Calculating the $\star$ connection

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This paper is a supplement to our paper [1]: it contains the calculations justifying the statements of Lemma 4.8 and Proposition 4.10 in [1].

1 The $\star$ connection

We have the following formulas:

- $[X^v, Y^v]_u = 0$
- $[X^h, Y^v]_u = (0, \nabla^\star_X Y)$
- $[X^h, Y^h]_u = ([X, Y], -R(X, Y)(u))$

Denote by V^h, JV^h, V^v, JV^v the horizontal and vertical lifts, respectively, of V and JV.

Lemma 1.1. *1.* $[V^v, JV^v]_u = [V^h, JV^v]_u = [V^h, V^v]_u = 0$,

2. $[JV^h, V^v]_u = (0, cJV)$

3. $[JV^h, V^h]_u = (cJV, -R(JV, V)u)$

4. $[JV^h, JV^v] = (0, -cV)$

Observe that $\|V^h\|_\star = \|JV^h\|_\star = \delta^{-1}$, and $\|V^v\|_\star = \|JV^v\|_\star = 1$. We have:

Lemma 1.2. *Let X and Y be arbitrary vector fields on S with $\|X\| = \|Y\| = 1$, and denote by X^h, X^v, Y^h, Y^v their horizontal and vertical lifts. Then*

$$\|\nabla^\star_{X^h} Y^h\|_\star = O(\delta^{-2}),\ \|\nabla^\star_{X^h} Y^v\|_\star = O(\delta^{-1}),\ \|\nabla^\star_{X^v} Y^h\|_\star = O(\delta^{-1}),$$

and

$$\nabla^{\star}_{X^v} Y^v = 0.$$

In particular, we have:

1. $\nabla^{\star}_{V^h} V^h = -\delta^{-1} V^h$,
2. $\nabla^{\star}_{V^h} JV^h = -\delta^{-1} JV^h + \frac{1}{2}\langle R(JV,V)u, V\rangle V^v + \frac{1}{2}\langle R(JV,V)u, JV\rangle JV^v$,
3. $\nabla^{\star}_{V^h} V^v = -\frac{\delta^2}{2}\langle R(JV,V)u, V\rangle JV^h$
4. $\nabla^{\star}_{V^h} JV^v = -\frac{\delta^2}{2}\langle R(JV,V)u, JV\rangle JV^h$
5. $\nabla^{\star}_{JV^h} JV^h = (\delta^{-1} - c) V^h$
6. $\nabla^{\star}_{JV^h} V^h = (-\delta^{-1} + c) JV^h - \frac{1}{2}\langle R(JV,V)u, V\rangle V^v - \frac{1}{2}\langle R(JV,V)u, JV\rangle JV^v$
7. $\nabla^{\star}_{JV^h} V^v = \frac{\delta^2}{2}\langle R(JV,V)u, V\rangle V^h + cJV^v$
8. $\nabla^{\star}_{JV^h} JV^v = \frac{\delta^2}{2}\langle R(JV,V)u, JV\rangle V^h - cV^v$
9. $\nabla^{\star}_{V^v} V^h = -\frac{\delta^2}{2}\langle R(JV,V)u, V\rangle JV^h$
10. $\nabla^{\star}_{V^v} JV^h = \frac{\delta^2}{2}\langle R(JV,V)u, V\rangle V^h$
11. $\nabla^{\star}_{V^v} V^v = \nabla^{\star}_{V^v} JV^v = 0$,
12. $\nabla^{\star}_{JV^v} V^h = -\frac{\delta^2}{2}\langle R(JV,V)u, JV\rangle JV^h$
13. $\nabla^{\star}_{JV^v} JV^h = \frac{\delta^2}{2}\langle R(JV,V)u), JV\rangle V^h$, *and*
14. $\nabla^{\star}_{JV^v} V^v = \nabla^{\star}_{JV^v} JV^v = 0$

Proof. Koszul's formula:

$$\begin{aligned} 2\langle \nabla^{\star}_X Y, Z\rangle_{\star} &= X(\langle Y, Z\rangle_{\star}) + Y(\langle X, Z\rangle_{\star}) - Z(\langle X, Y\rangle_{\star}) \\ &\quad + \langle [X,Y], Z\rangle_{\star} - \langle [X,Z], Y\rangle_{\star} - \langle [Y,Z], X\rangle_{\star}. \end{aligned}$$

In the sequel, we will systematically use the following fact:

$$P(\langle Q, R\rangle_{\star}) = 0$$

if $Q \in \{V^v, JV^v\}$, $R \in \{V^h, JV^h, V^v, JV^v\}$ (because $\langle Q, R\rangle_{\star} \equiv 0$ or 1 in this case) or $P \in \{V^v, JV^v\}$, $Q, R \in \{V^h, JV^h, V^v, JV^v\}$ (because $\langle Q, R\rangle_{\star} \equiv 0, 1$ or δ^{-2} in this case).

