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WELL POSEDNESS FOR THE 1D ZAKHAROV-RUBENCHIK SYSTEM

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Abstract. Local and global well posedness results are established for the initial-value problem associated to the 1D Zakharov-Rubenchik system. We show that our results are sharp in some situations by proving ill-posedness results otherwise. The global results allow us to study the norm growth of solutions corresponding to the Schrödinger equation term. We use ideas recently introduced to study the classical Zakharov systems.

1. Introduction

In this paper we will deal with issues concerning well posedness for the initial-value problem (IVP) associated to the Zakharov-Rubenchik system; that is,

$$\begin{cases} i\partial_t B + \omega\partial_x^2 B = \gamma(u - \frac{1}{2}\nu\,\rho + q\,|B|^2)B, \quad x, t \in \mathbb{R}, \\ \theta\,\partial_t\rho + \partial_x(u - \nu\,\rho) = -\gamma\partial_x(|B|^2), \\ \theta\,\partial_t u + \partial_x(\beta\,\rho - \nu\,u) = \frac{1}{2}\gamma\partial_x(|B|^2), \end{cases} \tag{1.1}$$

where B is a complex function, ρ and u are real functions, $\theta \neq 0$, γ, ω are real numbers, $\beta > 0$, $\beta - \nu^2 \neq 0$, and $q = \gamma + \nu(\gamma\nu - 1)/2(\beta - \nu^2)$. This system is the 1D version of the most general system deduced by Zakharov and Rubenchik [19] to describe the interaction of spectrally narrow high-frequency wave packets of small amplitude with low-frequency acoustic type oscillations. It has the following form:

$$\begin{cases} i\partial_t\psi + iv_g\partial_z\psi = -\frac{\omega''}{2}\,\partial_z^2\psi - \frac{v_g}{2k}\Delta_\perp\psi + (q|\psi|^2 + \beta\rho + \alpha\partial_z\varphi)\psi, \\ \partial_t\rho + \rho_0\Delta\varphi + \alpha\partial_z|\psi|^2 = 0, \\ \partial_t\varphi + \frac{c^2}{\rho_0}\,\rho + \beta|\psi|^2 = 0, \end{cases} \tag{1.2}$$

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where ψ is a complex function denoting the complex amplitude of the high-frequency carrywave, ρ, φ are real functions denoting the density fluctuation and the hydrodynamic potential respectively, α, β, $q \in \mathbb{R}$, and $\Delta_\perp = \partial_x^2 + \partial_y^2$.

Concerning well posedness for the IVP associated to (1.1), Oliveira [14] proved local and global well posedness for data in $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$. He also studied the existence and orbital stability of solitary wave solutions for (1.1). The method used in [14] to establish local well posedness follows the ideas of Tsutsumi and Ozawa [15] to treat the classical Zakharov systems,

$$\begin{cases} i\partial_t u + \Delta u = u\, N, \qquad x \in \mathbb{R}^n,\, t \in \mathbb{R}, \;\; n \geq 1, \\ \partial_t^2 N - \Delta N = \Delta(|u|^2). \end{cases} \tag{1.3}$$

For system (1.2), Ponce and Saut [16] proved that the Cauchy problem is locally well posed in $H^s(\mathbb{R}^n)$, $s > n/2$, in space dimension $n = 2$, 3. There are several open questions regarding this system such as global existence of solutions.

In the last few years progress has been made towards understanding the behavior of solutions for the Zakharov system (1.3). Here we will apply and extend some of the new techniques introduced to study system (1.3) to obtain the results we will describe next.

The first issue we investigate is related to the local well-posedness theory. Before stating our results in this direction, we make the following change of variables: Setting

$$\rho = \psi_1 + \psi_2, \quad \text{and} \quad u = \sqrt{\beta}\,(\psi_1 - \psi_2) \tag{1.4}$$

we write system (1.1) as

$$\begin{cases} i\partial_t B + \omega\partial_x^2 B = \gamma\left(\sqrt{\beta} - \frac{\nu}{2}\right)\psi_1 B - \gamma\left(\sqrt{\beta} + \frac{\nu}{2}\right)\psi_2 B + \gamma q |B|^2 B, \\ \theta\partial_t\psi_1 + (\sqrt{\beta} - \nu)\partial_x\psi_1 = \frac{\gamma}{2}\big(-1 + \frac{\nu}{2\sqrt{\beta}}\big)\partial_x(|B|^2), \\ \theta\partial_t\psi_2 - (\nu + \sqrt{\beta})\partial_x\psi_2 = \frac{\gamma}{2}\big(-1 - \frac{\nu}{2\sqrt{\beta}}\big)\partial_x(|B|^2). \end{cases} \tag{1.5}$$

Observe that, if (B, ψ_1, ψ_2) is a solution of (1.5) with initial data $(B_0, \psi_{10}, \psi_{20})$, then $(B^\lambda, \psi_1^\lambda, \psi_2^\lambda) = (\lambda\, B, \lambda^2\, \psi_1, \lambda^2\, \psi_2)$ is also a solution of (1.5) with data $(\lambda\, B_0, \lambda^2\, \psi_{10}, \lambda^2\, \psi_{20})$. Hence a scaling argument suggests local well posedness for the IVP (1.5) for data in $H^k(\mathbb{R}) \times H^s(\mathbb{R}) \times H^l(\mathbb{R})$ for $k > -1/2$, s, $l \geq -3/2$.

The local well-posedness theory for (1.5) is as follows.

Theorem 1.1. *The Zakharov-Rubenchik system* (1.5) *is locally well posed for initial data* $(B_0, \psi_{10}, \psi_{20}) \in H^k(\mathbb{R}) \times H^l(\mathbb{R}) \times H^s(\mathbb{R})$, *where*

$$-\tfrac{1}{2} < k - l \leq 1, \;\; 0 \leq l + \tfrac{1}{2} \leq 2k, \;\; -\tfrac{1}{2} < k - s \leq 1, \;\; 0 \leq s + \tfrac{1}{2} \leq 2k. \tag{1.6}$$

Remark 1.1. Notice that the "critical" indices of the Sobolev spaces where local well posedness is expected, i.e., $(s,k,l) = (-1/2,-3/2,-3/2)$, are not reached in our theorem.

Remark 1.2. Since solutions of system (1.1) satisfy the conserved quantities

$$I_1(t) = \int_{\mathbb{R}} |B|^2\,dx, \tag{1.7}$$

$$I_2(t) = \frac{\omega}{2}\int_{\mathbb{R}} |B_x|^2 + \frac{\gamma q}{4}\int_{\mathbb{R}} |B|^2 + \frac{\gamma}{2}\int_{\mathbb{R}} (u - \frac{\nu}{2})\rho|B|^2 + \frac{\beta}{4}\int_{\mathbb{R}} |\rho|^2 + \frac{1}{4}\int_{\mathbb{R}} |u|^2 - \frac{\nu}{2}\int_{\mathbb{R}} u\rho, \tag{1.8}$$

$$I_3(t) = \int_{\mathbb{R}} u\rho\,dx + \frac{i}{2}\int_{\mathbb{R}} (B\bar{B}_x - B_x\bar{B})\,dx, \tag{1.9}$$

and

$$I_4(t) = I_2(t) + \tfrac{\nu}{2\theta}I_3(t) = \tfrac{\omega}{2}\int |B_x|^2dx + \tfrac{\gamma q}{4}\int |B|^4dx + \tfrac{\gamma}{2}\int \left(u - \tfrac{\nu}{2}\rho\right)|B|^2dx + \tfrac{\beta}{4}\int |\rho|^2dx + \tfrac{1}{4}\int |u|^2dx + \tfrac{i\nu}{4\theta}\int (B\overline{B}_x - B_x\overline{B})dx \tag{1.10}$$

(see [14]), assuming $\omega > 0$ and $\beta - \nu^2 > 0$ we can deduce the global existence of solutions for data in the space $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$ (see Proposition 3.1 below). In particular, this result implies the stability of solitary wave solutions proved in [14] for data in the energy space.

To prove the local result we will follow the scheme used by Ginibre, Tsutsumi and Velo[9] to establish well posedness for the IVP associated to the Zakharov system (1.3). They used Bourgain and Kenig, Ponce and Vega arguments. Since system (1.5) also contains a cubic term of the function B we need to have good trilinear estimates in addition to bilinear estimates already used in [9].

Since solutions of system (1.5) have the L^2-norm of B invariant ((1.8)), a natural question regarding global well posedness arises. Can we extend B to any time? The answer is positive. Moreover, we have the following.

Theorem 1.2. *The Zakharov-Rubenchik system* (1.5) *is globally well posed for initial data* $(B_0, \psi_{10}, \psi_{20}) \in H^k(\mathbb{R}) \times H^l(\mathbb{R}) \times H^l(\mathbb{R})$, *where* $0 \le k = l + \frac{1}{2}$.

Remark 1.3. Notice that we establish global existence of solutions for data in $L^2(\mathbb{R}) \times H^{-1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$. Moreover, we can prove global well posedness for data with Sobolev indices satisfying $k = l + \frac{1}{2}$, $k \ge 0$. In

[7] global well posedness for (1.3) is only presented in the extremal point $(0, -1/2, -3/2)$.

To prove Theorem 1.2 we will use the arguments of Colliander, Holmer and Tzirakis in [7] recently put forward to construct global solutions for the 1D Zakharov system (1.3) in a similar situation. This is based on the conservation property (1.7) and the local theory.

Notice that we can write the system (1.5) in its integral equivalent form

$$\begin{cases} B(t) = U(t)(B_0) + i\displaystyle\int_0^t U(t-t')\,\big(|B|^2 + \psi_1 + \psi_2\big)(t')\,B(t')\,dt', \\ \psi_1(t) = W_+(t)(\psi_{10}) + \displaystyle\int_0^t W_+(t-t')\partial_x(|B|^2)(t')\,dt', \\ \psi_2(t) = W_-(t)(\psi_{10}) - \displaystyle\int_0^t W_-(t-t')\partial_x(|B|^2)(t')\,dt'. \end{cases} \tag{1.11}$$

The idea of the proof is to perform an iteration scheme. We describe the iteration process next only considering the extremal case $(0, -1/2, -1/2)$. One of the key observations is that the interaction of the functions ψ_1 and ψ_2 in the second and third equations are only with the function B. In other words, we can rewrite (1.11) as

$$B(t) = U(t)(B_0) + i\int_0^t U(t-t')\,\big(|B|^2 + \psi_1 + \psi_2\big)(t')\,B(t')dt', \tag{1.12}$$

where $\psi_1(t)$ and $\psi_1(t)$ are as in (1.11). So one will try to control the growth of the L^2-norm of B using the conserved quantity (1.7) and controlling the growths of ψ_1 and ψ_2 in the corresponding H^l-norms. The iteration scheme is as follows: given the time of local well posedness T_1 we denote $B(T_1) = B_1$, $\psi_1(T_1) = \psi_{11}$ and $\psi_1(T_1) = \psi_{21}$. Find T_2 via the local result and then iterate the local theory until time $T_j + T_{j-1} + \cdots + T_1$. These T_j may shrink due to the growth of $\|B_j\|_{L^2}$ and $\|\psi_{kj}\|_{H^{-1/2}}$, $k = 1, 2$. To iterate we have to remake the local theory and then use some spaces with less regularity to perform the iteration. Since $\|B(t)\|_{L^2} = \|B_0\|_{L^2}$ for all t, the reduction of T_j is then only forced through the growth of $\|\psi_{kj}\|_{H^{-1/2}}$, $k = 1, 2$. We then consider (1.11) posed at $t = T_j$, $(B, \psi_1, \psi_2)(0) = (B_j, \psi_{1j}, \psi_{2j})$. Here we may have two situations:

(1) $\|\psi_{1j}\|_{H^{-1/2}} \gg \|B_j\|_{L^2}$ and $\|\psi_{2j}\|_{H^{-1/2}} \gg \|B_j\|_{L^2}$ or
(2) $\min\{\|\psi_{1j}\|_{H^{-1/2}},\ \|\psi_{2j}\|_{H^{-1/2}}\} \lesssim \|B_j\|_{L^2}$.

