COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **8**, Number **3**, May **2009**

doi:10.3934/cpaa.2009.8.1

pp. –

GLOBAL WELL-POSEDNESS AND NON-LINEAR STABILITY OF PERIODIC TRAVELING WAVES FOR A SCHRÖDINGER-BENJAMIN-ONO SYSTEM

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(Communicated by Gigliola Staffilani)

Abstract. The objective of this paper is two-fold: firstly, we develop a local and global (in time) well-posedness theory for a system describing the motion of two fluids with different densities under capillary-gravity waves in a deep water flow (namely, a Schrödinger-Benjamin-Ono system) for *low-regularity* initial data in both periodic and continuous cases; secondly, a family of new periodic traveling waves for the Schrödinger-Benjamin-Ono system is given: by fixing a minimal period we obtain, via the implicit function theorem, a smooth branch of periodic solutions bifurcating a Jacobian elliptic function called *dnoidal*, and, moreover, we prove that all these periodic traveling waves are nonlinearly stable by perturbations with the same wavelength.

1. **Introduction.** In this paper we are interested in the study of the following Schrödinger-Benjamin-Ono (SBO) system

$$\begin{cases} iu_t + u_{xx} = \alpha v u, \\ v_t + \gamma D v_x = \beta(|u|^2)_x, \end{cases} \tag{1}$$

where $u = u(t,x)$ is a complex-valued function, $v = v(t,x)$ is a real-valued function, $t \in \mathbb{R}$, $x \in \mathbb{R}$ or $\mathbb{T}$, α, β and γ are real constants such that $\alpha \neq 0$ and $\beta \neq 0$, and $D\partial_x$ is a linear differential operator representing the dispersive term. Here $D = \mathcal{H}\partial_x$ where $\mathcal{H}$ denotes the *Hilbert transform* defined as

$$\widehat{\mathcal{H}f}(k) = -i\mathrm{sgn}(k)\widehat{f}(k),$$

where

$$\mathrm{sgn}(k) = \begin{cases} -1, & k < 0, \\ 1, & k > 0. \end{cases}$$

2000 *Mathematics Subject Classification.* Primary: 35Q55, 35Q51; Secondary: 76B25, 76B15.
Key words and phrases. Nonlinear PDE, initial value problem, traveling wave solutions.
The first author is partially supported by CNPq/Brazil.

Note that from these definitions we have that D is a linear positive Fourier operator with symbol $|k|$. The system (1) was deduced by Funakoshi and Oikawa ([22]). It describes the motion of two fluids with different densities under capillary-gravity waves in a deep water flow. The short surface wave is usually described by a Schrödinger type equation and the long internal wave is described by some sort of wave equation accompanied with a dispersive term (which is a Benjamin-Ono type equation in this case). This system is also of interest in the sonic-Langmuir wave interaction in plasma physics [30], in the capillary-gravity interaction wave [21], [27], and in the general theory of water wave interaction in a nonlinear medium [13], [14]. We note that the Hilbert transform considered in [22] for describing system (1) is given as $-\mathcal{H}$.

In studying an initial value problem, the first step is usually to investigate in which function space well-posedness occurs. In our case, smooth solutions of the SBO system (1) enjoy the following conserved quantities

$$\begin{cases} G(u,v) \equiv Im \displaystyle\int u(x)\overline{u_x(x)}\,dx + \frac{\alpha}{2\beta}\int |v(x)|^2\,dx, \\ E(u,v) \equiv \displaystyle\int |u_x(x)|^2\,dx + \alpha\int v(x)|u(x)|^2\,dx - \frac{\alpha\gamma}{2\beta}\int |D^{1/2}v(x)|^2\,dx, \\ H(u,v) \equiv \displaystyle\int |u(x)|^2\,dx, \end{cases} \tag{2}$$

where $D^{1/2}$ is the Fourier multiplier defined as $\widehat{D^{1/2}v}(k) = |k|^{\frac{1}{2}}\widehat{v}(k)$. Therefore, the natural spaces to study well-posedness are the Sobolev H^s-type spaces. Moreover, due to the scaling property of the SBO system (1) (see [9] Remark 2), we are led to investigate well-posedness in the spaces $H^s \times H^{s-1/2}$, $s \in \mathbb{R}$.

In the continuous case Bekiranov, Ogawa, and Ponce [10] proved local well-posedness for initial data in $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ when $|\gamma| \neq 1$ and $s \geq 0$. Thus, because of the conservation laws in (2), the solutions extend globally in time when $s \geq 1$, in the case $\frac{\alpha\gamma}{\beta} < 0$. Recently, Pecher [35] has shown local well-posedness in $H^s(\mathbb{R})\times H^{s-\frac{1}{2}}(\mathbb{R})$ when $|\gamma| = 1$ and $s > 0$. He also used the Fourier restriction norm method to extend the global well-posedness result when $1/3 < s < 1$, always in the case $\frac{\alpha\gamma}{\beta} < 0$. Here, we improve the global well-posedness result till $L^2(\mathbb{R})\times H^{-\frac{1}{2}}(\mathbb{R})$ in the case $\gamma \neq 0$ and $|\gamma| \neq 1$. Indeed, we refine the bilinear estimates of Bekiranov, Ogawa, and Ponce [10] in Bourgain spaces $X^{0,b_1} \times Y_\gamma^{-\frac{1}{2},b}$ with $b,\ b_1 < \frac{1}{2}$ (see Proposition 3.4). These estimates combined with the L^2-conservation law allow us to show that the size of the time interval provided by the local well-posedness theory depends only on the L^2-norm of u_0. It is worth while to point out that this scheme applies for other dispersive systems. In fact Colliander, Holmer, and Tzirakis [19] already applied this method to Zakharov and Klein-Gordon-Schrödinger systems. They also announced the above result for the SBO system (see Remark 1.5 in [19]). However they allowed us to include it in this paper since there were not planning to write it up anymore. Note that we also prove global well-posedness in $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ when $s > 0$ in the case $\gamma \neq 0$ and $|\gamma| \neq 1$. We take the opportunity to express our gratitude to Colliander, Holmer and Tzirakis for the fruitful interaction about the Schrödinger-Benjamin-Ono system.