Setting $X = V^h$:

$$\begin{aligned} 2\langle \nabla^\star_{V^h} Y, Z\rangle_\star &= V^h(\langle Y, Z\rangle_\star) + Y(\langle V^h, Z\rangle_\star) - Z(\langle V^h, Y\rangle_\star) \\ &+ \langle [V^h, Y], Z\rangle_\star - \langle [V^h, Z], Y\rangle_\star - \langle [Y, Z], V^h\rangle_\star. \end{aligned}$$

Assume first that $Y, Z \in \{JV^h, V^v, JV^v\}$, with $Y \neq Z$. Then:

$$2\langle \nabla^\star_{V^h} Y, Z\rangle_\star = \langle [V^h, Y], Z\rangle_\star - \langle [V^h, Z], Y\rangle_\star - \langle [Y, Z], V^h\rangle_\star.$$

• For $Y = JV^h$, $Z \in \{V^v, JV^v\}$, this reduces to

$$2\langle \nabla^\star_{V^h} JV^h, Z\rangle_\star = -\langle (cJV, -R(JV, V)u), Z\rangle_\star - \langle [V^h, Z], JV^h\rangle_\star - \langle [JV^h, Z], V^h\rangle_\star.$$

On the other hand, if $Z \in \{V^v, JV^v\}$, we have

$$\langle [V^h, Z], JV^h\rangle_\star = \langle [JV^h, Z], V^h\rangle_\star = 0,$$

and so

$$2\langle \nabla^\star_{V^h} JV^h, Z\rangle_\star = \langle (cJV, -R(JV, V)u), Z\rangle_\star.$$

When $Z = JV^h$, by Koszul's formula, we have

$$\begin{aligned} 2\langle \nabla^\star_{V^h} JV^h, JV^h\rangle_\star &= V^h(\langle JV^h, JV^h\rangle_\star) \\ &= V^h(\delta^{-2}) = -2\delta^{-3}. \end{aligned}$$

Finally, when $Z = V^h$, we have

$$\begin{aligned} 2\langle \nabla^\star_{V^h} JV^h, V^h\rangle_\star &= V^h(\langle JV^h, V^h\rangle_\star) + JV^h(\langle V^h, V^h\rangle_\star) - V^h(\langle V^h, JV^h\rangle_\star) \\ &+ \langle [V^h, JV^h], V^h\rangle_\star - \langle [V^h, V^h], JV^h\rangle_\star - \langle [JV^h, V^h], V^h\rangle_\star \\ &= JV^h(\langle V^h, V^h\rangle_\star) + \langle [V^h, JV^h], V^h\rangle_\star - \langle [JV^h, V^h], V^h\rangle_\star \\ &= JV^h(\delta^{-2}) + 2\langle [V^h, JV^h], V^h\rangle_\star, \end{aligned}$$

i.e.,

$$\begin{aligned} 2\langle \nabla^\star_{V^h} JV^h, V^h\rangle_\star &= JV^h(\delta^{-2}) - 2\langle (cJV, -R(JV, V)u), V^h\rangle_\star \\ &= JV^h(\delta^{-2}) = 0. \end{aligned}$$

We conclude that

$$\nabla^\star_{V^h} JV^h = -\delta^{-1} JV^h + \frac{1}{2}\langle R(JV, V)u, V\rangle V^v + \frac{1}{2}\langle R(JV, V)u, JV\rangle JV^v. \quad (1)$$

• Now suppose $Y = V^h$ Then

$$2\langle \nabla^\star_{V^h} V^h, Z\rangle_\star = 2V^h(\langle V^h, Z\rangle_\star) - Z(\langle V^h, V^h\rangle_\star) - 2\langle [V^h, Z], V^h\rangle_\star.$$

Thus, for $Z \in \{V^h, V^v, JV^v\}$, we have

$$2\langle \nabla^{\star}_{V^h} V^h, Z\rangle_{\star} \ = \ 2V^h(\langle V^h, Z\rangle_{\star}) - Z(\delta^{-2})$$

If $Z = V^h$, then

$$2\langle \nabla^{\star}_{V^h} V^h, V^h\rangle_{\star} \ = \ 2V^h(\langle V^h, V^h\rangle_{\star}) - V^h(\delta^{-2}) = V^h(\delta^{-2}) = -2\delta^{-3}$$

If $Z = V^v$ or JV^v, then

$$2\langle \nabla^{\star}_{V^h} V^h, Z\rangle_{\star} \ = 0$$

Finally, if $Z = JV^h$, then

$$\begin{aligned} 2\langle \nabla^{\star}_{V^h} V^h, JV^h\rangle_{\star} &= -JV^h(\delta^{-2}) - 2\langle [V^h, JV^h], V^h\rangle_{\star} \\ &= 2\langle (cJV, -R(JV, V)u), V^h\rangle_{\star} = 0. \end{aligned}$$

We conclude that

$$\nabla^{\star}_{V^h} V^h = -\delta^{-1} V^h \tag{2}$$

- Let $Y = V^v$. Then:

$$2\langle \nabla^{\star}_{V^h} V^v, Z\rangle_{\star} \ = \ -\langle [V^h, Z], V^v\rangle_{\star} - \langle [V^v, Z], V^h\rangle_{\star}.$$

If $Z = V^h$, we get:

$$2\langle \nabla^{\star}_{V^h} V^v, V^h\rangle_{\star} \ = \ -\langle [V^h, V^h], V^v\rangle_{\star} - \langle [V^v, V^h], V^h\rangle_{\star} \ = 0.$$

If $Z = JV^h$, we get:

$$\begin{aligned} 2\langle \nabla^{\star}_{V^h} V^v, JV^h\rangle_{\star} &= -\langle [V^h, JV^h], V^v\rangle_{\star} - \langle [V^v, JV^h], V^h\rangle_{\star} \\ &= \langle (cJV, -R(JV, V)u), V^v\rangle_{\star} + \langle (0, cJV), V^h\rangle_{\star} \\ &= -\langle R(JV, V)u), V\rangle. \end{aligned}$$