We will restrict ourselves to discussing possibility (1). On $[T_j, T_{j+1}]$ we write ψ_1 and ψ_2 in terms of ψ_{10}, ψ_{20} and B

$$\begin{cases}\psi_1(t) = W_+(t-T_j)(\psi_{10}) + \int_0^t W_+(t-t')\partial_x(|B|^2)(t')\,dt', \\ \psi_2(t) = W_-(t-T_j)(\psi_{20}) - \int_0^t W_-(t-t')\partial_x(|B|^2)(t')\,dt'. \end{cases} \tag{1.13}$$

As t moves from $t = T_j$ to $t = T_{j+1}$, the first terms in both equations in (1.13) stay the same size in the H^l-norm. Any growth in the H^k-norm as t evolves thus arises from the second terms. But these terms are intuitively controlled by the conserved size of B in the L^2-norm. Also these terms should be small if $[T_j, T_{j+1}]$ is small since it takes a while for B to contribute to the growth of ψ_1 and ψ_2. Indeed, the following estimates hold:

$$\begin{aligned}\|\psi_1\|_{H^{-1/2}} &\le c\|\psi_{1j}\|_{H^{-1/2}} + c\,\Delta T^\epsilon \|B_0\|^2_{L^2} \\ \|\psi_2\|_{H^{-1/2}} &\le c\|\psi_{2j}\|_{H^{-1/2}} + c\,\Delta T^\epsilon \|B_0\|^2_{L^2}.\end{aligned} \tag{1.14}$$

We then iterate the local well-posedness norm with uniform steps of size

$$\Delta T \sim \min\{\|\psi_{1j}\|^{-\alpha}_{H^{-1/2}}, \|\psi_{2j}\|^{-\alpha}_{H^{-1/2}}\} \tag{1.15}$$

over m steps with

$$m \sim \frac{\min\{\|\psi_{1j}\|_{H^{-1/2}}, \|\psi_{2j}\|_{H^{-1/2}}\}}{T^\epsilon \|B_0\|^2_{L^2}}. \tag{1.16}$$

We thus extend the solution past T_j to $[T_j, T_j + m\Delta T]$ without doubling the size of $\|\psi_{kj}\|_{H^{-1/2}}$, $k = 1, 2$. A simple computation reveals

$$m\,\Delta T \sim \|B_0\|^{-2} \min\{\|\psi_{10}\|, \|\psi_{20}\|\}^{1-\alpha+\epsilon\alpha}. \tag{1.17}$$

If $1 - \alpha + \epsilon\alpha \ge 0$, we obtain global well posedness by iterating the whole process. We notice that the factor ϵ in (1.14) can be obtained via a new local theory where the parameters b, c in the $X^{s,b}$ and $W^{l,c}_\pm$ space are not necessarily greater than 1/2. But we still can show that (1.7) holds.

The next issue that we are concerned with is the study of the growth of the H^s-norm for solutions of (1.5), corresponding to data in $H^s(\mathbb{R})$ for noninteger values of s. The presence of the conserved quantities (1.7)-(1.10) and the local existence theory allow us to obtain upper "polynomial" bounds, for the H^s-norm of these solutions.

Energy-type estimates were previously used to show that, for solutions of the IVP associated to the nonlinear Schrödinger equation, the H^s-norm of these solutions have an exponential bound; i.e.,

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} \le c^{|t|}. \tag{1.18}$$

This can be deduced right away from the local existence theory since

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} \leq \|\phi\|_{H^s} + c\,\|\phi\|_{H^s}. \tag{1.19}$$

Bourgain [4] proved polynomial bounds for certain Hamiltonian PDE's, including the nonlinear Schrödinger equation and the generalized KdV in the periodic setting. He observed that a slight improvement of (1.19)

$$\sup_{t\in[0,T]} \|u(t)\|_{H^s} \leq \|\phi\|_{H^s} + c\,\|\phi\|_{H^s}^{1-\delta}, \;\; 0<\delta<1, \tag{1.20}$$

implies the polynomial bound $\|u(t)\|_{H^s} \leq c\,|t|^{1/\delta}$. Staffilani [17] showed polynomial bounds for the nonlinear Schrödinger equation and its generalized form on the line. In [8] Colliander and Staffilani established polynomial bounds for solutions of the 1D Zakharov system (1.3). Following their arguments we prove the following:

Proposition 1.1. *For initial data $B_0, \psi_{10}, \psi_{20} \in \mathcal{S}$, the global solution of* (1.5) *satisfies*

$$\|B(t)\|_{H^s} \lesssim 1 + |t|^{(s-1)+}. \tag{1.21}$$

Finally, we would like to know whether the local well-posedness results obtained here are sharp. In this direction, we should present some ill-posedness results for the IVP associated to (1.5). Our results are inspired by those recently obtained by Holmer [10] for the 1D Zakharov system (1.3).

Our first result guarantees that the local result in Theorem 1.1 is the best possible when $0<k<1$, $l>2k-\frac{1}{2}$ or $k\leq 0$, $l>-1/2$. To do this we use the notion introduced by Christ, Colliander and Tao [6] called norm-inflation. More precisely, we have the next theorem.

Theorem 1.3. *Assume that either $0<k<1$, $l>2k-\frac{1}{2}$ or $k\leq 0$, $l>-1/2$. Then, there are some constants $0<T\ll 1$, $\alpha=\alpha(k,s)$ and a sequence $f_N\in\mathcal{S}$ with $\|f_N\|_{H^k}\leq 1$ for all $N\in\mathbb{N}$ such that the solution $(B_N,\phi_{1,N},\phi_{2,N})$ of the Zakharov-Rubenchik system* (1.5) *on $[0,T]$ with initial data $(f_N,0,0)$ satisfies*

$$\|\psi_{1,N}(t)\|_{H^l} \gtrsim t\cdot N^{\alpha}$$

for $0<t\leq T$ and $N\gtrsim \frac{1}{t}$.

Similarly, if either $0<k<1$, $s>2k-\frac{1}{2}$ or $k\leq 0$, $s>-1/2$, then we can find some constants $T>0$, $\alpha=\alpha(k,s)$ and a sequence $g_N\in\mathcal{S}$ with $\|g_N\|_{H^k}\leq 1$ for all N such that the solution $(B_N,\phi_{1,N},\phi_{2,N})$ of (1.5) *on $[0,T]$ with initial data $(g_N,0,0)$ satisfies*

$$\|\psi_{2,N}(t)\| \gtrsim t\cdot N^{\alpha} \quad \textit{for } t\in(0,T] \textit{ and } N\gtrsim \tfrac{1}{t}.$$

The second result is weaker than the previous one. It only affirms that a local result cannot be obtained by Picard iteration. This was first proved by Bourgain for the Korteweg-de Vries and nonlinear Schrödinger equation [3]. Several improvements have been obtained for other models. In particular, we use the ideas in [12], [13]. In our case, we show that the best local result suggested for the scaling argument cannot be attained.

Theorem 1.4. *For any $k \in \mathbb{R}$, l, s with $\min\{l, s\} < -1/2$ and $T > 0$, the map data-to-solution associated to the Zakharov-Rubenchik system* (1.5) *from $H^k(\mathbb{R}) \times H^l(\mathbb{R}) \times H^s(\mathbb{R})$ to $C([0,T]; H^k(\mathbb{R}) \times H^l(\mathbb{R}) \times H^s(\mathbb{R}))$ is not C^2 at the origin* $(0,0,0)$.

Finally, we show ill posedness for data in $H^k(\mathbb{R}) \times H^l(\mathbb{R}) \times H^s(\mathbb{R}))$, $k = 0$ and $\min\{l, s\} < -3/2$; this assures that below the indices suggested by the scaling argument the IVP is in fact ill posed. Results in this direction were introduced first by Birnir, Kenig, Ponce, Svanstedt, and Vega [1], and generalizations by Christ, Colliander and Tao [6]. The result reads as follows.

Theorem 1.5. *Assume that $k = 0$ and $\min\{l, s\} < -3/2$. Then, given $T > 0$ and $\delta > 0$, there exists a pair $(B_0, \psi_{+0}, \psi_{-0})$ and $(B'_0, \psi'_{+0}, \psi'_{-0})$ of initial data with*

$$\|B_0\|_{H^k}, \|\psi_{+0}\|_{H^l}, \|\psi_{-0}\|_{H^s}, \|B'_0\|_{H^k}, \|\psi'_{+0}\|_{H^l}, \|\psi'_{-0}\|_{H^s} \le 1$$

such that the associated solutions (B, ψ_+, ψ_-) and (B', ψ'_+, ψ'_-) of the Zakharov-Rubenchik system (1.5) *on the time interval $[0,T]$ start close:*

$$\|B_0 - B'_0\|_{H^k}, \|\psi_{+0} - \psi'_{+0}\|_{H^l} + \|\psi_{-0} - \psi'_{-0}\|_{H^s} \le \delta$$

but they become separated by time T (in the Schrödinger variable):

$$\|B - B'\|_{L^\infty_{[0,T]} H^k_x} \sim 1.$$

A final remark regarding the nature of the ill-posedness results can be found in the discussion presented by Tzvetkov in [18]. In this context, Theorems 1.4 and 1.5 show that the IVP is not semi-linearly well posed and Theorem 1.3 that well posedness is not possible for the IVP in that situation.

This paper is organized as follows: in Section 2, we will deal with the local and global theory for system (1.5). In Section 3, the growth rate of the H^s-norm will be established. Finally, we will show the sharpness of some of the results by establishing some ill-posedness results in Section 4.

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2. Well posedness of Zakharov-Rubenchik system

This section is devoted to the proof of the following results, Theorem 1.1 and Theorem 1.2.

We split the proof of these theorems into two steps: firstly, we verify that the linear and multilinear estimates obtained by Ginibre, Tsutsumi and Velo [9] directly imply that the Zakharov-Rubenchik system (1.1) is locally well-posed (Theorem 1.1); secondly, we modify this local well-posedness result and apply the conservation of the L^2-mass of B (along the lines of a recent work [7] of Colliander, Holmer and Tzirakis) to derive Theorem 1.2.

2.1. **Local well posedness.** We start with the linear and multilinear estimates derived in [9] for the Zakharov and Benney systems. Let $U(t) = e^{it\Delta}$ be the free Schrödinger linear group and $W_\pm(t) = e^{\pm t\partial_x}$ be the free linear group of the transport equations.

In the sequel, we use the following norms:

$$\|f\|_{H^s} := \Big(\int \langle\xi\rangle^{2s}|\widehat{f}(\xi)|^2 d\xi\Big)^{1/2},$$

$$\|u\|_{X^{k,b}} := \Big(\int \langle\xi\rangle^{2k}\langle\tau+\xi^2\rangle^{2b}|\widehat{u}(\xi,\tau)|^2 d\xi d\tau\Big)^{1/2},$$

$$\|v\|_{W_\pm^{l,c}} := \Big(\int \langle\xi\rangle^{2l}\langle\tau\pm\xi\rangle^{2c}|\widehat{v}(\xi,\tau)|^2 d\xi d\tau\Big)^{1/2},$$

where $\langle x\rangle := 1+|x|$.

Let κ be a smooth bump function so that $\kappa(t)=1$ for $|t|\leq 1$ and $\kappa(t)=0$ for $|t|\geq 2$. Define $\kappa_T(t) := \kappa(t/T)$. In this setting, Ginibre, Tsutsutmi and Velo [9] proved the following linear and multilinear estimates for the Zakharov and Benney systems:

Lemma 2.1 (Linear group estimates). *The following holds:*

- $\|\kappa(t)U(t)u_0\|_{X^{k,b}} \lesssim_b \|u_0\|_{H^k}$ *and*
- $\|\kappa(t)W_\pm(t)v_0\|_{W_\pm^{l,c}} \lesssim_c \|v_0\|_{H^l}$.

Moreover, if $0<T<1$ *and* $-\frac{1}{2}<b'\leq b<\frac{1}{2}$, *then*

- $\|\kappa_T(t)u\|_{X^{k,b'}} \lesssim_{b',b} T^{b-b'}\|u\|_{X^{k,b}}$,
- $\|\kappa_T(t)v\|_{W_\pm^{l,b'}} \lesssim_{b',b} T^{b-b'}\|v\|_{W_\pm^{l,b}}$.