In the periodic setting, there does not exist, as far as we know, any result about the well-posedness of the SBO system (1). Bourgain [16] proved well-posedness for the cubic nonlinear Schrödinger equation (NLS) (see (3)) in $H^s(\mathbb{T})$ for $s \geq 0$ using the Fourier transform restriction method. Unfortunately, this method does not apply directly for the Benjamin-Ono equation. Nevertheless, using an appropriate Gauge transformation introduced by Tao [37], Molinet [33] proved well-posedness in $L^2(\mathbb{T})$. Here we apply Bourgain's method for the SBO system to prove its local well-posedness in $H^s(\mathbb{T}) \times H^{s-\frac{1}{2}}(\mathbb{T})$ when $s \geq 1/2$ in the case $\gamma \neq 0$, $|\gamma| \neq 1$. The main tool is the new bilinear estimate stated in Proposition 3.11. Furthermore, by standard arguments based on the conservation laws, this leads to global well-posedness in the energy space $H^1(\mathbb{T}) \times H^{1/2}(\mathbb{T})$ in the case $\frac{\alpha\gamma}{\beta} < 0$. We also show that our results are sharp in the sense that the bilinear estimates on these Bourgain spaces fail whenever $s < 1/2$ and $|\gamma| \neq 1$ or $s \in \mathbb{R}$ and $|\gamma| = 1$. In fact, we use Dirichlet's Theorem on rational approximation to locate certain plane waves whose nonlinear interactions behave badly in low regularity.

In the second part of this paper, we turn our attention to another important aspect of dispersive nonlinear evolution equations: the traveling waves. The existence of these solutions imply a balance between the effects of nonlinearity and dispersion. Depending on the specific boundary conditions on the wave's shape, these special states of motion can arise as either solitary or periodic waves. The study of this special steady waveform is essential for the explanation of many wave phenomena observed in practice: in surface water waves propagating in a canal, in propagation of internal waves or in the interaction between long waves and short waves as in our case. In particular, some questions such as existence and stability of these traveling waves are very important in the understanding of the dynamic of the equation under investigation.

The solitary waves are in general a single crested, symmetric, localized traveling waves, with sech-profiles (see Ono [34] and Benjamin [11] for the existence of solitary waves of algebraic type or with a finite number of oscillations). The study of the nonlinear stability or instability of solitary waves has had a big development and refinement in recent years. The proofs have been simplified and sufficient conditions have been obtained to insure the stability to small localized perturbations in the waveform. Those conditions have showed to be effective in a variety of circumstances, see for example [1], [2], [3], [12], [15], [26], [38].

The situation regarding to the study of periodic traveling waves is very different. The stability and the existence of explicit formulas of these progressive wave trains have received comparatively little attention. Recently many research papers about this issue have appeared for specify dispersive equations, such as the existence and stability of *cnoidal waves* for the Korteweg-de Vries equation [5] and the stability of *dnoidal waves* for the one-dimensional cubic nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^2 u = 0, \tag{3}$$

where $u = u(t,x) \in \mathbb{C}$ and $x, t \in \mathbb{R}$ (Angulo [4], see also Angulo, and Linares [6] and Gallay, and Hărăgus [23], [24]).

In this paper we are also interested in giving a stability theory of periodic traveling waves solutions for the nonlinear dispersive system SBO (1). The periodic traveling waves solutions considered here will be of the general form

$$\begin{cases} u(x,t) = e^{i\omega t} e^{ic(x-ct)/2} \phi(x-ct), \\ v(x,t) = \psi(x-ct), \end{cases} \tag{4}$$

where $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are smooth, L-periodic functions (with a prescribed period L), $c > 0$, $\omega \in \mathbb{R}$ and we will suppose that there is a $q \in \mathbb{N}$ such that

$$4q\pi/c = L.$$

So, by replacing these permanent waves form into (1) we obtain the pseudo-differential system

$$\begin{cases} \phi'' - \sigma\phi = \alpha\,\psi\phi \\ \gamma\mathcal{H}\psi' - c\psi = \beta\phi^2 + A_{\phi,\psi} \end{cases} \tag{5}$$

where $\sigma = \omega - \frac{c^2}{4}$ and $A_{\phi,\psi}$ is an integration constant which we will set equal zero in our theory. Existence of analytic solutions of system (5) for $\gamma \neq 0$ is a difficult task. In the framework of traveling waves of type solitary waves, namely, the profiles ϕ, ψ satisfying $\phi(\xi), \psi(\xi) \to 0$ as $|\xi| \to \infty$, it is well known the existence of solutions for (5) in the form

$$\phi_{0,s}(\xi) = \sqrt{\frac{2c\sigma}{\alpha\beta}}\, sech(\sqrt{\sigma}\xi), \qquad \psi_{0,s}(\xi) = -\frac{\beta}{c}\phi_{0,s}^2(\xi) \tag{6}$$

when $\gamma = 0$, $\sigma > 0$, and $\alpha\beta > 0$. For $\gamma \neq 0$ a theory of even solutions of these permanent wave solutions has been established in [7] (see also [8]) by using the concentration-compactness method.

For $\gamma = 0$ and $\sigma > 2\pi^2/L^2$ we prove (along the lines of Angulo [4] with regard to (3)) the existence of a smooth curve of even periodic traveling wave solutions for (5) with $\alpha = 1, \beta = 1/2$; note that this restriction does not imply loss of generality. This construction is based on the *dnoidal* Jacobian elliptic function, namely,

$$\begin{cases} \phi_0(\xi) = \eta_1\, dn\Big(\dfrac{\eta_1}{2\sqrt{c}}\,\xi; k\Big) \\ \psi_0(\xi) = -\dfrac{\eta_1^2}{2c}\, dn^2\Big(\dfrac{\eta_1}{2\sqrt{c}}\,\xi; k\Big), \end{cases} \tag{7}$$

where η_1 and k are positive smooth functions depending of the parameter σ. We observe that the solution in (7) gives us " in the limit " the solitary waves solutions (6) when $\eta_1 \to \sqrt{4c\sigma}$ and $k \to 1^-$, because in this case the elliptic function dn converges, uniformly on compacts sets, to the hyperbolic function *sech*.

In the case of our main interest, $\gamma \neq 0$, the existence of periodic solutions is a delicate issue. Our approach for the existence of these solutions uses the implicit function theorem together with the explicit formulas in (7) and a detailed study of the periodic eigenvalue problem associated to the Jacobian form of Lame's equation

$$\begin{cases} \dfrac{d^2}{dx^2}\Psi + [\rho - 6k^2 sn^2(x;k)]\Psi = 0 \\ \Psi(0) = \Psi(2K(k)), \quad \Psi'(0) = \Psi'(2K(k)), \end{cases} \tag{8}$$

where $sn(\cdot;k)$ is the Jacobi elliptic function of type snoidal and $K = K(k)$ represents the complete elliptic integral of the first kind and defined for $k \in (0,1)$ as

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

So, by fixing a period L, and choosing c and ω such that $\sigma \equiv \omega - \frac{c^2}{4}$ satisfies $\sigma > 2\pi^2/L^2$, we obtain a smooth branch $\gamma \in (-\delta, \delta) \to (\phi_\gamma, \psi_\gamma)$ of periodic traveling wave solutions of (5) with a fundamental period L and bifurcating from (ϕ_0, ψ_0) in (7). Moreover, we obtain that for γ near zero $\phi_\gamma(x) > 0$ for all $x \in \mathbb{R}$ and $\psi_\gamma(x) < 0$ for $\gamma < 0$ and $x \in \mathbb{R}$.