If $Z = V^v$, we get:

$$2\langle \nabla^{\star}_{V^h} V^v, V^v\rangle_{\star} \ = \ -\langle [V^h, V^v], V^v\rangle_{\star} - \langle [V^v, V^v], V^h\rangle_{\star} = 0.$$

If $Z = JV^v$, we get:

$$2\langle \nabla^{\star}_{V^h} V^v, JV^v\rangle_{\star} \ = \ -\langle [V^h, JV^v], V^v\rangle_{\star} - \langle [V^v, JV^v], V^h\rangle_{\star} = 0.$$

We conclude that

$$\nabla^{\star}_{V^h} V^v = -\frac{\delta^2}{2}\langle R(JV, V)u), V\rangle JV^h \tag{3}$$

• Let $Y = JV^v$. Then:

$$2\langle \nabla^\star_{V^h} JV^v, Z\rangle_\star \;=\; -\langle [V^h, Z], JV^v\rangle_\star - \langle [JV^v, Z], V^h\rangle_\star.$$

If $Z = V^h$, we get:

$$2\langle \nabla^\star_{V^h} JV^v, V^h\rangle_\star \;=\; -\langle [V^h, V^h], JV^v\rangle_\star - \langle [JV^v, V^h], V^h\rangle_\star = 0.$$

If $Z = JV^h$, we get:

$$\begin{aligned} 2\langle \nabla^\star_{V^h} JV^v, JV^h\rangle_\star \;&=\; -\langle [V^h, JV^h], JV^v\rangle_\star - \langle [JV^v, JV^h], V^h\rangle_\star \\ &=\; \langle (cJV, -R(JV,V)u), JV^v\rangle_\star - \langle (0, cV), V^h\rangle_\star \\ &=\; -\langle R(JV,V)u, JV\rangle \end{aligned}$$

If $Z = V^v$, we get:

$$2\langle \nabla^\star_{V^h} JV^v, V^v\rangle_\star \;=\; -\langle [V^h, V^v], JV^v\rangle_\star - \langle [JV^v, V^v], V^h\rangle_\star = 0.$$

If $Z = JV^v$, we get:

$$2\langle \nabla^\star_{V^h} JV^v, JV^v\rangle_\star \;=\; -\langle [V^h, JV^v], JV^v\rangle_\star - \langle [JV^v, JV^v], V^h\rangle_\star. \;= 0$$

We conclude that

$$\nabla^\star_{V^h} JV^v = -\frac{\delta^2}{2}\langle R(JV,V)u, JV\rangle JV^h \tag{4}$$

Now take $X = JV^h$. We have

$$\begin{aligned} 2\langle \nabla_{JV^h} Y, Z\rangle_\star \;&=\; JV^h(\langle Y, Z\rangle_\star) + Y(\langle JV^h, Z\rangle_\star) - Z(\langle JV^h, Y\rangle_\star) \\ &\quad + \langle [JV^h, Y], Z\rangle_\star - \langle [JV^h, Z], Y\rangle_\star - \langle [Y, Z], JV^h\rangle_\star. \end{aligned}$$

• If $Y = JV^h$, we have

$$2\langle \nabla^\star_{JV^h} JV^h, Z\rangle_\star \;=\; 2JV^h(\langle JV^h, Z\rangle_\star) - Z(\delta^{-2}) - 2\langle [JV^h, Z], JV^h\rangle_\star.$$

For $Z = V^h$, this gives

$$\begin{aligned} 2\langle \nabla^\star_{JV^h} JV^h, V^h\rangle_\star \;&=\; -V^h(\delta^{-2}) - 2\langle [JV^h, V^h], JV^h\rangle_\star \\ &=\; -V^h(\delta^{-2}) - 2\langle (cJV, -R(JV,V)u), JV^h\rangle_\star \\ &=\; 2\delta^{-3} - 2c\delta^{-2}. \end{aligned}$$

For $Z = JV^h$, this gives

$$\langle \nabla^\star_{JV^h} JV^h, JV^h \rangle_\star = JV^h(\delta^{-2}) = 0$$

For $Z = V^v$, this gives

$$2\langle \nabla^\star_{JV^h} JV^h, V^v \rangle_\star = -2\langle [JV^h, V^v], JV^h \rangle_\star = -2\langle (0, cJV), JV^h \rangle_\star = 0.$$

For $Z = JV^v$, this gives

$$2\langle \nabla^\star_{JV^h} JV^h, JV^v \rangle_\star = -2\langle [JV^h, JV^v], JV^h \rangle_\star = -2\langle (0, -cV), JV^h \rangle_\star = 0.$$

We conclude:

$$\nabla^\star_{JV^h} JV^h = (\delta^{-1} - c)V^h \tag{5}$$

- Continue to take $X = JV^h$, but now with $Y = V^h$.
  We have

$$\begin{aligned} 2\langle \nabla^\star_{JV^h} V^h, Z \rangle_\star &= JV^h(\langle V^h, Z \rangle_\star) + V^h(\langle JV^h, Z \rangle_\star) \\ &+ \langle [JV^h, V^h], Z \rangle_\star - \langle [JV^h, Z], V^h \rangle_\star - \langle [V^h, Z], JV^h \rangle_\star. \end{aligned}$$