In the sequel, we define

$$U *_R F(t,x) := \int_0^t U(t-t')F(t',x)dt' \quad \text{and}$$

$$W_\pm *_R F(t,x) := \int_0^t W_\pm(t-t')F(t',x)dt'.$$

Lemma 2.2 (Duhamel estimates)**.** *Let* $-\frac{1}{2} < -a \le 0 \le b \le 1-a$. *The following holds:*

- $\|\kappa_T(t)U *_R F\|_{X^{k,b}} \lesssim T^{1-b-a}\|F\|_{X^{k,-a}}$,
- $\|\kappa_T(t)W_\pm *_R G\|_{W_\pm^{l,b}} \lesssim T^{1-b-a}\|G\|_{W_\pm^{l,-a}}$.

Next, we recall the following multilinear estimates derived by Bourgain [2] and Ginibre, Tsutsumi and Velo [9].

Lemma 2.3 (Multilinear estimates)**.** *The following holds:*

- *for any* $k \ge 0$ *and* $1/2 < b < 5/8$,

$$\begin{aligned}\|uv\overline{z}\|_{X^{k,b-1}} &\lesssim \|u\|_{X^{k,3/8+}}\|v\|_{X^{0,3/8+}}\|z\|_{X^{0,3/8+}} \\ &+ \|u\|_{X^{0,3/8+}}\|v\|_{X^{k,3/8+}}\|z\|_{X^{0,3/8+}} + \|u\|_{X^{0,3/8+}}\|v\|_{X^{0,3/8+}}\|z\|_{X^{k,3/8+}}\end{aligned}$$

- *for any* $k \ge 0$, $l \ge -\frac{1}{2}$, $k-l \le \min\{1, 2c\}$, $b, b', c > \frac{3}{8}$,

$$\|\psi_\pm u\|_{X^{k,-c}} \lesssim \|\psi_\pm\|_{W_\pm^{l,b'}}\|u\|_{X^{k,b}}$$

- *for any* $k \ge 0$, $k-l > -1/2$, $2k-l \ge 1/2$, $b, c > \frac{3}{8}$, $l-k+1 \le 2b$, $l+1-k < 2c+1/2$,

$$\|\partial_x(u\overline{v})\|_{W_\pm^{l,-c}} \lesssim \|u\|_{X^{k,b}}\|v\|_{X^{k,b}}.$$

Combining these lemmas, it is a standard matter to show the local well-posedness result of Theorem 1.1.

Proof of Theorem 1.1. We already know that it suffices to consider the equation (1.5). The integral formulation of (1.5) is

$$\begin{cases} B(t) = U(t)B_0 + iU *_R \left\{\psi_1 B + \psi_2 B + |B|^2 B\right\}(t), \\ \psi_1(t) = W_+(t)(\psi_{10}) + W_+ *_R \partial_x(|B|^2)(t), \\ \psi_2(t) = W_-(t)\psi_{20} - W_- *_R \partial_x(|B|^2)(t). \end{cases}$$

Fix $0 < T < 1$ and define

$$\begin{aligned}\Lambda_T(B)(t) :&= \kappa(t)U(t)B_0 \\ &+ i\kappa_T(t)U *_R \left\{(\kappa W_+(\psi_{10}) + W_+ *_R \partial_x(|B|^2))B\right\}(t) \\ &+ i\kappa_T(t)U *_R \left\{(\kappa W_-(\psi_{20}) - W_- *_R \partial_x(|B|^2))B\right\}(t) \\ &+ i\kappa_T(t)U *_R (|B|^2 B)(t).\end{aligned} \tag{2.1}$$

Our task is reduced to finding a fixed point $B = \Lambda_T(B)$ of Λ_T. Applying the linear and multilinear estimates of Lemmas 2.1, 2.2 and 2.3, we get

$$\|\Lambda_T(B)\|_{X^{k,1/2+}} \lesssim \|B_0\|_{H^k} + T^{3/8-}\left\{\|\psi_{10}\|_{H^l} + \|\psi_{20}\|_{H^s} + \|B\|^2_{X^{k,1/2+}}\right\} \cdot \|B\|_{X^{k,1/2+}}$$

and

$$\|\Lambda_T(B) - \Lambda_T(\widetilde{B})\|_{X^{k,1/2+}} \lesssim T^{3/8-}\Big\{\|\psi_{10}\|_{H^l} + \|\psi_{20}\|_{H^s} + (1+\|B\|_{X^{k,1/2+}})(1+\|\widetilde{B}\|_{X^{k,1/2+}})\Big\}\|B-\widetilde{B}\|_{X^{k,1/2+}}.$$

This implies that Λ_T is a contraction of a large ball of $X^{k,1/2+}$ when T is sufficiently small (depending only on $\|u_0\|_{H^k}, \|\psi_{10}\|_{H^l}, \|\psi_{20}\|_{H^s}$). In particular, (1.5) is locally well posed for some time interval $[0,T]$ with $T = T(\|u_0\|_{H^k}, \|\psi_{10}\|_{H^l}, \|\psi_{20}\|_{H^s}) > 0$. □

2.2. **Global well posedness.** Following the lines of Colliander, Holmer and Tzirakis [7], we begin with some refinements of the linear and multilinear estimates of Lemmas 2.1, 2.2 and 2.3

Lemma 2.4 (Linear estimates). *For $0 < T \leq 1$, $t \in \mathbb{R}$ and $0 \leq b, b_1 \leq 1/2$, the following holds:*

$$\|U(t)u_0\|_{H^s} = \|u_0\|_{H^s}, \quad \|\kappa_T(t)U(t)u_0\|_{X^{s,b}} \lesssim T^{\frac{1}{2}-b}\|u_0\|_{H^s};$$
$$\|W_\pm(t)v_0\|_{H^l} = \|v_0\|_{H^l}, \quad \|\kappa_T(t)W_\pm(t)v_0\|_{W_\pm^{l,b_1}} \lesssim T^{\frac{1}{2}-b}\|v_0\|_{H^l}.$$

Proof. See Lemma 2.1 of [7]. □

Lemma 2.5 (Duhamel estimates). *For $0 < T \leq 1$, $0 \leq c, c_1 < 1/2$, $0 \leq b \leq b + c \leq 1$ and $0 \leq b_1 \leq b_1 + c_1 \leq 1$, we have*

$$\|U *_R F\|_{C^0([0,T],H^k)} \lesssim T^{\frac{1}{2}-c}\|F\|_{X^{k,-c}},$$
$$\|\kappa_T U *_R F\|_{X^{k,b}} \lesssim T^{1-b-c}\|F\|_{X^{k,-c}};$$
$$\|W_\pm *_R G\|_{C^0([0,T],H^l)} \lesssim T^{\frac{1}{2}-c_1}\|G\|_{W_\pm^{l,-c_1}},$$
$$\|W_\pm *_R G\|_{W_\pm^{l,b_1}} \lesssim T^{1-b_1-c_1}\|G\|_{W_\pm^{l,-c_1}}.$$

Proof. See Lemma 2.3 of [7]. □

Lemma 2.6 (Multilinear estimates). *The following holds:*

a) for any $k \geq 0$ and $3/8 < c < 1/2$,

$$\|uv\overline{z}\|_{X^{k,-c}} \lesssim \|u\|_{X^{k,3/8+}}\|v\|_{X^{0,3/8+}}\|z\|_{X^{0,3/8+}}$$

$$+ \|u\|_{X^{0,3/8+}}\|v\|_{X^{k,3/8+}}\|z\|_{X^{0,3/8+}} + \|u\|_{X^{0,3/8+}}\|v\|_{X^{0,3/8+}}\|z\|_{X^{k,3/8+}};$$

b) for any $k \geq 0$, $k - l \leq 1/2$, $\frac{1}{4} < b, c, b_1 < \frac{1}{2}$ *and* $b + b_1 + c \geq 1$,

$$\|\psi_\pm u\|_{X^{k,-c}} \lesssim \|\psi_\pm\|_{W_\pm^{l,b_1}}\|u\|_{X^{0,b}} + \|\psi_\pm\|_{W_\pm^{-1/2,b_1}}\|u\|_{X^{k,b}};$$

c) for any $l \geq -1/2$, $k - l \geq 1/2$, $\frac{1}{4} < b, c_1 < \frac{1}{2}$ *and* $2b + c_1 \geq 1$,

$$\|\partial_x(u\overline{v})\|_{W_\pm^{l,-c_1}} \lesssim \|u\|_{X^{k,b}}\|v\|_{X^{0,b}} + \|u\|_{X^{0,b}}\|v\|_{X^{k,b}}.$$

Proof. From Lemma 2.3, we already know that a) holds. Furthermore, from lemma 3.1 of Colliander, Holmer and Tzirakis [7], the following holds:

$\|\psi_\pm u\|_{X^{0,-c}} \lesssim \|\psi_\pm\|_{W_\pm^{-1/2,b_1}}\|u\|_{X^{0,b}}$ if $\frac{1}{4} < b, c, b_1 < \frac{1}{2}$ and $b + b_1 + c \geq 1$;

$\|\partial_x(u_1\overline{u_2})\|_{W_\pm^{-1/2,-c_1}} \lesssim \|u_1\|_{X^{0,b}}\|u_2\|_{X^{0,b}}$ if $\frac{1}{4} < b, c_1 < \frac{1}{2}$ and $2b + c_1 \geq 1$.

Also, the triangle inequality implies $\langle\xi\rangle^r \leq \langle\xi\rangle^{r'}\langle\xi_1\rangle^{r-r'} + \langle\xi\rangle^{r'}\langle\xi_2\rangle^{r-r'}$, where $r \geq r'$ and $\xi = \xi_1 + \xi_2$. Denoting by $\widehat{J^s u}(\xi) := \langle\xi\rangle^s\widehat{u}(\xi)$ and putting together these facts, we obtain

$$\begin{aligned}
\|\psi_\pm u\|_{X^{k,-c}} &\lesssim \|u \cdot J^k\psi_\pm\|_{X^{0,-c}} + \|\psi_\pm \cdot J^k u\|_{X^{0,-c}} \\
&\lesssim \|J^k\psi_\pm\|_{W_\pm^{-1/2,b_1}}\|u\|_{X^{0,b}} + \|\psi_\pm\|_{W_\pm^{-1/2,b_1}}\|J^k u\|_{X^{0,b}} \\
&\lesssim \|\psi_\pm\|_{W_\pm^{l,b_1}}\|u\|_{X^{0,b}} + \|\psi_\pm\|_{W_\pm^{-1/2,b_1}}\|u\|_{X^{k,b}},
\end{aligned}$$

if $k \geq 0$, $k - l \leq 1/2$, $\frac{1}{4} < b, c, b_1 < \frac{1}{2}$, $b + b_1 + c \geq 1$, and

$$\begin{aligned}
\|\partial_x(u\overline{v})\|_{W_\pm^{l,-c}} &\lesssim \|\partial_x(J^{l+1/2}u \cdot \overline{v})\|_{W_\pm^{-1/2,-c}} + \|\partial_x(u \cdot J^{l+1/2}\overline{v})\|_{W_\pm^{-1/2,-c}} \\
&\lesssim \|J^{l+1/2}u\|_{X^{0,b}}\|v\|_{X^{0,b}} + \|u\|_{X^{0,b}}\|J^{l+1/2}v\|_{X^{0,b}} \\
&\lesssim \|u\|_{X^{k,b}}\|v\|_{X^{0,b}} + \|u\|_{X^{0,b}}\|v\|_{X^{k,b}},
\end{aligned}$$

if $k \geq 0$, $k - l \geq 1/2$, $\frac{1}{4} < b, c_1 < \frac{1}{2}$, $2b + c_1 \geq 1$. This completes the proof. □