Furthermore, concerning the non-linear stability of each point in this branch of periodic solutions, we extend the classical approach developed by Benjamin [12], Bona [15], and Weinstein [38] to the periodic case. In particular, using the conservation laws (2), we prove that the solutions $(\phi_\gamma, \psi_\gamma)$ are stable in $H^1_{per}([0,L]) \times H^{\frac{1}{2}}_{per}([0,L])$ at least when γ is negative near zero. We use essentially the Benjamin, Bona, and Weinstein's stability ideas because it gives us an easy form of manipulating with the required spectral conditions and the positivity property of the quantity $\frac{d}{d\sigma}\int \phi^2_\gamma(x)dx$, which are basic information in our stability analysis.

However, we do not use the abstract stability theory of Grillakis et al. [26] in our approach basically because of the two circumstances above. We recall that Grillakis *et al.* theory in general requires a study of the Hessian for the function

$$d(c,\omega) = L(e^{ic\xi/2}\phi_\gamma, \psi_\gamma) \equiv E(e^{ic\xi/2}\phi_\gamma, \psi_\gamma) + cG(e^{ic\xi/2}\phi_\gamma, \psi_\gamma) + \omega H(e^{ic\xi/2}\phi_\gamma, \psi_\gamma)$$

with $\gamma = \gamma(c,\omega)$, and a specific spectrum information of the matrix linear operator $H_{c,\omega} = L''(e^{ic\xi/2}\phi_\gamma, \psi_\gamma)$. In our case, these facts do not seem to be easily obtained.

So, for $\gamma < 0$ we reduce the required spectral information (see formula (100)) to the study of the self-adjoint operator $\mathcal{L}_\gamma$,

$$\mathcal{L}_\gamma = -\frac{d^2}{d\xi^2} + \sigma + \alpha\psi_\gamma - 2\alpha\beta\phi_\gamma \circ \mathcal{K}_\gamma^{-1} \circ \phi_\gamma,$$

where $\mathcal{K}_\gamma^{-1}$ is the inverse operator of $\mathcal{K}_\gamma = -\gamma D + c$. Hence we obtain via the min-max principle that $\mathcal{L}_\gamma$ has a simple negative eigenvalue and zero is a simple eigenvalue with eigenfunction $\frac{d}{dx}\phi_\gamma$ provide that γ is small enough.

Finally, we close this introduction with the organization of this paper: in Section 2, we introduce some notations to be used throughout the whole article; in Section 3, we prove the global well-posedness results in the periodic and continuous settings via some appropriate bilinear estimates; in Section 4, we show the existence of periodic traveling waves by the implicit function theorem; then, in Section 5, we derive the stability of these waves based on the ideas of Benjamin and Weinstein, that is, to manipulate the information from the spectral theory of certain self-adjoint operators and the positivity of some relevant quantities.

2. **Notation.** For any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant θ such that $a \le \theta b$. Here, θ may depend only on certain parameters related to the equation (1) such as γ, α, β. Also, we denote $a \sim b$ when, $a \lesssim b$ and $b \lesssim a$.

For $a \in \mathbb{R}$, we denote by $a+$ and $a-$ a number slightly larger and smaller than a, respectively.

In the sequel, we fix ψ a smooth function supported on the interval $[-2,2]$ such that $\psi(x) \equiv 1$ for all $|x| \le 1$ and, for each $T > 0$, $\psi_T(t) := \psi(t/T)$.

Let $L > 0$. The inner product of two functions in $L^2([0,L])$ is given by

$$< f, g >= \int_0^L f(x)\bar{g}(x)dx, \quad \forall f,\ g \in L^2([0,L]).$$

Now let $\mathcal{P}'_L$ be the set of periodic distributions of period L. For all $s \in \mathbb{R}$ we denote by $H^s_{per}([0,L]) = H^s_L(\mathbb{R})$ the set of all f in $\mathcal{P}'_L$ such that

$$\|f\|_{H^s_L} = \left(L \sum_{n=-\infty}^{+\infty} (1+|n|^2)^s |\widehat{f}(n)|^2 \right)^{\frac{1}{2}} < \infty,$$

where $(\widehat{f}(n))_{n\in\mathbb{Z}}$ denote the Fourier series of f (for further information see Iorio and Iorio [29]). Sometimes we also write $H^s(\mathbb{T})$ to denote the space $H^s_{per}([0,L])$ when the period L does not play a fundamental role.

Similarly, when $s\in\mathbb{R}$, we denote by $H^s(\mathbb{R})$ the set of all $f\in\mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{H^s}=\left(\int_{\mathbb{R}}(1+|\xi|^2)^s|\widehat{f}(\xi)|^2d\xi\right)^{\frac{1}{2}}<\infty,$$

where $\mathcal{S}'(\mathbb{R})$ is the set of tempered distributions and $\widehat{f}$ is the Fourier transform of f.

When the function u is of the two time-space variables $(t,x)\in\mathbb{R}\times\mathbb{R}$, periodic in space of period L, we define its Fourier transform by

$$\widehat{u}(\tau,n)=\frac{1}{(2\pi)^{1/2}L}\int_{\mathbb{R}\times[0,L]}u(t,x)e^{-i(\frac{xn\pi}{L}+t\tau)}dtdx,$$

and similarly, when $u:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$, we define

$$\widehat{u}(\tau,\xi)=\frac{1}{2\pi}\int_{\mathbb{R}^2}u(t,x)e^{-i(x\xi+t\tau)}dtdx.$$