For $Z = V^h$, this gives

$$2\langle \nabla^\star_{JV^h} V^h, V^h \rangle_\star = 0.$$

For $Z = JV^h$, this gives

$$2\langle \nabla^\star_{JV^h} V^h, JV^h \rangle_\star = V^h(\delta^{-2}) + 2\langle [JV^h, V^h], JV^h \rangle_\star$$

$$= -2\delta^{-3} + 2\langle (cJV, -R(JV, V)u), JV^h \rangle_\star = -2\delta^{-3} + 2c\delta^{-2}$$

For $Z = V^v$, this gives

$$2\langle \nabla^\star_{JV^h} V^h, V^v \rangle_\star = \langle (cJV, -R(JV, V)u), V^v \rangle_\star - \langle (0, cJV), V^h \rangle_\star$$

$$= \langle -R(JV, V)u, V \rangle$$

For $Z = JV^v$, this gives

$$2\langle \nabla^\star_{JV^h} V^h, JV^v \rangle_\star = \langle (cJV, -R(JV, V)u), JV^v \rangle_\star - \langle (0, -cV), V^h \rangle_\star$$

$$= \langle -R(JV, V)u, JV \rangle$$

We conclude:

$$\nabla^{\star}_{JV^h} V^h = (-\delta^{-1} + c) JV^h - \frac{1}{2}\langle R(JV,V)u, V\rangle V^v - \frac{1}{2}\langle R(JV,V)u, JV\rangle JV^v \tag{6}$$

• Let $Y = V^v$. Then:

$$2\langle \nabla^{\star}_{JV^h} V^v, Z\rangle_{\star} \;=\; \langle [JV^h, V^v], Z\rangle_{\star} - \langle [JV^h, Z], V^v\rangle_{\star} - \langle [V^v, Z], JV^h\rangle_{\star}.$$

If $Z = V^h$, we get:

$$\begin{aligned} 2\langle \nabla^{\star}_{JV^h} V^v, V^h\rangle_{\star} &= \langle [JV^h, V^v], V^h\rangle_{\star} - \langle [JV^h, V^h], V^v\rangle_{\star} - \langle [V^v, V^h], JV^h\rangle_{\star} \\ &= -\langle (cJV, -R(JV,V)u), V^v\rangle_{\star} = \langle R(JV,V)u, V\rangle. \end{aligned}$$

If $Z = JV^h$, we get:

$$\begin{aligned} 2\langle \nabla^{\star}_{JV^h} V^v, JV^h\rangle_{\star} &= \langle [JV^h, V^v], JV^h\rangle_{\star} - \langle [V^v, JV^h], JV^h\rangle_{\star} \\ &= 2\langle [JV^h, V^v], JV^h\rangle_{\star} = 2\langle (0, cJV), JV^h\rangle_{\star} = 0. \end{aligned}$$

If $Z = V^v$, we get:

$$2\langle \nabla^{\star}_{JV^h} V^v, V^v\rangle_{\star} \;=\; \langle [JV^h, V^v], V^v\rangle_{\star} - \langle [JV^h, V^v], V^v\rangle_{\star} - \langle [V^v, V^v], JV^h\rangle_{\star} = 0.$$

If $Z = JV^v$, we get:

$$\begin{aligned} 2\langle \nabla^{\star}_{JV^h} V^v, JV^v\rangle_{\star} &= \langle [JV^h, V^v], JV^v\rangle_{\star} - \langle [JV^h, JV^v], V^v\rangle_{\star} - \langle [V^v, JV^v], JV^h\rangle_{\star} \\ &= \langle (0, cJV), JV^v\rangle_{\star} - \langle (0, -cV), V^v\rangle_{\star} = 2c \end{aligned}$$

We conclude that

$$\nabla^{\star}_{JV^h} V^v = \frac{\delta^2}{2}\langle R(JV,V)u), V\rangle V^h + cJV^v \tag{7}$$

• Let $Y = JV^v$. Then:

$$2\langle \nabla^{\star}_{JV^h} JV^v, Z\rangle_{\star} \;=\; \langle [JV^h, JV^v], Z\rangle_{\star} - \langle [JV^h, Z], JV^v\rangle_{\star} - \langle [JV^v, Z], JV^h\rangle_{\star}.$$

If $Z = V^h$, we get:

$$\begin{aligned} 2\langle \nabla^{\star}_{JV^h} JV^v, V^h\rangle_{\star} &= \langle [JV^h, JV^v], V^h\rangle_{\star} - \langle [JV^h, V^h], JV^v\rangle_{\star} - \langle [JV^v, V^h], JV^h\rangle_{\star} \\ &= -\langle (cJV, -R(JV,V)u, JV^v\rangle_{\star} = \langle R(JV,V)u), JV\rangle. \end{aligned}$$

If $Z = JV^h$, we get:

$$\begin{aligned} 2\langle \nabla^{\star}_{JV^h} JV^v, JV^h\rangle_{\star} &= \langle [JV^h, JV^v], JV^h\rangle_{\star} - \langle [JV^h, JV^h], JV^v\rangle_{\star} - \langle [JV^v, JV^h], JV^h\rangle_{\star} \\ &= 2\langle [JV^h, JV^v], JV^h\rangle_{\star} = 2\langle (0, -cV), JV^h\rangle_{\star} = 0. \end{aligned}$$