Next, we recall that the L^2 mass of B is a conserved quantity; i.e.,

$$I_1(t) := \int |B(t)|^2 dx = \int |B_0|^2 dx$$

Now, we combine the previous lemmas and remarks to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. First, we treat system (1.5) for initial data $(B_0, \psi_{10}, \psi_{20})$ in the space $L^2(\mathbb{R}) \times H^{-1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$. Fix $0 < T < 1$ and define

$$\Phi_T(B, \psi_1, \psi_2)(t) := \kappa_T(t)U(t)B_0 + \kappa_T(t)U *_R \{(\psi_1 + \psi_2 + |B|^2)B\}(t)$$

$$\begin{aligned}\Psi_T^+(B)(t) &:= \kappa_T(t)W_+(t)(\psi_{10}) + \kappa_T(t)W_+ *_R \{\partial_x(|B|^2)\}(t) \\ \Psi_T^-(B)(t) &:= \kappa_T(t)W_-(t)(\psi_{20}) - \kappa_T(t)W_- *_R \{\partial_x(|B|^2)\}(t).\end{aligned} \tag{2.2}$$

We seek a fixed point $(B, \psi_1, \psi_2) = (\Phi_T(B, \psi_1, \psi_2), \Psi_T^+(B), \Psi_T^-(B))$. To do so, we estimate (2.2) in $X^{0,b} \times W_+^{-1/2,b_1} \times W_-^{-1/2,b_1}$ (with $b = c = 3/8 + \varepsilon$, $b_1 = c_1 = 1/4 + \varepsilon$) using Lemmas 2.4, 2.5, 2.6, so that

$$\begin{aligned}
&\|\Phi_T(B,\psi_1,\psi_2)\|_{X^{0,b}} \lesssim T^{\frac{1}{8}-\varepsilon}\|B_0\|_{L^2} + T^{\frac{1}{4}-2\varepsilon}(\|\psi_1\|_{W_+^{-1/2,b_1}} \\
&\qquad + \|\psi_2\|_{W_-^{-1/2,b_1}} + \|B\|^2_{X^{0,b}})\|B\|_{X^{0,b}} \\
&\|\Psi_T^+(B)\|_{W_+^{-1/2,b_1}} \lesssim T^{\frac{1}{4}-\varepsilon}\|\psi_{10}\|_{H^{-1/2}} + T^{\frac{1}{2}-2\varepsilon}\|B\|^2_{X^{0,b}} \\
&\|\Psi_T^-(B)\|_{W_-^{-1/2,b_1}} \lesssim T^{\frac{1}{4}-\varepsilon}\|\psi_{20}\|_{H^{-1/2}} + T^{\frac{1}{2}-2\varepsilon}\|B\|^2_{X^{0,b}} \\
&\|\Phi_T(B,\psi_1,\psi_2) - \Phi_T(\widetilde{B},\widetilde{\psi_1},\widetilde{\psi_2})\|_{X^{0,b}} \\
&\lesssim T^{\frac{1}{4}-2\varepsilon}(\|\psi_1\|_{W_+^{-1/2,b_1}} + \|\psi_2\|_{W_-^{-1/2,b_1}})\|B-\widetilde{B}\|_{X^{0,b}} \\
&+ T^{\frac{1}{4}-2\varepsilon}\|\widetilde{B}\|_{X^{0,b}}\|\psi_1 - \widetilde{\psi_1}\|_{W_+^{-1/2,b_1}} + T^{\frac{1}{4}-2\varepsilon}\|\widetilde{B}\|_{X^{0,b}}\|\psi_2 - \widetilde{\psi_2}\|_{W_-^{-1/2,b_1}} \\
&+ T^{\frac{1}{4}-2\varepsilon}(\|B\|_{X^{0,b}} + \|\widetilde{B}\|_{X^{0,b}})^2\|B-\widetilde{B}\|_{X^{0,b}}, \\
&\|\Psi_T^+(B) - \Psi_T^+(\widetilde{B})\|_{W_+^{-1/2,b_1}} \lesssim T^{\frac{1}{2}-2\varepsilon}(\|B\|_{X^{0,b}} + \|\widetilde{B}\|_{X^{0,b}})\|B-\widetilde{B}\|_{X^{0,b}}, \\
&\|\Psi_T^-(B) - \Psi_T^-(\widetilde{B})\|_{W_-^{-1/2,b_1}} \lesssim T^{\frac{1}{2}-2\varepsilon}(\|B\|_{X^{0,b}} + \|\widetilde{B}\|_{X^{0,b}})\|B-\widetilde{B}\|_{X^{0,b}}.
\end{aligned}$$

Thus, for any $0 < T < 1$ such that

$$T^{\frac{3}{8}-3\varepsilon}\|B_0\|_{L^2} \le 1, \qquad T^{\frac{1}{2}-4\varepsilon}\|B_0\|^2_{L^2} \le 1,$$

and

$$T^{\frac{1}{2}-3\varepsilon}\|\psi_{10}\|_{H^{-1/2}} \le 1, \qquad T^{\frac{1}{2}-3\varepsilon}\|\psi_{20}\|_{H^{-1/2}} \le 1, \tag{2.3}$$

$$T^{\frac{1}{2}-2\varepsilon} \cdot T^{\frac{1}{4}-2\varepsilon}\|B_0\|^2_{L^2} \le T^{\frac{1}{4}-\varepsilon}\|\psi_{10}\|_{H^{-1/2}}, \tag{2.4}$$

$$T^{\frac{1}{2}-2\varepsilon} \cdot T^{\frac{1}{4}-2\varepsilon}\|B_0\|^2_{L^2} \le T^{\frac{1}{4}-\varepsilon}\|\psi_{20}\|_{H^{-1/2}}$$

we obtain sufficient conditions for the application of a standard contraction argument yielding a unique fixed point $(B, \psi_1, \psi_2) \in X^{0,b} \times W_+^{-1/2,b_1} \times$

$W_-^{-1/2,b_1}$ for (2.2) satisfying

$$\begin{aligned}&\|B\|_{X^{0,b}} \lesssim T^{\frac{1}{8}-\varepsilon}\|B_0\|_{L^2}, \quad \|\psi_1\|_{W_+^{-1/2,b_1}} \lesssim T^{\frac{1}{4}-\varepsilon}\|\psi_{10}\|_{H^{-1/2}},\\ &\|\psi_2\|_{W_-^{-1/2,b_1}} \lesssim T^{\frac{1}{4}-\varepsilon}\|\psi_{20}\|_{H^{-1/2}}.\end{aligned} \tag{2.5}$$

Nevertheless, we can estimate $\Phi_T(B,\psi_1,\psi_2)$ in $C^0([0,T],L^2)$ using Lemmas 2.4, 2.5, 2.6 and (2.5) to prove that $B \in C^0([0,T],L^2)$. By the conservation of the L^2 norm of B (see (1.7)) the norm $\|B(t)\|_{L^2} = \|B_0\|_{L^2}$ is unchanged during the evolution. Therefore, it remains only to deal with the possible growth of $\|\psi_{10}\|_{H^{-1/2}}$ and $\|\psi_{20}\|_{H^{-1/2}}$. Assume that, after some iteration, we attain a time t such that either $\|\psi_1(t)\|_{H^{-1/2}} \gg \|B(t)\|_{L^2}^2 = \|B_0\|_{L^2}^2$ or $\|\psi_2(t)\|_{H^{-1/2}} \gg \|B(t)\|_{L^2}^2 = \|B_0\|_{L^2}^2$. In the sequel, the time position t will be the initial time $t=0$, so that either $\|\psi_{10}\|_{H^{-1/2}} \gg \|B_0\|_{L^2}^2$ or $\|\psi_{20}\|_{H^{-1/2}} \gg \|B_0\|_{L^2}^2$. We have two possibilities:

a) $\|\psi_{10}\|_{H^{-1/2}} \gg \|B_0\|_{L^2}^2$ and $\|\psi_{20}\|_{H^{-1/2}} \gg \|B_0\|_{L^2}^2$; it follows that (2.4) is *automatically* satisfied and we may choose

$$T \sim \min\{\|\psi_{10}\|^{-1/(\frac{1}{2}-3\varepsilon)}, \|\psi_{20}\|^{-1/(\frac{1}{2}-3\varepsilon)}\} \tag{2.6}$$

in the (local) iteration scheme (by (2.3)). Because

$$\psi_1(t) = W_+(\psi_{10})(t) + W_+ *_R (\partial_x(|B|^2)),$$

$$\psi_2(t) = W_-(\psi_{20})(t) - W_- *_R (\partial_x(|B|^2)),$$

a simple application of Lemmas 2.4, 2.5 and (2.5) yields

$$\|\psi_1(T)\|_{H^{-1/2}} \leq \|\psi_{10}\|_{H^{-1/2}} + C\cdot T^{\frac{1}{2}-3\varepsilon}\|B_0\|_{L^2}^2,$$

$$\|\psi_2(T)\|_{H^{-1/2}} \leq \|\psi_{20}\|_{H^{-1/2}} + C\cdot T^{\frac{1}{2}-3\varepsilon}\|B_0\|_{L^2}^2.$$

These estimates show that we can carry out m iterations on time intervals of size (2.6) before the quantities $\|\psi_1(t)\|_{H^{-1/2}}$ and $\|\psi_2(t)\|_{H^{-1/2}}$ double, where

$$m \sim \frac{\min\{\|\psi_{10}\|_{H^{-1/2}}, \|\psi_{20}\|_{H^{-1/2}}\}}{T^{\frac{1}{2}-3\varepsilon}\|B_0\|_{L^2}^2}. \tag{2.7}$$

Observe that, by (2.6) and (2.7), the amount of time that we advanced (after these m iterations) is

$$mT \sim \|B_0\|_{L^2}^{-2}.$$

This proves the global well-posedness result in case a) and it shows that

$$\|\psi_1(t)\|_{H^{-1/2}} \leq \exp(Ct\|B_0\|_{L^2}^2)\max\{\|\psi_{10}\|_{H^{-1/2}}, \|B_0\|_{L^2}^2\}$$

and

$$\|\psi_2(t)\|_{H^{-1/2}} \leq \exp(Ct\|B_0\|_{L^2}^2)\max\{\|\psi_{20}\|_{H^{-1/2}}, \|B_0\|_{L^2}^2\}.$$

b) $\min\{\|\psi_{10}\|_{H^{-1/2}}, \|\psi_{20}\|_{H^{-1/2}}\} \lesssim \|B_0\|_{L^2}^2$: this case is similar to the previous one, but we need to rework the entire argument above; by interchanging the roles of ψ_{10} and ψ_{20} if necessary, we can suppose that $\|\psi_{20}\|_{H^{-1/2}} \lesssim \|B_0\|_{L^2}^2$ and $\|\psi_{10}\|_{H^{-1/2}} \gg \|B_0\|_{L^2}^2$; in this case, for any $0 < T < 1$ such that $T^{\frac{1}{2}-4\varepsilon}\|B_0\|_{L^2}^2 \leq 1$,

$$T^{\frac{1}{2}-3\varepsilon}\|\psi_{10}\|_{H^{-1/2}} \leq 1, \tag{2.8}$$

$$T^{\frac{1}{2}-2\varepsilon} \cdot T^{\frac{1}{4}-2\varepsilon}\|B_0\|_{L^2}^2 \leq T^{\frac{1}{4}-\varepsilon}\|\psi_{10}\|_{H^{-1/2}} \tag{2.9}$$

we get a local-in-time solution (B, ψ_1, ψ_2) of (2.2) on the time interval T satisfying

$$\begin{aligned} &\|B\|_{X^{0,b}} \lesssim T^{\frac{1}{8}-\varepsilon}\|B_0\|_{L^2}, \quad \|\psi_1\|_{W_+^{-1/2,b_1}} \lesssim T^{\frac{1}{4}-\varepsilon}\|\psi_{10}\|_{H^{-1/2}}, \\ &\|\psi_2\|_{W_-^{-1/2,b_1}} \lesssim T^{\frac{1}{4}-\varepsilon}\|B_0\|_{L^2}^2. \end{aligned} \tag{2.10}$$

Note that (2.9) is fulfilled because $\|\psi_{10}\|_{H^{-1/2}} \gg \|B_0\|_{L^2}^2$ (by hypothesis). Hence, there exists a solution of the Zakharov-Rubenchik system on a time interval $[0, T]$ of size

$$T \sim \|\psi_{10}\|^{-1/(\frac{1}{2}-3\varepsilon)} \tag{2.11}$$

Again, using Lemmas 2.4, 2.5 and (2.10), it follows that

$$\|\psi_1(T)\|_{H^{-1/2}} \leq \|\psi_{10}\|_{H^{-1/2}} + C \cdot T^{\frac{1}{2}-3\varepsilon}\|B_0\|_{L^2}^2$$

$$\|\psi_2(T)\|_{H^{-1/2}} \leq \|\psi_{20}\|_{H^{-1/2}} + C \cdot T^{\frac{1}{2}-3\varepsilon}\|B_0\|_{L^2}^2$$

since

$$\psi_1(t) = W_+(\psi_{10})(t) + W_+ *_R (\partial_x(|B|^2))$$
$$\psi_2(t) = W_-(\psi_{20})(t) - W_- *_R (\partial_x(|B|^2)).$$

Thus, we can perform m iterations of this scheme before the quantity $\|\psi_1(t)\|_{H^{-1/2}}$ doubles, where

$$m \sim \frac{\|\psi_{10}\|_{H^{-1/2}}}{T^{\frac{1}{2}-3\varepsilon}\|B_0\|_{L^2}^2}. \tag{2.12}$$

On the other hand, during these m iterations, either $\|\psi_2(t)\|_{H^{-1/2}} \lesssim \|B_0\|_{L^2}^2$ for all $t \in [0, mT]$ or $\|\psi_2(jT)\|_{H^{-1/2}} \gg \|B_0\|_{L^2}^2$ at some stage $1 < j < m$:

In the first situation, it is clear that this iteration scheme allows us to produce a solution of the Zakharov-Rubenchik system on the time interval $[0, mT]$ with $mT \sim \|B_0\|_{L^2}^{-2}$ by (2.11) and (2.12).