Next, we introduce the Bourgain spaces related to the Schrödinger-Benjamin-Ono system in the periodic case

$$\|u\|_{X^{s,b}_{per}}:=\left(\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\langle\tau+n^2\rangle^{2b}\langle n\rangle^{2s}|\widehat{u}(\tau,n)|^2d\tau\right)^{1/2}, \tag{9}$$

$$\|v\|_{Y^{s,b}_{\gamma,per}}:=\left(\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\langle\tau+\gamma|n|n\rangle^{2b}\langle n\rangle^{2s}|\widehat{v}(\tau,n)|^2d\tau\right)^{1/2}, \tag{10}$$

and the continuous case

$$\|u\|_{X^{s,b}}:=\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\langle\tau+\xi^2\rangle^{2b}\langle\xi\rangle^{2s}|\widehat{u}(\tau,\xi)|^2d\xi d\tau\right)^{1/2}, \tag{11}$$

$$\|v\|_{Y^{s,b}_{\gamma}}:=\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\langle\tau+\gamma|\xi|\xi\rangle^{2b}\langle\xi\rangle^{2s}|\widehat{v}(\tau,\xi)|^2d\xi d\tau\right)^{1/2}, \tag{12}$$

where $\langle x\rangle:=1+|x|$. In the continuous case, we will also use the localized (in time) version of these spaces. Let I be a time interval. If $u:I\times\mathbb{R}\to\mathbb{C}$, then

$$\|u\|_{X^{s,b}_I}:=\inf\{\|\tilde{u}\|_{X^{s,b}}\ /\ \tilde{u}:\mathbb{R}\times\mathbb{R}\to\mathbb{C},\tilde{u}|_{I\times\mathbb{R}}=u\},$$

and if $v:I\times\mathbb{R}\to\mathbb{R}$, then

$$\|v\|_{Y^{s,b}_{\gamma,I}}:=\inf\{\|\tilde{v}\|_{Y^{s,b}_{\gamma}}\ /\ \tilde{v}:\mathbb{R}\times\mathbb{R}\to\mathbb{R},\tilde{v}|_{I\times\mathbb{R}}=v\}.$$

The relevance of these spaces are related to the fact that they are well-adapted to the linear part of the system, and after some time-localization, the coupling terms of (1) verify particularly nice bilinear estimates. Consequently, it will be a standard matter to conclude our local well-posedness results (via Picard fixed point method).

3. **Global well-posedness of the Schrödinger-Benjamin-Ono system.** This section is devoted to the proof of our well-posedness results for (1) in both continuous and periodic settings.

3.1. **Global well-posedness on $\mathbb{R}$.** The bulk of this subsection is the proof of the following theorem:

Theorem 3.1. *Let $0 < |\gamma| \neq 1$. Then, the SBO system is globally well-posed for initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$, when $s \geq 0$. Moreover the solution (u, v) satisfies*

$$\|v(t)\|_{H^{-1/2}} \leq e^{c\|u_0\|^2_{L^2} t} \max\{\|u_0\|_{L^2}, \|v_0\|_{H^{-1/2}}\},\ \forall t > 0. \tag{13}$$

In the rest of this section, we will denote by $U(t) := e^{it\partial_x^2}$ and $V_\gamma(t) := e^{-\gamma t \mathcal{H}\partial_x^2}$ the unitary groups associated to the linear parts of (1). The proof of Theorem 3.1 follows the lines of [19]. Let us first state the linear estimates:

Lemma 3.2. *Let $0 \leq b, b_1 \leq \frac{1}{2}$, $s \in \mathbb{R}$ and $0 < T \leq 1$. Then*

$$\|\psi_T U(t) u_0\|_{X^{s,b_1}} \lesssim T^{\frac{1}{2}-b_1} \|u_0\|_{H^s}, \tag{14}$$

and

$$\|\psi_T V_\gamma(t) v_0\|_{Y_\gamma^{s-\frac{1}{2},b}} \lesssim T^{\frac{1}{2}-b} \|v_0\|_{H^{s-\frac{1}{2}}}, \tag{15}$$

for $\gamma \in \mathbb{R}$.

Proof. The estimates (14) and (15) are proved as in [19] Lemma 2.1 (a), since $U(t)$ and $V_\gamma(t)$ are both unitary groups. □

Lemma 3.3. *(i) Let $s \in \mathbb{R}$, $0 < T \leq 1$, $0 \leq c_1 < \frac{1}{2}$ and $b_1 \geq 0$ such that $b_1 + c_1 \leq 1$. Then*

$$\|\psi_T \int_0^t U(t-t') z(t') dt'\|_{X^{s,b_1}} \lesssim T^{1-b_1-c_1} \|z\|_{X^{s,-c_1}}, \tag{16}$$

and

$$\|\int_0^t U(t-t') z(t') dt'\|_{C([0,T];H^s)} \lesssim T^{\frac{1}{2}-c_1} \|z\|_{X^{s,-c_1}}. \tag{17}$$

(ii) Let $s \in \mathbb{R}$, $\gamma \in \mathbb{R}$, $0 < T \leq 1$, $0 \leq c < \frac{1}{2}$ and $b \geq 0$ such that $b + c \leq 1$. Then

$$\|\psi_T \int_0^t V_\gamma(t-t') z(t') dt'\|_{Y_\gamma^{s-\frac{1}{2},b}} \lesssim T^{1-b-c} \|z\|_{Y_\gamma^{s-\frac{1}{2},-c}}, \tag{18}$$

and

$$\|\int_0^t V_\gamma(t-t') z(t') dt'\|_{C([0,T];H^{s-\frac{1}{2}})} \lesssim T^{\frac{1}{2}-c} \|z\|_{Y_\gamma^{s-\frac{1}{2},-c}}. \tag{19}$$

Proof. The proof of Lemma 3.3 is identical to the proof of Lemma 2.3 (a) in [19]. □

Once these linear estimates are established, our task is to prove the following bilinear estimates.

Proposition 3.4. *Let $\gamma \in \mathbb{R}$ such that $|\gamma| \neq 1$ and $\gamma \neq 0$. Then, we have for any $\frac{1}{4} < b,\ b_1,\ c,\ c_1 < \frac{1}{2}$*

$$\|uv\|_{X^{0,-c_1}} \lesssim \|u\|_{Y_\gamma^{-\frac{1}{2},b}} \|v\|_{X^{0,b_1}}, \quad \text{if} \quad b + b_1 + c_1 \geq 1, \tag{20}$$

$$\|\partial_x(u\bar{v})\|_{Y_\gamma^{-\frac{1}{2},-c}} \lesssim \|u\|_{X^{0,b_1}} \|v\|_{X^{0,b_1}}, \quad \text{if} \quad 2b_1 + c \geq 1, \tag{21}$$

where the implicit constants depend on γ.

For the proof of these bilinear estimates, we need the following standard Bourgain-Strichartz estimates.