If $Z = V^v$, we get:

$$\begin{aligned} 2\langle \nabla^\star_{JV^h} JV^v, V^v\rangle_\star &= \langle [JV^h, JV^v], V^v\rangle_\star - \langle [JV^h, V^v], JV^v\rangle_\star - \langle [JV^v, V^v], JV^h\rangle_\star \\ &= \langle [JV^h, JV^v], V^v\rangle_\star - \langle [JV^h, V^v], JV^v\rangle_\star \\ &= \langle (0, -cV), V^v\rangle_\star - \langle (0, cJV), JV^v\rangle_\star = -2c. \end{aligned}$$

If $Z = JV^v$, we get:

$$\begin{aligned} 2\langle \nabla^\star_{JV^h} JV^v, JV^v\rangle_\star &= \langle [JV^h, JV^v], JV^v\rangle_\star - \langle [JV^h, JV^v], JV^v\rangle_\star - \langle [JV^v, JV^v], JV^h\rangle_\star \\ &= 0. \end{aligned}$$

We conclude that

$$\nabla^\star_{JV^h} JV^v = \frac{\delta^2}{2}\langle R(JV, V)u), JV\rangle V^h - cV^v. \tag{8}$$

Setting $X = V^v$, we get:

$$\begin{aligned} 2\langle \nabla^\star_{V^v} Y, Z\rangle_\star &= V^v(\langle Y, Z\rangle_\star) + Y(\langle V^v, Z\rangle_\star) - Z(\langle V^v, Y\rangle_\star) \\ &\quad + \langle [V^v, Y], Z\rangle_\star - \langle [V^v, Z], Y\rangle_\star - \langle [Y, Z], V^v\rangle_\star \\ &= \langle [V^v, Y], Z\rangle_\star - \langle [V^v, Z], Y\rangle_\star - \langle [Y, Z], V^v\rangle_\star, \end{aligned}$$

for any $Y, Z \in \{V^h, JV^h, V^v, JV^v\}$.

• Now set $Y = V^h$. We get

$$2\langle \nabla^\star_{V^v} V^h, Z\rangle_\star = -\langle [V^v, Z], V^h\rangle_\star - \langle [V^h, Z], V^v\rangle_\star$$

For $Z = V^h$, this gives

$$2\langle \nabla^\star_{V^v} V^h, V^h\rangle_\star = -\langle [V^v, V^h], V^h\rangle_\star - \langle [V^h, V^h], V^v\rangle_\star = 0$$

For $Z = JV^h$, this gives

$$\begin{aligned} 2\langle \nabla^\star_{V^v} V^h, JV^h\rangle_\star &= \langle (0, cJV), V^h\rangle_\star + \langle (cJV, -R(JV, V)u), V^v\rangle_\star \\ &= -\langle (R(JV, V)u), V\rangle \end{aligned}$$

For $Z = V^v$, this gives

$$2\langle \nabla^\star_{V^v} V^h, V^v\rangle_\star = -\langle [V^v, V^v], V^h\rangle_\star - \langle [V^h, V^v], V^v\rangle_\star = 0$$

For $Z = JV^v$, this gives

$$2\langle \nabla^\star_{V^v} V^h, JV^v\rangle_\star = -\langle [V^v, JV^v], V^h\rangle_\star - \langle [V^h, JV^v], V^v\rangle_\star = 0$$

We conclude:

$$\nabla^{\star}_{V^v} V^h = -\frac{\delta^2}{2}\langle (R(JV,V)u), V\rangle JV^h \tag{9}$$

- Now set $Y = JV^h$. We get

$$2\langle \nabla^{\star}_{V^v} JV^h, Z\rangle_\star \;=\; \langle [V^v, JV^h], Z\rangle_\star - \langle [V^v, Z], JV^h\rangle_\star - \langle [JV^h, Z], V^v\rangle_\star$$

For $Z = V^h$, this gives

$$\begin{aligned} 2\langle \nabla^{\star}_{V^v} JV^h, V^h\rangle_\star &= \langle [V^v, JV^h], V^h\rangle_\star - \langle [V^v, V^h], JV^h\rangle_\star - \langle [JV^h, V^h], V^v\rangle_\star \\ &= -\langle (0, cJV), V^h\rangle_\star - \langle (cJV, -R(JV,V)u), V^v\rangle_\star \\ &= \langle R(JV,V)u), V\rangle \end{aligned}$$

For $Z = JV^h$, this gives

$$\begin{aligned} 2\langle \nabla^{\star}_{V^v} JV^h, JV^h\rangle_\star &= \langle [V^v, JV^h], JV^h\rangle_\star - \langle [V^v, JV^h], JV^h\rangle_\star - \langle [JV^h, JV^h], V^v\rangle_\star \\ &= 0. \end{aligned}$$

For $Z = V^v$, this gives

$$\begin{aligned} 2\langle \nabla^{\star}_{V^v} JV^h, V^v\rangle_\star &= \langle [V^v, JV^h], V^v\rangle_\star - \langle [V^v, V^v], JV^h\rangle_\star - \langle [JV^h, V^v], V^v\rangle_\star \\ &= 2\langle [V^v, JV^h], V^v\rangle_\star = 2\langle (0, -cJV), V^v\rangle_\star = 0. \end{aligned}$$

For $Z = JV^v$, this gives

$$\begin{aligned} 2\langle \nabla^{\star}_{V^v} JV^h, JV^v\rangle_\star &= \langle [V^v, JV^h], JV^v\rangle_\star - \langle [V^v, JV^v], JV^h\rangle_\star - \langle [JV^h, JV^v], V^v\rangle_\star \\ &= \langle [V^v, JV^h], JV^v\rangle_\star - \langle [JV^h, JV^v], V^v\rangle_\star \\ &= \langle (0, -cJV), JV^v\rangle_\star + \langle (0, cV), V^v\rangle_\star = -c + c = 0 \end{aligned}$$