In the second situation, we observe that $(B(jT), \psi_1(jT), \psi_2(jT))$ fits the assumptions of the previous case a).

Therefore, we are able to conclude (in any situation) the global well-posedness result in case b) and the estimate

$$\|\psi_1(t)\|_{H^{-1/2}} \leq \exp(Ct\|B_0\|_{L^2}^2) \max\{\|\psi_{10}\|_{H^{-1/2}}, \|B_0\|_{L^2}^2\}$$

and

$$\|\psi_2(t)\|_{H^{-1/2}} \leq \exp(Ct\|B_0\|_{L^2}^2) \max\{\|\psi_{20}\|_{H^{-1/2}}, \|B_0\|_{L^2}^2\}.$$

This shows that the global well-posedness result of Theorem 1.2 for initial data $(B_0, \psi_{10}, \psi_{20})$ in $L^2 \times H^{-1/2} \times H^{-1/2}$ holds and, moreover, we have the estimates

$$\|\psi_1(t)\|_{H^{-1/2}} \leq \exp(Ct\|B_0\|_{L^2}^2) \max\{\|\psi_{10}\|_{H^{-1/2}}, \|B_0\|_{L^2}^2\} \tag{2.13}$$

$$\|\psi_2(t)\|_{H^{-1/2}} \leq \exp(Ct\|B_0\|_{L^2}^2) \max\{\|\psi_{20}\|_{H^{-1/2}}, \|B_0\|_{L^2}^2\}. \tag{2.14}$$

Once the case $k = 0$ and $l = -1/2$ of Theorem 1.2 is proved, we can deal with (1.5) for initial data $(B_0, \psi_{10}, \psi_{20}) \in H^k \times H^l \times H^l$ (where $0 \leq k = l + \frac{1}{2}$) as follows: estimating (2.2) in $X^{k,b} \times W_+^{l,b_1} \times W_-^{l,b_1}$ (for $b = 3/8+$, $b_1 = 1/4+$), we obtain

$$\begin{aligned} &\|\Phi_T(B, \psi_1, \psi_2)\|_{X^{k,b}} \qquad (2.15)\\ &\lesssim T^{\frac{1}{8}-}\|B_0\|_{H^k} + T^{\frac{1}{4}-}(\|\psi_1\|_{W_+^{-1/2,b_1}} + \|\psi_2\|_{W_-^{-1/2,b_1}} + \|B\|_{X^{0,b}}^2)\|B\|_{X^{k,b}}\\ &+ T^{\frac{1}{4}-}\|B\|_{X^{0,b}}(\|\psi_1\|_{W_+^{l,b_1}} + \|\psi_2\|_{W_-^{l,b_1}})\\ &\|\Psi_T^+(B)\|_{W_+^{l,b_1}} \lesssim T^{\frac{1}{4}-}\|\psi_{10}\|_{H^l} + T^{\frac{1}{2}-}\|B\|_{X^{0,b}}\|B\|_{X^{k,b}}\\ &\|\Psi_T^-(B)\|_{W_-^{-1/2,b_1}} \lesssim T^{\frac{1}{4}-}\|\psi_{20}\|_{H^l} + T^{\frac{1}{2}-}\|B\|_{X^{0,b}}\|B\|_{X^{k,b}} \qquad (2.16) \end{aligned}$$

by Lemmas 2.4, 2.5, 2.6. On the other hand, since $0 \leq k = l + \frac{1}{2}$, we know that $(B_0, \psi_{10}, \psi_{20}) \in L^2 \times H^{-1/2} \times H^{-1/2}$ and the corresponding solution (B, ψ_1, ψ_2) satisfy the *a priori* estimates (2.13) and (2.14) (besides the upper bound $\|B\|_{X^{0,b}} \lesssim \|B_0\|_{L^2}$) on a time interval $[0, T]$ for some $T = T(\|B_0\|_{L^2}^{-2})$. Thus, using these facts in (2.15), we get uniform estimates for $\|B\|_{X^{k,b}}$, $\|\psi_1\|_{W_+^{l,b_1}}$ and $\|\psi_2\|_{W_-^{l,b_1}}$ on a time interval of length $T = T(\|B_0\|_{L^2}^{-2})$. This completes the proof. □

3. Polynomial growth of higher Sobolev norms

The goal of this section is the study of certain upper bounds for the higher Sobolev norms of the solutions (B, ψ_1, ψ_2) of the Zakharov-Rubenchik system (1.5). The strategy adopted here follows the lines of Colliander and Staffilani [8]. In particular, we construct global-in-time solutions of (1.5) for initial data $(B_0, \psi_{10}, \psi_{20}) \in H^1 \times L^2 \times L^2$ and we show that the Sobolev norm $\|B\|_{H^s}$ of the Schrödinger part B of the Zakharov-Rubenchik system satisfies certain *polynomial* upper bounds for all $s \gg 1$. More precisely, we prove the following result.

Theorem 3.1. *The Zakharov-Rubenchik* (1.5) *is globally well posed for initial data* $(B_0, \psi_{10}, \psi_{20}) \in H^1 \times L^2 \times L^2$. *Furthermore, if* $B_0, \psi_{10}, \psi_{20} \in \mathcal{S}$, *where* $\mathcal{S}$ *is the Schwartz class, then this global solution* (B, ψ_1, ψ_2) *satisfies*

$$\|B(t)\|_{H^s} \lesssim 1 + |t|^{(s-1)+}. \tag{3.1}$$

Remark 3.1. Since we can write

$$\begin{cases} \psi_1(t) = W_+(t)\psi_{10} + W_+ *_R \partial_x(|B|^2)(t) \\ \psi_2(t) = W_-(t)\psi_{20} - W_- *_R \partial_x(|B|^2)(t), \end{cases}$$

it is not hard to infer regularity bounds (in H^{s-1}) for ψ_1 and ψ_2 from the estimate (3.1).

In the sequel, we subdivide this section into two parts: the first subsection is dedicated to the global well-posedness statement of Theorem 3.1 and the second subsection contains the proof of the estimate (3.1).

3.1. Global well posedness in $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

Using the conservation laws $I_1(t)$, $I_2(t)$, $I_3(t)$ and $I_4(t)$ in (1.7)-(1.10) the following *a priori* estimate can be established.

Lemma 3.1. *Solutions of the IVP associated to* (1.5) *satisfy*

$$\|(B(t), \psi_1(t), \psi_2(t))\|_{H^1\times L^2\times L^2} \lesssim \|(B_0, \psi_{10}, \psi_{20})\|^2_{H^1\times L^2\times L^2} + I_1(0)^3.$$

Proof. This follows using the conserved quantities (see Lemma 3.3 of [14]) and (1.4). □

On the other hand, the local well-posedness result of Theorem 1.1 ensures the existence of a local-in-time solution (B, ψ_1, ψ_2) for initial data $(B_0, \psi_{10}, \psi_{20}) \in H^1 \times L^2 \times L^2$ in a time interval $[0, T]$, where

$$T > c\|(B_0, \psi_{10}, \psi_{20})\|^{-\alpha}_{H^1\times L^2\times L^2} \tag{3.2}$$

for some $\alpha > 0$. Combining this fact with Lemma 3.1, we obtain the following result.

Proposition 3.1. *The Zakharov-Rubenchik system* (1.5) *is globally well-posed in* $H^1 \times L^2 \times L^2$.

3.2. **Polynomial upper bounds for** $\|B(t)\|_{H^s}$**.** In view of Proposition 3.1 above, the proof of Theorem 3.1 is complete once we show the following result.

Proposition 3.2. *For initial data* $B_0, \psi_{10}, \psi_{20} \in \mathcal{S}$*, the global solution of* (1.5) *satisfies*

$$\|B(t)\|_{H^s} \lesssim 1 + |t|^{(s-1)+}.$$

Proof. We begin by noticing that it suffices to bound $\|B(t)\|_{H^s}$ for $t \in [0, T]$ (with T defined by (3.2)). Since $\|B(t)\|_{L^2} = \|B_0\|_{L^2}$ for all t, it remains only to compute $\|A^s B(t)\|_{L^2}$, where $A^s := \sqrt{-\Delta}$. To avoid technical issues involving fractional derivatives, we assume $s = 2m$, a large even integer ($m \gg 1$). Denoting by $\langle ., . \rangle$ the usual L^2 inner product, we obtain

$$\begin{aligned}\|A^s B(t)\|_{L^2} &= \|A^s B_0\|_{L^2} + \int_0^t \frac{d}{d\mu} \langle A^s B(\mu), A^s B(\mu) \rangle d\mu \\ &= \|A^s B_0\|_{L^2} + \Re \int_0^t \frac{d}{d\mu} \langle A^s B(\mu), A^s B(\mu) \rangle d\mu.\end{aligned}$$

This reduces matters to the computation of the term

$$I := \Re \int_0^t \frac{d}{d\mu} \langle A^s B(\mu), A^s B(\mu) \rangle d\mu = 2\Re \int_0^t \langle A^s \dot{B}(\mu), A^s B(\mu) \rangle d\mu.$$

Because the Schrödinger part B of the Zakharov-Rubenchik system (1.1) satisfies[1]

$$\begin{aligned}\partial_t B &= i\Delta B - i\psi_1 B + i\psi_2 B - i|B|^2 B \\ &= i\Delta B - i\Big(W_+(\psi_{10}) - W_-(\psi_{20})\Big)B \\ &\quad - i\Big(W_+ *_R \partial_x(|B|^2) - W_- *_R \partial_x(|B|^2)\Big)B - i|B|^2 B,\end{aligned} \tag{3.3}$$

we conclude that

$$\begin{aligned}I = &-2\Im \int_0^t \langle A^s \Delta B(\mu), A^s B(\mu) \rangle d\mu \\ &+ 2\Im \int_0^t \langle A^s \Big(W_\pm(\psi_{10}, \psi_{20})(\mu) B(\mu)\Big), A^s B(\mu) \rangle d\mu\end{aligned}$$

[1]Here we are omitting the irrelevant constants since our task is to show a polynomial upper bound for $\|B(t)\|_{H^s}$.

$$
\begin{aligned}
&+ 2\Im \int_0^t \langle A^s\Big(W_\pm *_R \partial_x(|B|^2)(\mu)B(\mu)\Big), A^s B(\mu)\rangle d\mu \\
&+ 2\Im \int_0^t \langle A^s\Big(|B(\mu)|^2 B(\mu)\Big), A^s B(\mu)\rangle d\mu \\
&:= I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where $W_\pm(\psi_{10},\psi_{20})(t) := W_+(\psi_{10})(t) - W_-(\psi_{20})(t)$ and

$$W_\pm *_R \partial_x(|B|^2)(t) := W_+ *_R \partial_x(|B|^2)(t) - W_- *_R \partial_x(|B|^2)(t).$$