Proposition 3.5. *Let $\gamma \in \mathbb{R}$ such that $\gamma \neq 0$. Then*

$$\|u\|_{L^3_{t,x}} \lesssim \|u\|_{X^{0,1/4+}}, \tag{22}$$

and

$$\|u\|_{L^3_{t,x}} \lesssim \|u\|_{Y^{0,1/4+}_\gamma}. \tag{23}$$

Proof. The estimate (22) is obtained by interpolating halfway between Strichartz's estimate $\|u\|_{L^6_{x,t}} \lesssim \|u\|_{X^{0,1/2+}}$ and Plancherel's identity $\|u\|_{L^2_{x,t}} = \|u\|_{X^{0,0}}$. The estimate (23) can be proved by a similar argument. □

Finally, we recall the two following technical lemmas proved in [25].

Lemma 3.6. *Let $f \in L^q(\mathbb{R})$, $g \in L^{q'}(\mathbb{R})$ with $1 \le q,\ q' \le +\infty$ and $\frac{1}{q}+\frac{1}{q'}=1$. Assume that f and g are nonnegative, even and nonincreasing for positive argument. Then $f*g$ enjoys the same property. In particular $f*g$ takes its maximum at zero.*

Lemma 3.7. *Let $0 \le a_1, a_2 < \frac12$ such that $a_1+a_2 > \frac12$. Then*

$$\int_{\mathbb{R}} \langle y-\alpha\rangle^{-2a_1}\langle y-\beta\rangle^{-2a_2} \lesssim \langle \alpha-\beta\rangle^{1-2(a_1+a_2)}, \quad \forall\, \alpha,\beta \in \mathbb{R}.$$

Proof of Proposition 3.4. Without loss of generality we can suppose that $|\gamma|<1$ in the rest of the proof.

We first begin with the proof of the estimate (20). Letting $f(\tau,\xi) = \langle \xi\rangle^{-1/2}\langle \tau + \gamma\xi|\xi|\rangle^b \widehat{u}(\tau,\xi)$, $g(\tau,\xi) = \langle \tau+\xi^2\rangle^{b_1}\widehat{v}(\tau,\xi)$ and using duality, we deduce that (20) is equivalent to

$$I \lesssim \|f\|_{L^2_{\tau,\xi}}\|g\|_{L^2_{\tau,\xi}}\|h\|_{L^2_{\tau,\xi}}, \tag{24}$$

where

$$I = \int_{\mathbb{R}^4} \frac{h(\tau,\xi)\langle \xi_1\rangle^{1/2} f(\tau_1,\xi_1) g(\tau_2,\xi_2)}{\langle \sigma\rangle^{c_1}\langle\sigma_1\rangle^b\langle\sigma_2\rangle^{b_1}} d\xi d\xi_1 d\tau d\tau_1, \tag{25}$$

with $\xi_2 = \xi-\xi_1$, $\tau_2 = \tau-\tau_1$, $\sigma = \tau+\xi^2$, $\sigma_1 = \tau_1+\gamma|\xi_1|\xi_1$ and $\sigma_2 = \tau_2+\xi_2^2$. The algebraic relation associated to (25) is given by

$$-\sigma+\sigma_1+\sigma_2 = -\xi^2+\gamma|\xi_1|\xi_1+\xi_2^2. \tag{26}$$

We split the integration domain $\mathbb{R}^4$ in the following regions

$$\begin{aligned}
\mathcal{A} &= \{(\tau,\tau_1,\xi,\xi_1)\in\mathbb{R}^4 \ :\ |\xi_1|\le 1\},\\
\mathcal{B} &= \{(\tau,\tau_1,\xi,\xi_1)\in\mathbb{R}^4 \ : |\xi_1|>1 \text{ and } |\sigma_1| = \max(|\sigma|,|\sigma_1|,|\sigma_2|)\},\\
\mathcal{C} &= \{(\tau,\tau_1,\xi,\xi_1)\in\mathbb{R}^4 \ : |\xi_1|>1 \text{ and } |\sigma| = \max(|\sigma|,|\sigma_1|,|\sigma_2|)\},\\
\mathcal{D} &= \{(\tau,\tau_1,\xi,\xi_1)\in\mathbb{R}^4 \ : |\xi_1|>1 \text{ and } |\sigma_2| = \max(|\sigma|,|\sigma_1|,|\sigma_2|)\},
\end{aligned}$$

and denote by $I_{\mathcal{A}}$, $I_{\mathcal{B}}$, $I_{\mathcal{C}}$ and $I_{\mathcal{D}}$ the restriction of I to each one of these regions.

Estimate for $I_{\mathcal{A}}$. In this region $\langle \xi_1\rangle \le 1$, then we deduce using Plancherel's identity and Hölder's inequality that

$$\begin{aligned}
I_{\mathcal{A}} &\lesssim \int_{\mathbb{R}^2} \Big(\frac{h(\tau,\xi)}{\langle\tau+\xi^2\rangle^{c_1}}\Big)^\vee \Big(\frac{f(\tau,\xi)}{\langle\tau+\gamma|\xi|\xi\rangle^{b}}\Big)^\vee \Big(\frac{g(\tau,\xi)}{\langle\tau+\xi^2\rangle^{b_1}}\Big)^\vee dtdx\\
&\lesssim \Big\|\Big(\frac{h(\tau,\xi)}{\langle\tau+\xi^2\rangle^{c_1}}\Big)^\vee\Big\|_{L^3_{t,x}} \Big\|\Big(\frac{f(\tau,\xi)}{\langle\tau+\gamma|\xi|\xi\rangle^{b}}\Big)^\vee\Big\|_{L^3_{t,x}} \Big\|\Big(\frac{g(\tau,\xi)}{\langle\tau+\xi^2\rangle^{b_1}}\Big)^\vee\Big\|_{L^3_{t,x}}.
\end{aligned}$$

This implies, together with (22) and (23), that

$$I_{\mathcal{A}} \lesssim \|f\|_{L^2_{\tau,\xi}}\|g\|_{L^2_{\tau,\xi}}\|h\|_{L^2_{\tau,\xi}}, \tag{27}$$

since $b,\ b_1,\ c_1 > \frac{1}{4}$.

