquad \textit{and} \qquad v_{R_0} = (1,5)(2)(3,6)(4)$$

is (cuspidal) Lagrangian and its Kontsevich–Zorich spectrum is multiple:

$$\mathscr{L}(R_0, \omega_{R_0}) = \{1, 1/3, 1/3\}.$$

Proof. We begin by showing that μ_{R_0} is Lagrangian. The vertical foliation of R_0 has 4 cylinders $C_{(1,5)}$, $C_{(2)}$, $C_{(3,6)}$, and $C_{(4)}$ (where the subindices is composed of the labellings of all squares forming the corresponding cylinders). Denoting by $\gamma_{(1,5)}, \gamma_{(2)}, \gamma_{(3,6)}, \gamma_{(4)}$ the homology classes of the waist curves of these cylinders, it is easy to see that they generate a 3-dimensional subspace of $H_1(R_0,\mathbb{R})$ (because $\gamma_{(1,5)} = \gamma_{(3,6)}$ and $\gamma_{(1,5)}, \gamma_{(2)}, \gamma_{(4)}$ are linearly independent by direct calculation). Since the genus of R is 3, we are done.

Next, we compute the Kontsevich–Zorich spectrum of (R_0, ω_{R_0}). We will accomplish this task with the aid of the following formula of A. Eskin, M. Kontsevich, and A. Zorich [12] for the *sum* of Kontsevich–Zorich exponents associated to square-tiled surfaces

$$\lambda_1 + \cdots + \lambda_g = \frac{1}{12}\sum_{i=1}^{n} \frac{m_i(m_i+2)}{m_i+1} + \frac{1}{\#\mathrm{SL}(2,\mathbb{Z})\cdot P_0} \sum_{P_i \in \mathrm{SL}(2,\mathbb{Z})\cdot P_0} \left(\sum_{P_i = \cup \mathrm{cyl}_{ij}} \frac{h_{ij}}{w_{ij}} \right).$$

Here, $\lambda_1,\dots,\lambda_g$ are the Kontsevich–Zorich exponents of a square-tiled surface P_0 of genus g belonging to the stratum $\mathscr{H}(m_1,\dots,m_n)$. Also, $\mathrm{SL}(2,\mathbb{Z})\cdot P_0$ denotes the (finite) orbit of P_0 under the action of $\mathrm{SL}(2,\mathbb{Z})$. For each $P_i \in \mathrm{SL}(2,\mathbb{Z})\cdot P_0$, the decomposition of P_i into *maximal* horizontal cylinders is denoted by $P_i = \cup \mathrm{cyl}_{ij}$. Furthermore, h_{ij} denotes the height of the cylinder cyl_{ij} and w_{ij} is the length of the waist curve of the cylinder cyl_{ij}.

In the case of the genus 3 square-tiled surface $(R_0,\omega_{R_0}) \in \mathcal{H}(2,2)$, we combine this formula for the sum of Lyapunov exponents together with our knowledge of two Lyapunov exponents of (R_0,ω_0) (from Corollary A.3) to get

$$1+\lambda_2(R_0)+\frac{1}{3}=\frac{4}{9}+\frac{1}{\#\mathrm{SL}(2,\mathbb{Z})\cdot R_0}\sum_{R_i\in SL(2,\mathbb{Z})\cdot R_0}\left(\sum_{R_i=\cup \mathrm{cyl}_{ij}}\frac{h_{ij}}{w_{ij}}\right), \tag{A.1}$$

where $\lambda_2(R_0)$ is the second Kontsevich–Zorich exponent of (R_0,ω_{R_0}). This reduces our task to the computation of $\mathrm{SL}(2,\mathbb{Z})\cdot R_0$. Keeping this goal in mind, we will work with the generators $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $J=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ of $\mathrm{SL}(2,\mathbb{Z})$. Their actions on a square-tiled surface presented as a pair of permutations (h,v) is given by the Nielsen transformations $T(h,v)=(h,vh^{-1})$ and $J(h,v)=(v^{-1},h)$. Put

- $h_{R_1}:=h_{R_2}:=h_{R_3}:=h_{R_0}=(1,2,3,4)(5,6)$;
- $v_{R_1}:=(1,4,6)(2,5,3)$, $v_{R_2}:=(1,6,3,5)(2,4)$, and $v_{R_3}:=(1,2,6)(3,4,5)$;
- $h_{R_4}:=h_{R_5}:=(1,5)(2)(3,6)(4)$;
- $v_{R_4}:=h_{R_0}$ and $v_{R_5}:=(1,6,4)(2,3,5)$;
- $h_{R_6}:=h_{R_7}:=h_{R_8}:=(1,6,4)(2,3,5)$;
- $v_{R_6}:=h_{R_0}$, $v_{R_7}:=(1)(2,6)(3)(4,5)$, and $v_{R_8}:=(1,5,3,6)(2,4)$.