Consider the term I_1. Since $-\Delta = A^2$, we get from integration by parts that the integrand of I_1 is a real number, so that $I_1 = 0$. On the other hand, the term I_2 involves the expression $A^s(W_\pm(\psi_{10},\psi_{20})\cdot B)$. This expression can be expanded using the Leibnitz rule. The term with the highest derivative on B is $W_\pm(\psi_{10},\psi_{20})A^s B$, but it does not contribute to the computation of I_2 since $W_\pm(\psi_{10},\psi_{20})$ is a real-valued function so that the corresponding integrand is a real number. Thus, I_2 becomes a sum of terms of the form

$$C\Im \int_0^t \langle A^{s_1} W_\pm(\psi_{10},\psi_{20})(\mu)\cdot A^{s_2}B(\mu), A^s B(\mu)\rangle d\mu,$$

where $s = s_1 + s_2$, $s_1, s_2 \in \mathbb{N}$, $1 \le s_1 \le s$ and $0 \le s_2 \le s-1$. Multiplying by a smooth localized-in-time cutoff function κ_T supported on the interval $[0,T]$ and using the Hölder inequality, we can estimate each of these terms by

$$\|A^{s_1} W_\pm(\psi_{10},\psi_{20})\|_{L^2_{xt}} \|A^{s_2}B\|_{L^4_{xt}} \|A^s B\|_{L^4_{xt}}.$$

Applying the Bourgain-Strichartz estimate $\|f\|_{L^4_{xt}} \lesssim \|f\|_{X^{0,3/8+}}$ and the local well-posedness result $\|B\|_{X^{m,b}} \lesssim \|B_0\|_{H^m}$ (for $b = 1/2+$ and $m \ge 0$), we obtain that I_2 can be estimated by a sum of terms of the form

$$(\|\psi_{10}\|_{H^{s_1}} + \|\psi_{20}\|_{H^{s_1}})\|B_0\|_{H^{s_2}}\|B_0\|_{H^s}.$$

Interpolating $\|B_0\|_{H^{s_2}}$ between $\|B_0\|_{H^1}$ and $\|B_0\|_{H^s}$, we get the following inequality:

$$|I_2| \lesssim \sum_{s_2=1}^{s-1} \|B_0\|_{H^s}^{1+\frac{s_2-1}{s-1}} \lesssim \|B_0\|_{H^s}^{2-\frac{1}{s-1}}. \qquad (3.4)$$

Similarly, I_4 is a sum of terms of the form

$$I_4(s_0,s_1,s_2) := C\Im \int_0^t \langle A^{s_0}B(\mu)\overline{A^{s_1}B(\mu)}A^{s_2}B(\mu), A^s B(\mu)\rangle d\mu$$

where $s_0 + s_1 + s_2 = s$, $s_0, s_1, s_2 \in \mathbb{N}$. Since the integrand of $I_4(s,0,0)$ and $I_4(0,0,s)$, we have $I_4(s,0,0) = I_4(0,0,s) = 0$. On the other hand, the bilinear estimates provided by Lemmas 3.3, 3.4 and 3.7 of [17] give the estimate

$$|I_4(0,s,0)| \lesssim \|B\|^2_{X^{1,1/2+}} \|B\|^2_{X^{s-3/4+,1/2+}}.$$

Hence, from the local well-posedness theory, we deduce that

$$|I_4(0,s,0)| \lesssim \|B_0\|^2_{H^1} \|B_0\|^2_{H^{s-3/4+}}.$$

Next, we estimate $I_4(s_0,s_1,s_2)$ for $1 \le s_0, s_1, s_2 \le s-1$, $s_0+s_1+s_2 = s$. Denoting by $j_2 = \max\{s_0,s_1,s_2\}$, $j_0 = \min\{s_0,s_1,s_2\}$ and $j_1 \in \{s_0,s_1,s_2\} - \{j_0,j_2\}$ and using again Lemmas 3.3 and 3.4 of [17] (plus the local well-posedness theory), we obtain

$$\begin{aligned} |I_4(s_0,s_1,s_2)| &\lesssim \|B\|_{X^{j_0+1,1/2+}} \|B\|_{X^{j_2+1,1/2+}} \|B\|_{X^{j_2-3/4+,1/2+}} \|B\|_{X^{s-3/4+,1/2+}} \\ &\lesssim \|B_0\|_{H^{j_0+1}} \|B_0\|_{H^{j_1+1}} \|B_0\|_{H^{j_2-3/4+}} \|B_0\|_{H^{s-3/4+}}. \end{aligned}$$

Therefore, summing up the bounds for the terms $I_4(s_0,s_1,s_2)$ and interpolating $\|B_0\|_{H^{\tilde{s}}}$ between H^1 and H^s, we see that

$$|I_4| \lesssim \|B_0\|^{(2s-2-3/2+)/(s-1)}_{H^s}. \tag{3.5}$$

Finally, it remains to deal with I_3. Using the Leibnitz rule, we can write I_3 as a sum of terms of the form

$$I_3(s_0,s_1,s_2) = C\int_0^t \langle W_\pm *_R (\partial_x(A^{s_0}B(\mu)A^{s_1}\overline{B}(\mu))) \cdot A^{s_2}B(\mu), A^sB(\mu)\rangle d\mu,$$

where $s_0+s_1+s_2 = s$. Note that the integrand of $I_3(0,0,s)$ is real so that $I_3(0,0,s) = 0$. Now we treat the terms $I_3(s_0,s_1,s_2)$ with $s_0+s_1+s_2 = s$ and $s_2 \le s-1$. By the Cauchy-Schwarz inequality we get

$$|I_3(s_0,s_1,s_2)| \lesssim \left\|W_\pm *_R (\partial_x(A^{s_0}B(\mu)A^{s_1}\overline{B}(\mu)))\right\|_{L^2_{xt}} \left\|A^{s_2}B(\mu)A^s\overline{B}(\mu)\right\|_{L^2_{xt}}.$$

Using the bilinear estimate of Lemma 2.3 (with $l=0$, $k=1/4$), the identity $L^2_{xt} = W^{0,0}_\pm$, Bourgain's refinement of Strichartz estimate [2] and the local well-posedness theory, it follows that

$$\begin{aligned} |I_3(s_0,s_1,s_2)| &\lesssim \|B\|_{X^{s_0+1/4,1/2+}} \|B\|_{X^{s_1+1/4,1/2+}} \|B\|^2_{X^{s-1/2+,1/2+}} \\ &\lesssim \|B_0\|_{H^{s_0+1/4}} \|B_0\|_{H^{s_1+1/4}} \|B_0\|_{H^{s_3-1/2+}} \|B_0\|_{H^{s-1/2}}. \end{aligned}$$

By interpolation, we infer that

$$|I_3(s_0,s_1,s_2)| \lesssim \|B_0\|^{(2s-2-5/2+)/(s-1)}_{H^s}. \tag{3.6}$$

Combining the estimates (3.4), (3.5) and (3.4) with the fact that $I_1 = 0$, we obtain that

$$\|B(t)\|_{H^s} \le \|B_0\|_{H^s} + C\|B_0\|^{1-\frac{1}{s-1}+}_{H^s}. \tag{3.7}$$

This ends the proof of Proposition 3.2. ☐

4. Ill posedness of the Zakharov-Rubenchik

In this section we establish Theorems 1.3-1.5. They are concerned with the lack of smoothness of the Zakharov-Rubenchik evolution for initial data $(B_0, \psi_{10}, \psi_{20}) \in H^k \times H^l \times H^s$ when (k, l, s) stays outside the local well-posedness region established in Theorem 1.1.

4.1. **Proof of Theorem 1.3.** In the sequel, we follow closely the "norm-inflation" scheme for the Zakharov system discussed in [10]. In particular, since the evolutions of the Zakharov system and the Zakharov-Rubenchik system are somewhat similar[2], we will just point out certain modifications of the arguments of section 3 of [10] in order to get the result of Theorem 1.3.

Let $0 < k < 1$, $l > 2k - \frac{1}{2}$. Firstly, we suppose also that l is *near* $2k - \frac{1}{2}$; i.e.,

- if $0 < k \leq \frac{1}{4}$, l is restricted to the region $2k - \frac{1}{2} < l \leq 4k - \frac{1}{2}$;
- if $\frac{1}{4} < k < 1$, l is restricted to the region $2k - \frac{1}{2} < l \leq \frac{4k}{3} + \frac{1}{6}$.

Under these assumptions, we fix

$$\sigma := \begin{cases} k, & \text{if } 0 < k \leq \frac{1}{4} \text{ and } 2k - \frac{1}{2} < l \leq 4k - \frac{1}{2} \\ \frac{k}{3} + \frac{1}{6}, & \text{if } \frac{1}{4} < k < 1 \text{ and } 2k - \frac{1}{2} < l \leq \frac{4k}{3} + \frac{1}{6}. \end{cases}$$

Next, we put $f_N = f_{N,A} + f_{N,B}$ where

$$\widehat{f_{N,A}}(\xi) := N^{\frac{1}{2}-k} \chi_{[-N-\frac{1}{N}, -N]}(\xi) \quad \text{and} \quad \widehat{f_{N,B}}(\xi) := N^{\frac{1}{2}-k} \chi_{[N+1, N+1+\frac{1}{N}]}(\xi).$$

Recall that the solution $(B_N, \psi_{1,N}, \psi_{2,N})$ of the Zakharov-Rubenchik system with initial data $(f_N, 0, 0)$ satisfies

$$\begin{cases} B_N(t) = \kappa(t) U(t) f_N + i\kappa_T(t) U *_R \{|B_N|^2 \\ \qquad\qquad + (W_+ - W_-) *_R \partial_x(|B_N|^2)\} \cdot B_N(t), \\ \psi_{1,N}(t) = \kappa_T(t) W_+ *_R \partial_x(|B_N|^2)(t), \\ \psi_{2,N}(t) = \kappa_T(t) W_- *_R \partial_x(|B_N|^2)(t). \end{cases}$$

During section 3 of [10], Holmer proved

$$\|W_+ *_R \partial_x(|U f_N|^2)(t)\|_{H^l} \sim t N^{l-(2k-\frac{1}{2})}$$

for $N \gtrsim t^{-1}$ (see estimate (3.7) of [10]). On the other hand, the linear and multilinear estimates of Lemma 2.3 can be used in the same manner as

[2] in the sense that the linear groups involved in the analysis of both systems (namely Schrödinger and transport) are equal.

Holmer [10] (pages 10, 11) to give us that

$$\begin{aligned}\|B_N - \kappa(t)U(t)f_N\|_{X^{k+\sigma,b_1}} &\lesssim \|W_+ *_R \partial_x(|B_N|^2)\cdot B_N\|_{X^{k+\sigma,b_1}}\\ &+ \|W_- *_R \partial_x(|B_N|^2)\cdot B_N\|_{X^{k+\sigma,b_1}} + \||B_N|^2 B_N\|_{X^{k+\sigma,b_1}}\\ &\lesssim \|f_N\|^2_{H^{k'}}\|f_N\|_{H^k} \sim N^{2(k'-k)+\sigma},\end{aligned}$$

where $k'=0$, $b_1 = \frac{3}{4} - \frac{k+\sigma}{2}$ if $0 < k+\sigma < \frac{1}{2}$ and $k' = \frac{k+\sigma}{2} - \frac{1}{4}$, $b_1 = \frac{1}{2}$ if $\frac{1}{2} \le k+\sigma < \frac{5}{2}$. Combining these estimates and reasoning as Holmer [10] (pages 11, 12) lead us to

$$\|\psi_{1,N}(t)\|_{H^l} \gtrsim t \cdot N^{l-(2k-\frac{1}{2})}$$

for $N \gtrsim t^{-1}$. Therefore, we showed the first part of Theorem 1.3 under the assumptions $0 < k < 1$, $l > 2k - \frac{1}{2}$ and l near $2k - \frac{1}{2}$ (in the sense above). Finally, the general case, i.e., either $0<k<1$, $l > 2k-\frac{1}{2}$ or $k<0$, $l > -1/2$, can be reduced to this previous particular case by decreasing l and increasing k appropriately (since $\|F\|_{H^{q'}} \le \|F\|_{H^q}$ whenever $q' \le q$).