Estimate for $I_{\mathcal{B}}$. Using the Cauchy-Schwarz inequality twice, we deduce that

$$I_{\mathcal{B}} \lesssim \left(\sup_{\xi_1,\sigma_1} \langle \sigma_1 \rangle^{-2b} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma \rangle^{2c_1} \langle \sigma_2 \rangle^{2b_1}} d\xi d\sigma \right)^{\frac{1}{2}} \|f\|_{L^2_{\tau,\xi}} \|g\|_{L^2_{\tau,\xi}} \|h\|_{L^2_{\tau,\xi}}. \tag{28}$$

Recalling the algebraic relation (26), we have for $\xi_1,\ \sigma,\ \sigma_1$ fixed that $d\sigma_2 = -2\xi_1 d\xi$. Thus we obtain, by change of variables in the inner integral of the right-hand side of (28),

$$\begin{aligned} &\langle \sigma_1 \rangle^{-2b} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma \rangle^{2c_1} \langle \sigma_2 \rangle^{2b_1}} d\xi d\sigma \\ &\qquad \lesssim \langle \sigma_1 \rangle^{-2b} \left(\int_{|\sigma| \le |\sigma_1|} \frac{d\sigma}{\langle \sigma \rangle^{2c_1}} \right) \left(\int_{|\sigma_2| \le |\sigma_1|} \frac{d\sigma_2}{\langle \sigma_2 \rangle^{2b_1}} \right) \lesssim \langle \sigma_1 \rangle^{2(1-(b+b_1+c_1))} \lesssim 1, \end{aligned}$$

since $b + b_1 + c_1 \ge 1$. Combining this estimate with (28), we have

$$I_{\mathcal{B}} \lesssim \|f\|_{L^2_{\tau,\xi}} \|g\|_{L^2_{\tau,\xi}} \|h\|_{L^2_{\tau,\xi}}. \tag{29}$$

Estimate for $I_{\mathcal{C}}$. By the Cauchy-Schwarz inequality (applied two times) it is sufficient to bound

$$\langle \sigma \rangle^{-2c_1} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b_1}} d\xi_1 d\sigma_1 \tag{30}$$

independently of ξ and σ to obtain that

$$I_{\mathcal{C}} \lesssim \|f\|_{L^2_{\tau,\xi}} \|g\|_{L^2_{\tau,\xi}} \|h\|_{L^2_{\tau,\xi}}. \tag{31}$$

Now following [10], we first treat the subregion $|2((1+\gamma \mathrm{sgn}(\xi_1))\xi_1 - \xi)| \ge \frac{1-|\gamma|}{2}|\xi_1|$. When ξ, σ and σ_1 are fixed, the identity (26) implies that

$$d\sigma_2 = 2((1+\gamma \mathrm{sgn}(\xi_1))\xi_1 - \xi) d\xi_1.$$

Hence we deduce that

$$\begin{aligned} &\langle \sigma \rangle^{-2c_1} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b_1}} d\xi_1 d\sigma_1 \\ &\qquad \lesssim \langle \sigma \rangle^{-2c_1} \left(\int_{|\sigma_1| \le |\sigma|} \frac{d\sigma_1}{\langle \sigma_1 \rangle^{2b}} \right) \left(\int_{|\sigma_2| \le |\sigma|} \frac{d\sigma_2}{\langle \sigma_2 \rangle^{2b_1}} \right) \lesssim \langle \sigma \rangle^{2(1-(b+b_1+c_1))}, \end{aligned}$$

which is bounded since $b + b_1 + c_1 \ge 1$.

In the subregion of $\mathcal{C}$ where $|2((1+\gamma \mathrm{sgn}(\xi_1))\xi_1 - \xi)| < \frac{1-|\gamma|}{2}|\xi_1|$, we have from (26) that

$$|\xi_1|^2 \lesssim |\xi_1^2 + \gamma|\xi_1|\xi_1 - 2\xi\xi_1| \lesssim |\sigma|.$$

Then, we obtain (by applying Lemma 3.7):

$$\begin{aligned} \langle \sigma \rangle^{-2c_1} \int_{\mathbb{R}^2} \frac{|\xi_1|}{\langle \sigma_1 \rangle^{2b} \langle \sigma_2 \rangle^{2b_1}} d\xi_1 d\sigma_1 &\lesssim \langle \sigma \rangle^{\frac{1}{2}-2c_1} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \langle \sigma_1 \rangle^{-2b} \langle \sigma_2 \rangle^{-2b_1} d\sigma_1 \right) d\xi_1 \\ &\lesssim \langle \sigma \rangle^{\frac{1}{2}-2c_1} \int_{\mathbb{R}} \langle \sigma + \xi_1^2 + \gamma|\xi_1|\xi_1 - 2\xi\xi_1 \rangle^{1-2(b+b_1)} d\xi_1. \end{aligned}$$

Performing the change of variable $y=(\theta\xi_1-\theta^{-1}\xi)^2$, where $\theta=(1+\text{sgn}(\xi_1)\gamma)^{\frac12}$ and noticing that $|y|\lesssim|\sigma|$ and $dy=2\theta|y|^{\frac12}d\xi_1$ we deduce that

$$\begin{aligned}\langle\sigma\rangle^{-2c_1}&\int_{\mathbb{R}^2}\frac{|\xi_1|}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b_1}}d\xi_1d\sigma_1\\&\lesssim\langle\sigma\rangle^{\frac12-2c_1}\int_{|y|\lesssim|\sigma|}\frac{dy}{|y|^{\frac12}\langle y-\theta^{-2}\xi^2+\sigma\rangle^{-(1-2(b+b_1))}}.\end{aligned}$$

Now we use Lemma 3.6 to bound the right-hand side integral by

$$\int_{|y|\lesssim|\sigma|}|y|^{-\frac12}\langle y\rangle^{1-(2(b+b_1))}dy\lesssim\langle\sigma\rangle^{[\frac32-2(b+b_1)]_+},$$

where $[\alpha]_+=\alpha$ if $\alpha>0$, $[\alpha]_+=\epsilon$ arbitrarily small if $\alpha=0$, and $[\alpha]_+=0$ if $\alpha<0$. Therefore

$$\langle\sigma\rangle^{-2c_1}\int_{\mathbb{R}^2}\frac{|\xi_1|}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b_1}}d\xi_1d\sigma_1\lesssim\langle\sigma\rangle^{\frac12-2c_1+[\frac32-2(b+b_1)]_+}$$

which is always bounded using the assumptions on b, b_1 and c_1.
Estimate for $I_{\mathcal{D}}$. By the Cauchy-Schwarz inequality it suffices to bound

$$\langle\sigma_2\rangle^{-2b_1}\int_{\mathbb{R}^2}\frac{|\xi_1|}{\langle\sigma_1\rangle^{2b}\langle\sigma\rangle^{2c_1}}d\xi_1d\sigma_1 \tag{32}$$

independently of ξ_2 and σ_2.

We first treat the subregion $|2((1-\gamma\text{sgn}(\xi_1))\xi_1+\xi_2)|\geq\frac{1-|\gamma|}{2}|\xi_1|$. When ξ_2, σ_2 and σ are fixed, the identity (26) implies that

$$d\sigma=2(\xi_1+\xi_2-\gamma\text{sgn}(\xi_1)\xi_1)d\xi_1$$

Thus we can estimate (32) by

$$\langle\sigma_2\rangle^{-2b_1}\left(\int_{|\sigma_1|\leq|\sigma_2|}\frac{d\sigma_1}{\langle\sigma_1\rangle^{2b}}\right)\left(\int_{|\sigma|\leq|\sigma_2|}\frac{d\sigma}{\langle\sigma\rangle^{2c_1}}\right)\lesssim\langle\sigma\rangle^{2(1-(b+b_1+c_1))},$$

which is bounded since $b+b_1+c_1\geq1$.