We conclude:

$$\nabla^{\star}_{V^v} JV^h = \frac{\delta^2}{2}\langle R(JV,V)u), V\rangle V^h \tag{10}$$

- Let $Y = V^v$. Then:

$$\begin{aligned} 2\langle \nabla^{\star}_{V^v} V^v, Z\rangle_\star &= \langle [V^v, V^v], Z\rangle_\star - \langle [V^v, Z], V^v\rangle_\star - \langle [V^v, Z], V^v\rangle_\star \\ &= -2\langle [V^v, Z], V^v\rangle_\star \end{aligned}$$

If $Z = V^h$, we get:

$$2\langle \nabla^{\star}_{V^v} V^v, Z\rangle_\star \;=\; -2\langle [V^v, V^h], V^v\rangle_\star = 0.$$

If $Z = JV^h$, we get:

$$2\langle \nabla^{\star}_{V^v} V^v, Z\rangle_\star = -2\langle [V^v, JV^h], V^v\rangle_\star = 2\langle (0, cJV), V^v\rangle_\star = 0.$$

If $Z = V^v$, we get:

$$2\langle \nabla^{\star}_{V^v} V^v, Z\rangle_\star = -2\langle [V^v, V^v], V^v\rangle_\star = 0.$$

If $Z = JV^v$, we get:

$$2\langle \nabla^{\star}_{V^v} V^v, Z\rangle_\star = -2\langle [V^v, JV^v], V^v\rangle_\star = 0.$$

We conclude that

$$\nabla^{\star}_{V^v} V^v = 0. \tag{11}$$

- Let $Y = JV^v$. Then:

$$\begin{aligned} 2\langle \nabla^{\star}_{V^v} JV^v, Z\rangle_\star &= \langle [V^v, JV^v], Z\rangle_\star - \langle [V^v, Z], JV^v\rangle_\star - \langle [JV^v, Z], V^v\rangle_\star \\ &= -\langle [V^v, Z], JV^v\rangle_\star - \langle [JV^v, Z], V^v\rangle_\star \end{aligned}$$

If $Z = V^h$, we get:

$$2\langle \nabla^{\star}_{V^v} JV^v, V^h\rangle_\star = -\langle [V^v, V^h], JV^v\rangle_\star - \langle [JV^v, V^h], V^v\rangle_\star = 0.$$

If $Z = JV^h$, we get:

$$\begin{aligned} 2\langle \nabla^{\star}_{V^v} JV^v, JV^h\rangle_\star &= -\langle [V^v, JV^h], JV^v\rangle_\star - \langle [JV^v, JV^h], V^v\rangle_\star \\ &= \langle (0, cJV), JV^v\rangle_\star + \langle (0, -cV), V^v\rangle_\star = c - c = 0. \end{aligned}$$

If $Z = V^v$, we get:

$$2\langle \nabla^{\star}_{V^v} JV^v, V^v\rangle_\star = -\langle [V^v, V^v], JV^v\rangle_\star - \langle [JV^v, V^v], V^v\rangle_\star = 0.$$

If $Z = JV^v$, we get:

$$2\langle \nabla^{\star}_{V^v} JV^v, JV^v\rangle_\star = -\langle [V^v, JV^v], JV^v\rangle_\star - \langle [JV^v, JV^v], V^v\rangle_\star = 0.$$

We conclude that

$$\nabla^{\star}_{V^v} JV^v = 0. \tag{12}$$

Setting $X = JV^v$, we get:

$$2\langle \nabla^{\star}_{JV^v} Y, Z\rangle_\star = \langle [JV^v, Y], Z\rangle_\star - \langle [JV^v, Z], Y\rangle_\star - \langle [Y, Z], JV^v\rangle_\star,$$

for any $Y, Z \in \{V^h, JV^h, V^v, JV^v\}$.

- Now set $Y = V^h$. We get

$$2\langle \nabla^\star_{JV^v} V^h, Z\rangle_\star = -\langle [JV^v, Z], V^h\rangle_\star - \langle [V^h, Z], JV^v\rangle_\star$$

For $Z = V^h$, this gives

$$2\langle \nabla^\star_{JV^v} V^h, V^h\rangle_\star = -\langle [JV^v, V^h], V^h\rangle_\star - \langle [V^h, V^h], JV^v\rangle_\star = 0.$$

For $Z = JV^h$, this gives

$$\begin{aligned} 2\langle \nabla^\star_{JV^v} V^h, JV^h\rangle_\star &= -\langle [JV^v, JV^h], V^h\rangle_\star - \langle [V^h, JV^h], JV^v\rangle_\star \\ &= -\langle R(JV, V)u, JV\rangle_\star \end{aligned}$$

For $Z = V^v$, this gives

$$2\langle \nabla^\star_{JV^v} V^h, V^v\rangle_\star = -\langle [JV^v, V^v], V^h\rangle_\star - \langle [V^h, V^v], JV^v\rangle_\star = 0.$$

For $Z = JV^v$, this gives

$$2\langle \nabla^\star_{JV^v} V^h, JV^v\rangle_\star = -\langle [JV^v, JV^v], V^h\rangle_\star - \langle [V^h, JV^v], JV^v\rangle_\star = 0$$

We conclude:

$$\nabla^\star_{JV^v} V^h = -\frac{\delta^2}{2}\langle R(JV, V)u, JV\rangle JV^h. \tag{13}$$