Let R_i be the square-tiled surface associated to (h_{R_i},v_{R_i}), $i=0,\dots,8$. A straightforward calculation shows that $\mathrm{SL}(2,\mathbb{Z})\cdot R_0=\{R_0,\dots,R_8\}$ and it is organized as follows:

- $T(R_0)=R_1$, $T(R_1)=R_2$, $T(R_2)=R_3$, and $T(R_3)=R_0$;
- $J(R_0)=R_4$ and $J(R_3)=R_0$;
- $T(R_4)=R_5$ and $T(R_5)=R_4$;
- $J(R_1)=R_6$ and $J(R_6)=R_1$;
- $T(R_6)=R_7$, $T(R_7)=R_8$, and $T(R_8)=R_6$;
- $J(R_7)=R_5$ and $J(R_5)=R_7$;
- $J(R_8)=R_4$ and $J(R_4)=R_8$;
- $J(R_2)=R_2$.

In the literature, each T-orbit is called a *cusp*. The above description says that $\mathrm{SL}(2,\mathbb{Z})\cdot R_0=\mathscr{C}_1\cup\mathscr{C}_2\cup\mathscr{C}_3$ is the disjoint union of 3 cusps $\mathscr{C}_1=\{R_0,R_1,R_2,R_3\}$, $\mathscr{C}_2=\{R_4,R_5\}$, and $\mathscr{C}_3=\{R_6,R_7,R_8\}$. The contribution of two square-tiled surfaces belonging to a fixed cusp to the sum appearing in the right-hand side of (A.1) are equal (because their horizontal permutations are the same). Therefore, it suffices to compute this contribution at an arbitrarily chosen surface inside a fixed cusp, multiply it by the length (size) of this cusp and then sum up over all cusps in order to determine the sum of the right-hand side of (A.1). In the case at hand, we have:

- each surface in the 1st cusp $\mathscr{C}_1$ contributes with $\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$ and $\#\mathscr{C}_1=4$;
- each surface in the 2nd cusp $\mathscr{C}_2$ contributes with $2\cdot\frac{1}{2}+2\cdot 1=3$ and $\#\mathscr{C}_2=2$;
- each surface in the 3rd cusp $\mathscr{C}_3$ contributes with $\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$ and $\mathscr{C}_3=3$.

Plugging this information into (A.1) (and noting that $\#\mathrm{SL}(2,\mathbb{Z})=9$), we obtain:

$$\frac{4}{3}+\lambda_2(R_0)=\frac{4}{9}+\frac{1}{9}\left(4\cdot\frac{3}{4}+2\cdot 3+3\cdot\frac{2}{3}\right)=\frac{5}{3},$$

i.e., $\lambda_2(R_0) = 1/3$ and, *a fortiori*, $\mathscr{L}(R_0, \omega_{R_0}) = \{1, 1/3, 1/3\}$. □

A.2. **Non-zero exponents beyond the homological dimension.** In §3.1 of [32], the authors introduced a family of closed $SL(2,\mathbb{R})$-orbits, indexed by odd integers $q \geq 3$, of square-tiled surfaces (M_q, ω_q) isomorphic to the desingularization of the algebraic curve

$$w^{2q} = z^{q-2}(z^2 - 1)$$

equipped with the (unit area) abelian differential

$$\omega_q = \frac{2\sqrt{2\pi}}{(\Gamma(1/4))^2} \frac{z^{\frac{q-3}{2}} dz}{w^q}.$$