In the subregion $|2((1-\gamma\text{sgn}(\xi_1))\xi_1+\xi_2)|<\frac{1-|\gamma|}{2}|\xi_1|$, using (26), we deduce that

$$|\xi_1|^2\lesssim|\xi_1^2-\gamma|\xi_1|\xi_1+2\xi_2\xi_1|\lesssim|\sigma_2|,$$

where the implicit constant depends on γ. Then, Lemma 3.7 implies that

$$\begin{aligned}\langle\sigma_2\rangle^{-2b_1}\int_{\mathbb{R}^2}\frac{|\xi_1|}{\langle\sigma_1\rangle^{2b}\langle\sigma\rangle^{2c_1}}d\xi_1d\sigma_1&\lesssim\langle\sigma_2\rangle^{\frac12-2b_1}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\langle\sigma_1\rangle^{-2b}\langle\sigma\rangle^{-2c_1}d\sigma_1\right)d\xi_1\\&\lesssim\langle\sigma_2\rangle^{\frac12-2b_1}\int_{\mathbb{R}}\langle\sigma_2+(1-\gamma\text{sgn}(\xi_1))\xi_1^2+2\xi_2\xi_1\rangle^{1-2(b+c_1)}d\xi_1.\end{aligned}$$

We perform the change of variable $y=(\theta\xi_1+\theta^{-1}\xi_2)^2$ where $\theta=(1-\gamma\text{sgn}(\xi_1))^{\frac12}$ in the last integral and we use Lemma 3.6 plus the assumptions on b, b_1 and c_1 to bound (32) by

$$\langle\sigma_2\rangle^{\frac12-2b_1}\int_{|y|\lesssim|\sigma_2|}|y|^{-\frac12}\langle y-\theta^{-2}\xi_2^2+\sigma_2\rangle^{1-2(b+c_1)}\lesssim\langle\sigma_2\rangle^{\frac12-2b_1+[\frac32-2(b+c_1)]_+}\lesssim1.$$

Therefore, we deduce that

$$I_{\mathcal{D}}\lesssim\|f\|_{L^2_{\tau,\xi}}\|g\|_{L^2_{\tau,\xi}}\|h\|_{L^2_{\tau,\xi}},$$

which, combined with (27), (29), and (31), implies (20).

The proof of (21) is actually identical to that of (20). Indeed, letting $f(\tau,\xi) = \langle \tau+\xi^2\rangle^{b_1}\widehat{u}(\tau,\xi)$ and $g(\tau,\xi) = \langle \tau-\xi^2\rangle^{b_1}\widehat{v}(\tau,\xi)$, we conclude that (21) is equivalent to

$$J \lesssim \|f\|_{L^2_{\tau,\xi}}\|g\|_{L^2_{\tau,\xi}}\|h\|_{L^2_{\tau,\xi}},$$

where

$$J = \int_{\mathbb{R}^4} \frac{|\xi|\langle\xi\rangle^{-\frac{1}{2}}h(\tau,\xi)f(\tau_1,\xi_1)g(\tau_2,\xi_2)}{\langle\sigma\rangle^c\langle\sigma_1\rangle^{b_1}\langle\sigma_2\rangle^{b_1}}d\xi d\xi_1 d\tau d\tau_1,$$

with $\xi_2 = \xi-\xi_1$, $\tau_2 = \tau-\tau_1$, $\sigma = \tau+\gamma|\xi|\xi$, $\sigma_1 = \tau_1+\xi_1^2$ and $\sigma_2 = \tau_2-\xi_2^2$ and where the algebraic relation associated to this integral is given by

$$-\sigma+\sigma_1+\sigma_2 = -\gamma|\xi|\xi+\xi_1^2-\xi_2^2.$$

□

We now slightly modify the bilinear estimates of Proposition 3.4.

Corollary 3.8. *Let $\gamma \in \mathbb{R}$ such that $|\gamma| \neq 1$ and $\gamma \neq 0$. Then, we have for any $\frac{1}{4} < b,\ b_1,\ c,\ c_1 < \frac{1}{2}$ and $s \geq 0$.*

$$\|uv\|_{X^{s,-c_1}} \lesssim \|u\|_{Y_\gamma^{s-\frac{1}{2},b}}\|v\|_{X^{0,b_1}} + \|u\|_{Y_\gamma^{-\frac{1}{2},b}}\|v\|_{X^{s,b_1}}, \quad \text{if} \quad b+b_1+c_1 \geq 1, \quad (33)$$

$$\|\partial_x(u\bar{v})\|_{Y_\gamma^{s-\frac{1}{2},-c}} \lesssim \|u\|_{X^{s,b_1}}\|v\|_{X^{0,b_1}} + \|u\|_{X^{0,b_1}}\|v\|_{X^{s,b_1}}, \quad \text{if} \quad 2b_1+c \geq 1. \quad (34)$$

Proof. For all $s \geq 0$, we have from the triangle inequality $\langle\xi\rangle^s \lesssim \langle\xi_1\rangle^s + \langle\xi-\xi_1\rangle^s$. Thus we obtain, denoting $(J^s\phi)^\wedge(\xi) = \langle\xi\rangle^s\widehat{\phi}(\xi)$ and using (20), that

$$\begin{aligned}\|uv\|_{X^{s,-c_1}} &\lesssim \|J^s uv\|_{X^{0,-c_1}} + \|uJ^s v\|_{X^{0,-c_1}} \\ &\lesssim \|J^s u\|_{Y_\gamma^{-\frac{1}{2},b}}\|v\|_{X^{0,b_1}} + \|u\|_{Y_\gamma^{-\frac{1}{2},b}}\|J^s v\|_{X^{0,b_1}} \\ &= \|u\|_{Y_\gamma^{s-\frac{1}{2},b}}\|v\|_{X^{0,b_1}} + \|u\|_{Y_\gamma^{-\frac{1}{2},b}}\|v\|_{X^{s,b_1}}.\end{aligned}$$

This proves the estimate (33). The estimate (34) follows using similar arguments with (21) instead of (20). □

Finally, we conclude this subsection with the proof of Theorem 3.1.