- Now set $Y = JV^h$. We get

$$\begin{aligned} 2\langle \nabla^\star_{JV^v} JV^h, Z\rangle_\star &= \langle [JV^v, JV^h], Z\rangle_\star - \langle [JV^v, Z], JV^h\rangle_\star - \langle [JV^h, Z], JV^v\rangle_\star \\ &= \langle (0, cV), Z\rangle_\star - \langle [JV^v, Z], JV^h\rangle_\star - \langle [JV^h, Z], JV^v\rangle_\star \end{aligned}$$

For $Z = V^h$, this gives

$$\begin{aligned} 2\langle \nabla^\star_{JV^v} JV^h, V^h\rangle_\star &= \langle (0, cV), V^h\rangle_\star - \langle [JV^v, V^h], JV^h\rangle_\star - \langle (cJV, -R(JV, V)u), JV^v\rangle_\star \\ &= \langle R(JV, V)u), JV\rangle. \end{aligned}$$

For $Z = JV^h$, this gives

$$2\langle \nabla^\star_{JV^v} JV^h, JV^h\rangle_\star = \langle (0, cV), JV^h\rangle_\star - \langle [JV^v, JV^h], JV^h\rangle_\star - \langle [JV^h, JV^h], JV^v\rangle_\star = 0.$$

For $Z = V^v$, this gives

$$\begin{aligned} 2\langle \nabla^\star_{JV^v} JV^h, V^v\rangle_\star &= \langle (0, cV), V^v\rangle_\star - \langle [JV^v, V^v], JV^h\rangle_\star - \langle [JV^h, V^v], JV^v\rangle_\star \\ &= c - \langle (0, cJV), JV^v\rangle_\star = c - c = 0. \end{aligned}$$

For $Z = JV^v$, this gives

$$\begin{aligned} 2\langle \nabla^\star_{JV^v} JV^h, JV^v\rangle_\star &= \langle (0, cV), JV^v\rangle_\star - \langle [JV^v, JV^v], JV^h\rangle_\star - \langle [JV^h, JV^v], JV^v\rangle_\star \\ &= -\langle [JV^h, JV^v], JV^v\rangle_\star = \langle (0, cV), JV^v\rangle_\star = 0. \end{aligned}$$

We conclude:

$$\nabla^\star_{JV^v} JV^h = \frac{\delta^2}{2} \langle R(JV, V)u), JV\rangle V^h \tag{14}$$

- Let $Y = V^v$. Then:

$$\begin{aligned} 2\langle \nabla^\star_{JV^v} V^v, Z\rangle_\star &= \langle [JV^v, V^v], Z\rangle_\star - \langle [JV^v, Z], V^v\rangle_\star - \langle [V^v, Z], JV^v\rangle_\star \\ &= -\langle [JV^v, Z], V^v\rangle_\star - \langle [V^v, Z], JV^v\rangle_\star. \end{aligned}$$

If $Z = V^h$, we get:

$$2\langle \nabla^\star_{JV^v} V^v, V^h\rangle_\star = -\langle [JV^v, V^h], V^v\rangle_\star - \langle [V^v, V^h], JV^v\rangle_\star = 0.$$

If $Z = JV^h$, we get:

$$\begin{aligned} 2\langle \nabla^\star_{JV^v} V^v, JV^h\rangle_\star &= -\langle [JV^v, JV^h], V^v\rangle_\star - \langle [V^v, JV^h], JV^v\rangle_\star \\ &= \langle (0, -cV), V^v\rangle_\star + \langle (0, cJV), JV^v\rangle_\star = -c + c = 0. \end{aligned}$$

If $Z = V^v$, we get:

$$2\langle \nabla^\star_{JV^v} V^v, V^v\rangle_\star = -\langle [JV^v, V^v], V^v\rangle_\star - \langle [V^v, V^v], JV^v\rangle_\star = 0.$$

If $Z = JV^v$, we get:

$$2\langle \nabla^\star_{JV^v} V^v, JV^v\rangle_\star = -\langle [JV^v, JV^v], V^v\rangle_\star - \langle [V^v, JV^v], JV^v\rangle_\star = 0.$$

We conclude that

$$\nabla^\star_{JV^v} V^v = 0. \tag{15}$$

- Let $Y = JV^v$. Then:

$$\begin{aligned} 2\langle \nabla^\star_{JV^v} JV^v, Z\rangle_\star &= \langle [JV^v, JV^v], Z\rangle_\star - \langle [JV^v, Z], JV^v\rangle_\star - \langle [JV^v, Z], JV^v\rangle_\star \\ &= -2\langle [JV^v, Z], JV^v\rangle_\star. \end{aligned}$$

If $Z = V^h$, we get:

$$2\langle \nabla^\star_{JV^v} JV^v, V^h\rangle_\star = -2\langle [JV^v, V^h], JV^v\rangle_\star = 0.$$

If $Z = JV^h$, we get:

$$2\langle \nabla^\star_{JV^v} JV^v, JV^h\rangle_\star = -2\langle [JV^v, JV^h], JV^v\rangle_\star = 2\langle (0, -cV), JV^v\rangle_\star = 0.$$

If $Z = V^v$, we get:

$$2\langle \nabla^\star_{JV^v} JV^v, V^v\rangle_\star \;=\; -2\langle [JV^v, V^v], JV^v\rangle_\star = 0.$$