Analogously, when dealing with the second part of Theorem 1.3, it suffices to replace l by s and f_N by $g_N = g_{N,A} + g_{N,B}$ where

$$\widehat{g_{N,A}}(\xi) := N^{\frac{1}{2}-k}\chi_{[-N-\frac{1}{N},-N]}(\xi) \quad \text{and} \quad \widehat{g_{N,B}}(\xi) := N^{\frac{1}{2}-k}\chi_{[N-1,N-1+\frac{1}{N}]}(\xi)$$

in the previous considerations so that the same argument applies. The details are left to the reader. This completes the proof of Theorem 1.3.

4.2. **Proof of Theorem 1.4.** For the sake of simplicity, we assume that $k \in \mathbb{R}$ and $l < -1/2$ (since the case $s < -1/2$ is similar). We take N to be a large integer and we put $\widehat{B_0}(\xi) := N^{\frac{1}{2}-k}\chi_{[0,\frac{1}{N}]}(\xi)$, $\widehat{\psi_{10}}(\xi) := N^{\frac{1}{2}-l}\chi_{[-\frac{1}{N},\frac{1}{N}]}(\xi)$ and $\psi_{20} \equiv 0$. Let $\gamma \in \mathbb{R}$ be a parameter and denote by $F(B_0,\psi_{10},\psi_{20}) = (B(t),\psi_1(t),\psi_2(t))$ the data-to-solution map of the Zakharov-Rubenchik system. Assume that F is C^2 at the origin $(0,0,0)$ and consider the path $G(\gamma) = (\gamma B_0, \gamma\psi_{10}, \gamma\psi_{20}) = (\gamma B_0, \gamma\psi_{10}, 0)$.

Note that the solution $(B,\psi_1,\psi_2) = F\circ G(\gamma)$ of (1.5) satisfies

$$\begin{cases} B(t) = U(t)(\gamma B_0) + i\displaystyle\int_0^t U(t-t')\{|B|^2+\psi_1+\psi_2\}\cdot B(t')dt',\\ \psi_1(t) = W_+(t)(\gamma\psi_{10}) + \displaystyle\int_0^t W_+(t-t')\partial_x(|B|^2)(t')dt',\\ \psi_2(t) = -\displaystyle\int_0^t W_-(t-t')\partial_x(|B|^2)(t')dt'.\end{cases}$$

Thus, we get

$$\begin{aligned}\partial_\gamma B(t) &= U(t)B_0 + i\int_0^t U(t-t')\{\partial_\gamma B(\psi_1+\psi_2+|B|^2)\\ &+ B(\partial_\gamma\psi_1+\partial_\gamma\psi_2+\partial_\gamma B\cdot\overline{B}+B\overline{\partial_\gamma B})\}(t')dt'\end{aligned}$$

and

$$\begin{cases}\partial_\gamma\psi_1(t) = W_+(t)\psi_{10} + \int_0^t W_+(t-t')\partial_x(\partial_\gamma B\cdot\overline{B}+B\cdot\overline{\partial_\gamma B})(t')dt',\\ \partial_\gamma\psi_2(t) = -\int_0^t W_-(t-t')\partial_x(\partial_\gamma B\cdot\overline{B}+B\cdot\overline{\partial_\gamma B})(t')dt'.\end{cases}$$

Using the fact that $F\circ G(0)=(0,0,0)$, it follows that

$$\partial_\gamma B|_{\gamma=0}(t) = U(t)B_0, \quad \partial_\gamma\psi_1(t) = W_+(t)\psi_{10}, \quad \partial_\gamma\psi_2(t)=0.$$

Taking the derivative with respect to γ leads us to

$$\begin{aligned}\partial_\gamma^2 B(t) = i\int_0^t U(t-t')\{&\partial_\gamma^2 B(\psi_1+\psi_2+|B|^2)\\ &+2\partial_\gamma B(\partial_\gamma\psi_1+\partial_\gamma\psi_2+\partial_\gamma B\cdot\overline{B}+B\overline{\partial_\gamma B})\\ &+B(\partial_\gamma^2\psi_1+\partial_\gamma^2\psi_2+\partial_\gamma^2 B\cdot\overline{B}+2|\partial_\gamma B|^2+B\overline{\partial_\gamma^2 B})\}(t')dt'\end{aligned}$$

and

$$\partial_\gamma^2\psi_1(t) = \int_0^t W_+(t-t')\partial_x(\partial_\gamma^2 B\cdot\overline{B}+2|\partial_\gamma B|^2+B\cdot\overline{\partial_\gamma^2 B})(t')dt',$$

$$\partial_\gamma^2\psi_2(t) = -\int_0^t W_-(t-t')\partial_x(\partial_\gamma^2 B\cdot\overline{B}+2|\partial_\gamma B|^2+B\cdot\overline{\partial_\gamma^2 B})(t')dt'.$$

Hence,

$$\partial_\gamma^2 B|_{\gamma=0}(t) = 2i\int_0^t U(t-t')\{U(t')B_0\cdot W_+(t')\psi_{10}\}dt'.$$

If F is C^2 at the origin, we can use the fact that $D^2F(0,0,0)$ is a bounded bilinear operator to conclude that the following bilinear estimate holds:

$$\left\|\int_0^t U(t-t')\{U(t')B_0\cdot W_+(t')\psi_{10}\}dt'\right\|_{H^k} \lesssim \|B_0\|_{H^k}\|\psi_{10}\|_{H^l}.$$

Since $\|B_0\|_{H^k}=\|\psi_{10}\|_{H^l}=1$, we get

$$\left\|\int_0^t U(t-t')\{U(t')B_0\cdot W_+(t')\psi_{10}\}dt'\right\|_{H^k} \lesssim 1. \tag{4.1}$$

On the other hand, we know that

$$L(x,t) := \int_0^t U(t-t')\{U(t')B_0 \cdot W_+(t')\psi_{10}\}dt'$$

satisfies

$$\widehat{L}(\xi,t) = \int_0^t e^{-i(t-t')\xi^2} \int e^{-it'\xi_1^2}\widehat{B_0}(\xi_1)e^{-it'(\xi-\xi_1)}\widehat{\psi_{10}}(\xi-\xi_1)d\xi_1 dt'.$$

Thus,

$$\widehat{L}(\xi,t) = e^{-it\xi^2}\int \widehat{B_0}(\xi_1)\widehat{\psi_{10}}(\xi-\xi_1)\frac{e^{it(\xi-\xi_1)(\xi+\xi_1-1)-1}}{i(\xi-\xi_1)(\xi+\xi_1-1)}d\xi_1.$$

Because the support of $\widehat{B_0}(\xi_1)\widehat{\psi_{10}}(\xi-\xi_1)$ is contained in the region $\xi_1 \in [0, \frac{1}{N}]$ and $\xi \in [-\frac{1}{N}, \frac{2}{N}]$, we can use the expansion of e^z to obtain

$$\widehat{L}(\xi,t) = e^{-it\xi^2}(t+O(t^2))N^{-l-k}.$$

In particular,

$$\Big\| \int_0^t U(t-t')\{U(t')B_0 \cdot W_+(t')\psi_{10}\}dt'\Big\|_{H^k} \gtrsim t\cdot N^{-l-\frac{1}{2}}. \tag{4.2}$$

Combining the estimates (4.1) and (4.2), we get a contradiction for N sufficiently large (since $l < -1/2$). This ends the proof of Theorem 1.4.

4.3. **Proof of Theorem 1.5.** The idea is to find a suitable class of initial data so that the Schrödinger variables of the Zakharov-Rubenchik evolution eventually exhibit completely different phases.

For the sake of simplicity, we will normalize the constants depending on $\omega, k, \beta, \nu, q, \theta$ from the system (1.5) so that the ZR system becomes

$$\begin{cases} i\partial_t B + \partial_x^2 B = \psi_+ B + \psi_- B + |B|^2 B, \\ \partial_t\psi_+ + \partial_x\psi_+ = \partial_x(|B|^2), \\ \partial_t\psi_- - \partial_x\psi_- = \partial_x(|B|^2). \end{cases} \tag{4.3}$$

Also, up to exchanging the roles of ψ_+ and ψ_- in the arguments below, it suffices to prove Theorem 1.5 in the case $k=0$ and $l < -3/2$.

Next, we fix four parameters $L \gg 1$, $0 < \mu \ll 1$, $0 < c < 1$ and $0 < \Theta \ll 1$. Consider the following modified ZR system:

$$\begin{cases} i\partial_t \widetilde{B} + \mu^2\partial_x^2\widetilde{B} = \widetilde{\psi}_{+0}\big(x - \frac{\mu(1-c)}{L}t\big)\widetilde{B} + \widetilde{\psi}_{-0}\big(x + \frac{\mu(1+c)}{L}t\big)\widetilde{B} \\ \qquad\qquad +(\widetilde{\psi}_+ + \widetilde{\psi}_-)\widetilde{B} + \Theta^2|\widetilde{B}|^2\widetilde{B}, \\ \frac{L}{\mu(1-c)}\partial_t\widetilde{\psi}_+ + \partial_x\widetilde{\psi}_+ = \frac{\Theta^2}{1-c}\partial_x(|\widetilde{B}|^2), \\ \frac{L}{\mu(1+c)}\partial_t\widetilde{\psi}_- - \partial_x\widetilde{\psi}_- = \frac{\Theta^2}{1+c}\partial_x(|\widetilde{B}|^2), \end{cases} \tag{4.4}$$

with initial conditions

$$\widetilde{B}(0,x)=\widetilde{B}_0(x), \quad \widetilde{\psi}_+(0,x)=0, \quad \widetilde{\psi}_-(0,x)=0.$$

Proposition 4.1. *If* $k\geq 1$ *and*

$$T_0\lesssim|\log\mu|, \quad L\gtrsim\mu^{-5}, \quad \Theta^2\sim\mu L^{-1}, \tag{4.5}$$

then the following holds:

$$\|\widetilde{B}\|_{L^\infty_{[0,T_0]}H^k_x}\lesssim\mu^{-1/2}, \tag{4.6}$$

$$\|\widetilde{\psi}_\pm\|_{L^\infty_{[0,T_0]}H^{k-1}_x}\lesssim\frac{\Theta^2}{L}. \tag{4.7}$$

Proof. Since

$$\psi_\pm=\tfrac{\Theta^2}{(1\mp c)}\int_0^{t\alpha_\pm}\partial_x(|\widetilde{B}|^2)(x-t',t-\tfrac{t'}{\alpha_\pm})dt',$$

where $\alpha_\pm=L/\mu(1\mp c)$, it follows that

$$\|\widetilde{\psi}_\pm\|_{L^\infty_{[0,T_0]}H^{k-1}_x}\leq\frac{\Theta^2\mu T}{L}\|\widetilde{B}\|^2_{L^\infty_{[0,T_0]}H^k_x}. \tag{4.8}$$

On the other hand, using the energy method for $\widetilde{B}$, we get

$$\begin{aligned}\|\partial_x^k\widetilde{B}(T_0)\|^2_{L^2_x}-\|\partial_x^k\widetilde{B}(0)\|^2_{L^2_x}&=-\Re\Big(i\int_0^{T_0}\int\partial_x^k(\psi_{\pm0}(\ldots)\widetilde{B})\overline{\partial_x^k\widetilde{B}}dxdt\Big)\\ &\quad-\Re\Big(i\int_0^{T_0}\int\partial_x^k(\psi_\pm\widetilde{B})\overline{\partial_x^k\widetilde{B}}dxdt\Big)-\Theta^2\Re\Big(i\int_0^{T_0}\int\partial_x^k(|\widetilde{B}|^2\widetilde{B})\overline{\partial_x^k\widetilde{B}}dxdt\Big)\\ &=(I)+(II)+(III).\end{aligned}$$