For a pictorial description of these square-tiled surfaces (together with the zeroes of ω_q), see Figure 3 (for the general case), Figure 4 (for $q = 3$) and Figure 5 (for $q = 5$) of [32]. As pointed out in this article, (M_3, ω_3) corresponds to the totally degenerate genus 4 example of [19] (see also [20]), so that this family maybe considered as a natural generalization of the exceptionally symmetric case $q = 3$. Generally speaking, $(M_q, \omega_q) \in \mathscr{H}(q-1, q-1, q-1)$ (*i.e.*, ω_q has 3 zeroes of order $(q-1)$, and, a fortiori, the genus of M_q is $(3q-1)/2$). However, the similarities between the cases $q = 3$ and $q \geq 5$ end here; for instance, it was proved in [32] that the Veech group of (M_3, ω_3) is $SL(2,\mathbb{Z})$, while the Veech group of (M_q, ω_q) is

$$SL(M_q, \omega_q) = \{M \in SL(2,\mathbb{Z}) \mid M \equiv I \text{ or } M \equiv J \bmod 2\} \tag{A.2}$$

for every (odd) $q \geq 5$. Also, the Kontsevich–Zorich spectrum of (M_3, ω_3) is maximally degenerate (*i.e.*, all but one of the Kontsevich–Zorich exponents are zero), while (M_q, ω_q) are never totally degenerate when $q \geq 5$ (see Remark 3.2 of [32]).

The next result shows that the non-maximally degenerate examples associated to (M_q, ω_q), $q \geq 5$, have homological dimension equal to 1.

THEOREM A.5. *For every odd $q \geq 5$, the canonical (absolutely continuous) ergodic $SL(2,\mathbb{R})$-invariant probability measure μ_q supported on the closed orbit $SL(2,\mathbb{R}) \cdot (M_q, \omega_q)$ has homological dimension 1, and the number of strictly positive Kontsevich–Zorich exponents is*

- $1 + (q-1)/2$ *when* $q \equiv 1 \pmod 4$ *or*
- $1 + (q-3)/2$ *when* $q \equiv 3 \pmod 4$.

Proof. Fix $q \geq 5$ odd. The description (A.2) of the Veech group of (M_q, ω_q) shows that it is a index 3 subgroup of $SL(2,\mathbb{Z})$ and

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad ST$$

are distinct representatives of the cosets of $SL(2,\mathbb{Z}) / SL(M_q, \omega_q)$. Therefore, we can compute the homological dimension of μ_q by calculating the dimensions of the subspaces of $H_1(M_q, \omega_q)$ generated by the homology classes of the waist curves of cylinders in the horizontal and main diagonal (*i.e.*, slope 1) directions. Using the notations from §3.1 of [32], the waist curves of the two cylinders of

(M_q,ω_q) along the horizontal direction are homologous to $\sum_{i\in\mathbb{Z}/q}(\sigma_i+\sigma'_i) := \sigma$, and the waist curves of the two cylinders of (M_q,ω_q) along the main diagonal direction are homologous to $\sum_{i\in\mathbb{Z}/q}(\sigma_i+\sigma'_i+\zeta_{i-1}+\zeta'_{i+1}) := \sigma+\zeta$. It follows that the homological dimension of μ_q is equal to 1.

The computation of the number of positive Kontsevich–Zorich exponents can be done with the aid of the results of [11]. In fact, (M_q,ω_q) is a square-tiled cyclic cover of type $M_{2q}(1,1,q-2,q)$ (in the notation of [20]), so that we can use the (action T^* of the) automorphism $T(z,w)=(z,\zeta w)$, ζ a primitive $2q$-th root of unit, of the algebraic curve M_q to decompose the complex cohomology $H^1(M_q,\mathbb{C})$ into a direct sum of eigenspaces $V_k=\ker(T^*-\zeta^k\,\mathrm{Id})$, $0<k<2q$. Using this decomposition, one can determine the number of positive Kontsevich–Zorich exponents. The number of positive Kontsevich–Zorich exponents is

$$\#\{0<k<2q \mid \dim_{\mathbb{C}} V_k^{1,0}=\dim_{\mathbb{C}} V_{N-k}^{1,0}=1\}, \tag{A.3}$$

where $V^{1,0}$ denotes the $(1,0)$-part of $V\subset H^1(M_q,\mathbb{C})$. On the other hand, for a general cyclic cover of type $M_N(a_1,a_2,a_3,a_4)$,

$$\dim_{\mathbb{C}} V_{N-k}^{1,0}=\sum_{\mu=1}^{4}\left\{\frac{k a_\mu}{N}\right\}-1,$$

where $\{x\}$ is the fractional part of x. See, *e.g.*, , [11] for more details. In the particular case of (M_q,ω_q), *i.e.*, $N=2q$, $a_1=a_2=1$ and $a_3=q-2$, $a_4=q$, we get

$$\dim_{\mathbb{C}} V_{2q-k}^{1,0}=2\left\{\frac{k}{2q}\right\}+\left\{\frac{(q-2)k}{2q}\right\}+\left\{\frac{k}{2}\right\}-1.$$