If $Z = JV^v$, we get:

$$2\langle \nabla^\star_{JV^v} JV^v, JV^v\rangle_\star \;=\; -2\langle [JV^v, JV^v], JV^v\rangle_\star = 0.$$

We conclude that

$$\nabla^\star_{JV^v} JV^v = 0. \tag{16}$$

◇

Lemma 1.3. *Let $a(w) = \langle w, V(w)\rangle, b(w) = \langle w, JV(w)\rangle$ be defined as above. Then*

1. $V^h(a) = V^h(b) = 0,$
2. $JV^h(a) = bc$, *and* $JV^h(b) = -ac,$
3. $V^v(a) = 1$, *and* $V^v(b) = 0$
4. $JV^v(a) = 0$, *and* $JV^v(b) = 1$

Proposition 1.4. *Let X be any vector field on T^1S of unit $\star$ length. Then $\|\nabla^\star_X \dot\varphi\|_\star = O(\delta^{-1})$. In particular:*

1. $\nabla^\star_{V^h}\dot\varphi = -a\delta^{-1}V^h - b\delta^{-1}JV^h - \frac{b^2K}{2}V^v + \frac{abK}{2}JV^v$
2. $\nabla^\star_{JV^h}\dot\varphi = b\delta^{-1}V^h - a\delta^{-1}JV^h + \frac{1}{2}KabV^v - \frac{1}{2}Ka^2JV^v$
3. $\nabla^\star_{V^v}\dot\varphi = \frac{Kab\delta^2}{2}JV^h + \left(\frac{-Kb^2\delta^2}{2} + 1\right)V^h$
4. $\nabla^\star_{JV^v}\dot\varphi = \left(1 - \frac{Ka^2\delta^2}{2}\right)JV^h + \frac{Kab\delta^2}{2}V^h$

Proof. 1.

$$\nabla^\star_{V^h}\dot\varphi = \nabla^\star_{V^h}(aV^h + bJV^h) = a\nabla^\star_{V^h}V^h + b\nabla^\star_{V^h}JV^h$$
$$= -a\delta^{-1}V^h - b\delta^{-1}JV^h + \frac{b}{2}\langle R(JV,V)u, V\rangle V^v + \frac{b}{2}\langle R(JV,V)u, JV\rangle JV^v,$$

where $u = aV + bJV$. So, $\nabla^\star_{V^h}\dot\varphi$ equals

$$-a\delta^{-1}V^h - b\delta^{-1}JV^h + \frac{b}{2}\langle R(JV,V)bJV, V\rangle V^v + \frac{b}{2}\langle R(JV,V)aV), JV\rangle JV^v,$$

$$= -a\delta^{-1}V^h - b\delta^{-1}JV^h - \frac{b^2K}{2}V^v + \frac{abK}{2}JV^v.$$

2.

$$\nabla^\star_{JV^h}\dot{\varphi} = \nabla^\star_{JV^h}(aV^h + bJV^h) = bcV^h - acJV^h + a\nabla^\star_{JV^h}V^h + b\nabla^\star_{JV^h}JV^h$$
$$= bcV^h - acJV^h + a\left((-\delta^{-1}+c)JV^h - \frac{1}{2}\langle R(JV,V)u, V\rangle V^v - \frac{1}{2}\langle R(JV,V)u, JV\rangle JV^v\right)$$
$$+b(\delta^{-1}-c)V^h$$

where $u = aV + bJV$. So, $\nabla^\star_{JV^h}\dot{\varphi}$ equals

$$= b\delta^{-1}V^h - a\delta^{-1}JV^h + \frac{1}{2}KabV^v - \frac{1}{2}Ka^2JV^v$$

3.

$$\nabla^\star_{V^v}\dot{\varphi} = \nabla^\star_{V^v}(aV^h + bJV^h) = V^h + a\nabla^\star_{V^v}V^h + b\nabla^\star_{V^v}JV^h$$
$$= V^h + a\left(-\frac{\delta^2}{2}\langle (R(JV,V)u), V\rangle JV^h\right) + b\left(\frac{\delta^2}{2}\langle R(JV,V)u), V\rangle V^h\right)$$
$$= V^h + a\left(-\frac{\delta^2}{2}\langle (R(JV,V)bJV), V\rangle JV^h\right) + b\left(\frac{\delta^2}{2}\langle R(JV,V)bJV), V\rangle V_h\right)$$
$$= \frac{Kab\delta^2}{2}JV^h + \left(\frac{-Kb^2\delta^2}{2} + 1\right)V^h$$

4.

$$\nabla^\star_{JV^v}\dot{\varphi} = \nabla^\star_{JV^v}(aV^h + bJV^h) = JV^h + a\nabla^\star_{JV^v}V^h + b\nabla^\star_{JV^v}JV^h$$
$$= JV^h + b\frac{\delta^2}{2}\langle R(JV,V)aV), JV\rangle V^h + a\left(-\frac{\delta^2}{2}\langle R(JV,V)aV, JV\rangle\right)JV^h$$
$$= \left(1 - \frac{Ka^2\delta^2}{2}\right)JV^h + \frac{Kab\delta^2}{2}V^h$$

◇

References

[1] K. Burns, H. Masur, C. Matheus, and A. Wilkinson, *Rates of mixing for the Weil-Petersson geodesic flow II: Exponential mixing in exceptional moduli spaces,* preprint (2016) available at `http://www.math.uchicago.edu/~wilkinso/papers/bmmwfinal.pdf`.