Note that

$$|(I)|\leq\|\psi_{\pm0}\|_{H^k}\int_0^{T_0}\|\widetilde{B}(t)\|^2_{H^k}dt.$$

Now, following the calculations of Holmer [10] (page 16), it is not hard to see that

$$|(II)|\lesssim\Big(\frac{\Theta^2\mu T^2}{L}+\frac{\Theta^2T}{\mu L}+\frac{\Theta^2T^2}{L^2}\Big)\|\widetilde{B}\|^4_{L^\infty_{[0,T_0]}H^k_x}.$$

Also,

$$|(III)|\leq T\Theta^2\|\widetilde{B}\|^4_{L^\infty_{[0,T_0]}H^k_x}.$$

Combining these estimates, we obtain

$$\|\widetilde{B}(T_0)\|^2_{H^k}\leq\|\widetilde{B}_0\|^2_{H^k}+(\|\psi_{+0}\|_{H^k}+\|\psi_{-0}\|_{H^k})\int_0^{T_0}\|\widetilde{B}(t)\|^2_{H^k}dt+\varepsilon\|\widetilde{B}\|^4_{L^\infty_{[0,T_0]}H^k_x},$$

where $\varepsilon = \frac{\Theta^2\mu T^2}{L} + \frac{\Theta^2 T}{\mu L} + \frac{\Theta^2 T^2}{L^2} + \Theta^2 T$. Using our assumptions on T_0, L and Θ, we see that $\varepsilon \le L^{-1/2} \lesssim e^{-(\|\psi_{+0}\|_{H^k}+\|\psi_{-0}\|_{H^k})T}\|\widetilde{B_0}\|_{H^k}^{-2}$. This fact combined with the Gronwall inequality and a continuity argument allows us to conclude that

$$\|\widetilde{B}\|^2_{L^\infty_{[0,T_0]}H^k_x} \le 2e^{(\|\psi_{+0}\|_{H^k}+\|\psi_{-0}\|_{H^k})T}\|\widetilde{B}_0\|^2_{H^k}.$$

This completes the proof of the proposition. □

Denote by $\widetilde{A}$ the solution of the small-dispersion limit (i.e., $\mu = 0$) of (4.4):

$$i\partial_x \widetilde{A} = (\psi_{+0} + \psi_{-0})\widetilde{A} \tag{4.9}$$

with $\widetilde{A}(0,x) = \widetilde{B}_0(x)$. In other words,

$$\widetilde{A}(x,t) = e^{-it(\psi_{+0}(x)+\psi_{-0}(x))}\widetilde{B}_0(x).$$

Proposition 4.2. $\|\widetilde{B} - \widetilde{A}\|_{L^\infty_{[0,T_0]}H^k} \lesssim \mu$.

Proof. The energy method applied to $\widetilde{B} - \widetilde{A}$ yields

$$\begin{aligned}
&\|\partial_x^k(\widetilde{B} - \widetilde{A})(T_0)\|_{L^2_x} = -2\Re\Big(i\mu^2\int_0^{T_0}\int \partial_x^{k+2}\widetilde{B}\overline{\partial_x^k(\widetilde{B}-\widetilde{A})}\Big)\\
&- 2\Re\Big(i\int_0^{T_0}\int \partial_x^k(\psi_{\pm 0}(x \mp \tfrac{\mu(1\mp c)}{L}t)\widetilde{B} - \psi_{\pm 0}(x)\widetilde{A})\overline{\partial_x^k(\widetilde{B}-\widetilde{A})}\Big)\\
&- 2\Re\Big(i\int_0^{T_0}\int \partial_x^k(\psi_\pm B)\overline{\partial_x^k(\widetilde{B}-\widetilde{A})}\Big) - 2\Re\Big(i\Theta^2\int_0^{T_0}\int \partial_x^k(|\widetilde{B}|^2\widetilde{B})\overline{\partial_x^k(\widetilde{B}-\widetilde{A})}\Big)\\
&= (I) + (II) + (III) + (IV).
\end{aligned}$$

Using Proposition 4.1, we get

$$\begin{aligned}
|(I)| &\le T^2\mu^4\|\widetilde{B}\|^2_{L^\infty_{[0,T_0]}H^{k+2}_x} + \tfrac{1}{4}\|\partial_x^k(\widetilde{B}-\widetilde{A})\|^2_{L^\infty_{[0,T_0]}L^2_x}\\
&\le cT^2\mu^3 + \tfrac{1}{4}\|\partial_x^k(\widetilde{B}-\widetilde{A})\|^2_{L^\infty_{[0,T_0]}L^2_x},\\
|(III)| &\le T^2\|\widetilde{\psi}_\pm\|^2_{L^\infty_{[0,T_0]}H^k_x}\|\widetilde{B}\|^2_{L^\infty_{[0,T_0]}H^k_x} + \tfrac{1}{4}\|\partial_x^k(\widetilde{B}-\widetilde{A})\|^2_{L^\infty_{[0,T_0]}L^2_x}\\
&\le \tfrac{T^2\Theta^4}{L^2\mu} + \tfrac{1}{4}\|\partial_x^k(\widetilde{B}-\widetilde{A})\|^2_{L^\infty_{[0,T_0]}L^2_x}\\
|(IV)| &\le T^2\Theta^4\|\widetilde{B}\|^6_{L^\infty_{[0,T_0]}H^k_x} + \tfrac{1}{4}\|\partial_x^k(\widetilde{B}-\widetilde{A})\|^2_{L^\infty_{[0,T_0]}L^2_x}\\
&\le \tfrac{T^2\Theta^4}{\mu^3} + \tfrac{1}{4}\|\partial_x^k(\widetilde{B}-\widetilde{A})\|^2_{L^\infty_{[0,T_0]}L^2_x}.
\end{aligned}$$

Furthermore, we can rewrite

$$\widetilde{\psi}_{\pm 0}(x \mp \tfrac{\mu(1\mp c)}{L}t)\widetilde{B} - \widetilde{\psi}_{\pm 0}(x)\widetilde{A}$$
$$= \Big(\int_0^{\frac{\mp\mu(1\mp c)}{L}t} \partial_x\widetilde{\psi}_{\pm 0}(x+t')dt'\Big)\widetilde{B} + \widetilde{\psi}_{\pm 0}(x)(\widetilde{B}-\widetilde{A}).$$

In particular,

$$|(II)| \leq \tfrac{\mu(1\mp c)T^2}{L}\|\psi_{\pm 0}\|_{H^{k+1}}\|\widetilde{B}\|_{L^\infty_{[0,T_0]}H^k_x}\|\widetilde{B}-\widetilde{A}\|_{L^\infty_{[0,T_0]}H^k_x}$$
$$+ \|\psi_{\pm 0}\|_{H^k}\int_0^{T_0}\|(\widetilde{B}-\widetilde{A})(t)\|^2_{H^k}dt.$$

Putting these facts together and using the Gronwall inequality, we conclude

$$\|\widetilde{B}-\widetilde{A}\|^2_{L^\infty_{[0,T_0]}H^k_x} \lesssim e^{cT}\cdot\Big(T^2\mu^3 + \frac{\mu(1\mp c)^2T^4}{L^2} + \frac{T^2\Theta^4}{L^2\mu} + \frac{T^2\Theta^4}{\mu^3}\Big).$$

This ends the proof of the proposition. □

Define

$$B(x,t) := L\Theta e^{-ic^2t}e^{icx}\widetilde{B}(L\mu(x-ct), L^2t),$$
$$\psi_\pm(x,t) = L^2\widetilde{\psi}_{\pm 0}(L\mu(x\mp t)) + L^2\widetilde{\psi}_\pm(L\mu(x-ct), L^2t),$$

where $(\widetilde{B}, \widetilde{\psi}_\pm)$ is a solution of (4.4). It is quite straightforward to check that $(B, \psi_\pm)$ solves the Zakharov-Rubenchik system (4.3) with initial data $B(x,0) = L\Theta e^{icx}\widetilde{B}_0(L\mu x)$, $\psi_\pm(x,0) = L^2\widetilde{\psi}_{\pm 0}(L\mu x)$.

Since $\|\widetilde{B}(x,t)\|_{L^2_x} = \|\widetilde{B_0}\|_{L^2_x}$ for all t, we get

$$\|B(x,t)\|_{L^2_x} = \frac{L^{1/2}\Theta}{\mu^{1/2}}\|\widetilde{B_0}\|_{L^2_x}.$$

Also, if $\widehat{\psi_{+0}}(\xi) = 0$ for $|\xi| \leq 1$ and $\mu L \geq 1$, we see

$$\|\psi_+(x,0)\|_{H^l_x} \leq L^{\frac{3}{2}+l}\mu^{l-\frac{1}{2}}\|\psi_{+0}\|_{H^l_x}.$$

Therefore, if $l < -3/2$ and $L \geq \mu^{-\alpha}$, where $\alpha = \max\{5, \frac{1/2-l}{-l-3/2}\}$, we obtain

$$\|\psi_+(x,0)\|_{H^l_x} \leq \|\widetilde{\psi}_{+0}\|_{H^l_x}.$$

At this point, we are ready to complete the proof of Theorem 1.5. We fix $M \gg 1$ and $0 < \mu \ll 1$ to be chosen later and we put $T = |\log\mu|\cdot M^{-2}$,

$$L_1 = M, \quad L_2 = \sqrt{\frac{\pi}{2T} + M^2}.$$

Observe that $e^{iT(L_2^2-L_1^2)} = i$ and $L_2/L_1 = \sqrt{\frac{\pi}{2|\log\mu|}+1} \to 1$ (uniformly on M). Take $\Theta^2 = \mu \cdot M^{-1}$. Fix $\widetilde{B}_0$ a Schwartz function such that $\widetilde{B}_0(x) = 1$ for $|x| \leq 1$, and $\widetilde{\psi}_{+0}(x) = \cos(3x)\sin(x)/x$ and $\widetilde{\psi}_{-0}(x) = 0$.

Next we consider $\widetilde{B}_1$ and $\widetilde{B}_2$ solutions of (4.4) with parameters L_1 and L_2 (respectively) and the same initial data $\widetilde{B}_0$, $\widetilde{\psi}_{\pm 0}$. From the previous discussion, for $j = 1, 2$,

$$\|B_j(x,t)\|_{L_x^2} \sim 1, \quad \|\psi_{+,j}(x,t)\|_{H_x^l} \leq 1, \quad \|\psi_{-,j}(x,t)\|_{H_x^s} \leq 1,$$

whenever $M \geq \mu^{-\alpha}$.

On the other hand, using the definition of Θ,

$$\|B_2(x,t) - B_1(x,t)\|_{L_x^2} = \|\tfrac{L_2}{L_1}\widetilde{B}_2(\tfrac{L_2}{L_1}x, L_2^2 t) - \widetilde{B}_1(x, L_1^2 t)\|_{L_x^2}.$$

Since $L_2/L_1 \to 1$ uniformly on M and $\|\widetilde{B}_j(x,t)\|_{L_x^2} = \|\widetilde{B}_0\|_{L_x^2}$ for all t, we can select $\mu = \mu(\delta) > 0$ sufficiently small so that

$$\|B_2(x,t) - B_1(x,t)\|_{L_x^2} = \|\widetilde{B}_2(x, L_2^2 t) - \widetilde{B}_1(x, L_1^2 t)\|_{L_x^2} + O(\delta).$$

Using Proposition 4.2, it follows that

$$\|B_2(x,t) - B_1(x,t)\|_{L_x^2} = \|(e^{i(L_2^2-L_1^2)t\widetilde{\psi}_{+0}(x)} - 1)\widetilde{B}_0\|_{L_x^2} + O(\delta).$$

Hence, $\|B_2(x,0) - B_1(x,0)\|_{L_x^2} \lesssim \delta$ but $\|B_2(x,T) - B_1(x,T)\|_{L_x^2} \sim 1$ (because $e^{iT(L_2^2-L_1^2)} = i$). Since $T = |\log\mu| \cdot M^{-2} \leq |\log\mu| \cdot \mu^{10} \to 0$ as $\mu \to 0$, the proof of Theorem 1.5 is complete.

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