In particular,

- $\dim_{\mathbb{C}} V_{2q-k}^{1,0}=0$ when $0<k<q$ and k is *even*;
- $\dim_{\mathbb{C}} V_{2q-k}^{1,0}=1$ when $q<k<2q$ and k is *even*;
- $\dim_{\mathbb{C}} V_{2q-k}^{1,0}=0$ when $0<k<q/2$ and k is *odd*;
- $\dim_{\mathbb{C}} V_{2q-k}^{1,0}=2$ when $3q/2<k<2q$ and k is *odd*;
- $\dim_{\mathbb{C}} V_{2q-k}^{1,0}=1$ when $q/2<k<3q/2$ and k is *odd*.

Inserting this information into formula (A.3), we see that the number of positive Kontsevich–Zorich exponents of (M_q,ω_q) is

$$\#\{q/2<k<3q/2 \mid k \text{ is odd}\}=\begin{cases}1+(q-1)/2 & \text{if } q\equiv 1 \pmod 4,\\ 1+(q-3)/2 & \text{if } q\equiv 3 \pmod 4.\end{cases}$$

This completes the argument. □

REMARK A.6. An alternative way of counting the number of positive exponents of (M_q,ω_q) uses Remark 3.2 of [32]. Using the notation of that paper, the two-dimensional tautological subspace H_1^{st} contributes with the exponent $+1$ and the other exponents come from $H_\tau\oplus\breve{H}$. The $(q-1)$-dimensional subspace H_τ contains only zero exponents (since the action of the affine group is through a finite group) and $\breve{H}=\bigoplus_{j=1}^{q-1}\breve{H}(\rho^j)$ where $\breve{H}(\rho^j)$ are 2-dimensional invariant subspaces (under the action of the affine group) naturally indexed by the powers ρ^j of the primitive qth root of unit $\rho := \exp(2\pi i/q)$. The trace of the actions of the generators $\widetilde{S}^2$, $\widetilde{T}^2$ of $\mathrm{Aff}_{(1)}(M_q,\omega_q)$ (the subgroup of the affine group fixing the zeroes of ω_q) as well as the trace of $\widetilde{S}^2\widetilde{T}^2$ on $\check{H}(\rho^j)$ were calculated in [32, §3.2]. The outcome of this calculation is the fact that these traces are less than 2 when $q/4 < j < 3q/4$, while the trace of the action of $\widetilde{S}^2\widetilde{T}^2$ is greater than 2 when $1 \le j < q/4$ or $3q/4 < j < q$. This implies that the affine group of (M_q,ω_q) acts on $\check{H}(\rho^j)$ by a compact subgroup (of elliptic matrices) when $q/4 < j < 3q/4$, while it acts nontrivially (with some hyperbolic elements) on $\check{H}(\rho^j)$ when $1 \le j < q/4$ or $3q/4 < j < q$. Consequently, the quantity of positive exponents is

$$1 + 2\cdot\#\{j \mid 1 \le j < q/4\} = \begin{cases} 1+(q-1)/2 & \text{if } q \equiv 1 \pmod 4, \\ 1+(q-3)/2 & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

REMARK A.7. By the recent work of A. Eskin, M. Kontsevich and A. Zorich [11], it is possible to determine the precise value of the Kontsevich–Zorich exponents of (M_q,ω_q). Indeed, going back to the notation of the proof of the theorem above, the main result of [11] implies that every $0 < k < 2q$ with $\dim_{\mathbb{C}} V_k^{1,0} = \dim_{\mathbb{C}} V_{2q-k}^{1,0} = 1$ gives rise to a (positive) Kontsevich–Zorich exponent

$$\lambda(k) = 2\min\big\{\{k/2q\},\, 1-\{k/2q\}\big\}.$$

Taking into account the symmetry $\lambda(k) = \lambda(2q-k)$, this means that positive Kontsevich–Zorich exponents of (M_q,ω_q) are

$$\big\{k/q \mid 0<k<q,\ \dim_{\mathbb{C}} V_k^{1,0} = \dim_{\mathbb{C}} V_{2q-k}^{1,0} = 1\big\},$$

where each value k/q appears with multiplicity 2.

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GIOVANNI FORNI <gforni@math.umd.edu>: Department of Mathematics, University of Maryland, College Park, MD, USA