Proof of Theorem 3.1. Case $s = 0$. For $|t| \leq T \leq 1$, the system (1) is, at least formally, equivalent to the integral system

$$\begin{cases} u(t) = F_T^1(u,v) := \psi_T U(t)u(0) - i\alpha\psi_T \displaystyle\int_0^t U(t-t')u(t')v(t')dt', \\ v(t) = F_T^2(u) := \psi_T V_\gamma(t)v(0) + \beta\psi_T \displaystyle\int_0^t V_\gamma(t-t')\partial_x(|u(t')|^2)dt'. \end{cases} \quad (35)$$

Let $(u(0),v(0)) \in L^2(\mathbb{R}) \times H^{-\frac{1}{2}}(\mathbb{R})$, we want to use a contraction argument to solve (35) in a product of balls

$$X^{0,b_1}(a_1) \times Y_\gamma^{-\frac{1}{2},b}(a_2) = \{(u,v) \in X^{0,b_1} \times Y_\gamma^{-\frac{1}{2},b} \ / \ \|u\|_{X^{0,b_1}} \leq a_1,\ \|v\|_{Y_\gamma^{-\frac{1}{2},b}} \leq a_2\}. \quad (36)$$

We deduce combining the estimates (14), (15), (16), (18), (20) and (21) that

$$\|F_T^1(u,v)\|_{X^{0,b_1}} \lesssim T^{\frac{1}{2}-b_1}\|u(0)\|_{L^2} + T^{1-b_1-c_1}\|u\|_{X^{0,b_1}}\|v\|_{Y_\gamma^{-\frac{1}{2},b}},$$

and

$$\|F_T^2(u)\|_{Y_\gamma^{-\frac{1}{2},b}} \lesssim T^{\frac{1}{2}-b}\|v(0)\|_{H^{-\frac{1}{2}}} + T^{1-b-c}\|u\|_{X^{0,b_1}}^2,$$

for $\frac{1}{4} < b,\ b_1,\ c,\ c_1 < \frac{1}{2}$ such that $b + b_1 + c_1 \geq 1$ and $2b_1 + c \geq 1$. In the sequel, we fix $b = b_1 = c = c_1 = \frac{1}{3}$. Therefore we deduce by taking $a_1 \sim T^{\frac{1}{6}}\|u(0)\|_{L^2}$ and $a_2 \sim T^{\frac{1}{6}}\|v(0)\|_{H^{-\frac{1}{2}}}$ that (F_T^1, F_T^2) is a contraction in $X^{0,\frac{1}{3}}(a_1) \times Y_\gamma^{-\frac{1}{2},\frac{1}{3}}(a_2)$ if

$$T^{\frac{1}{2}}\|v(0)\|_{H^{-\frac{1}{2}}} \lesssim 1, \tag{37}$$

and

$$T^{\frac{1}{2}}\|u(0)\|_{L^2}^2 \lesssim \|v(0)\|_{H^{-\frac{1}{2}}}. \tag{38}$$

This leads to a solution (u, v) of (1) in $C([0,T]; L^2(\mathbb{R})) \times C([0,T]; H^{-\frac{1}{2}}(\mathbb{R}))$ satisfying

$$\|u\|_{X^{0,\frac{1}{3}}_{[0,T]}} \lesssim T^{\frac{1}{6}}\|u(0)\|_{L^2} \quad \text{and} \quad \|v\|_{Y^{-\frac{1}{2},\frac{1}{3}}_{\gamma,[0,T]}} \lesssim T^{\frac{1}{6}}\|v(0)\|_{H^{-\frac{1}{2}}}, \tag{39}$$

whenever T satisfies (37) and (38).

Since the L^2-norm of u is a conserved quantity by the SBO flow, we have two possibilities. Either $\|v(t)\|_{H^{-\frac{1}{2}}} \lesssim \|u(0)\|_{L^2}^2$ for all t, and then we can iterate the local argument indefinitely (with an iteration time $T_0 \sim \frac{1}{1+\|u(0)\|_{L^2}^2}$). Or there exists a time t such that $\|v(t)\|_{H^{-\frac{1}{2}}} \gg \|u(0)\|_{L^2}^2$. Take this time as the initial time $t = 0$. Hence the condition (38) is automatically satisfied and the condition (37) implies that the iteration time T must be $T \sim \|v(0)\|_{H^{-\frac{1}{2}}}^{-2}$. Then we deduce from (19), (21), (35) and (39) that there exists a positive constant C such that

$$\|v(T)\|_{H^{-\frac{1}{2}}} \leq \|v(0)\|_{H^{-\frac{1}{2}}} + CT^{\frac{1}{2}}\|u(0)\|_{L^2}^2,$$

so that we obtain after m iterations of time T where $m \sim \frac{\|v(0)\|_{H^{-\frac{1}{2}}}}{T^{\frac{1}{2}}(1+\|u(0)\|_{L^2}^2)}$ that

$$\|v(T_0)\|_{H^{-\frac{1}{2}}} = \|v(mT)\|_{H^{-\frac{1}{2}}} \leq 2\|v(0)\|_{H^{-\frac{1}{2}}}, \quad \text{where} \quad T_0 \sim \frac{1}{1+\|u(0)\|_{L^2}^2}.$$

Since T_0 only depends on $\|u(0)\|_{L^2}$, we can repeat the above argument and extend the solution (u, v) of (1) globally in time satisfying (13).

Case $s > 0$. Let $(u(0), v(0)) \in H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$. Fix an arbitrary time $\widetilde{T} > 0$. Applying the case $s = 0$, we know that there exists a global solution (u, v) to (1) satisfying

$$(u, v) \in C(\mathbb{R}; L^2(\mathbb{R})) \times C(\mathbb{R}; H^{-\frac{1}{2}}(\mathbb{R})).$$

We observe that (u, v) is a solution to the integral equation (35) in the time interval $[0, T_0]$ (where $T_0 \sim \frac{1}{1+\|u_0\|_{L^2}^2}$ is defined in the above case). Furthermore, by using the estimate (39), we see that (u, v) verifies

$$\|u\|_{X^{0,\frac{1}{3}}_{[0,T_0]}} \leq Cm \max_{1\leq j\leq m} \|u\|_{X^{0,\frac{1}{3}}_{[(j-1)T,jT]}} \lesssim C(\|u(0)\|_{L^2}, \|v(0)\|_{H^{-1/2}}), \tag{40}$$

and

$$\|v\|_{Y^{-\frac{1}{2},\frac{1}{3}}_{\gamma,[0,T_0]}} \leq Cm \max_{1\leq j\leq m} \|v\|_{Y^{-\frac{1}{2},\frac{1}{3}}_{\gamma,[(j-1)T,jT]}} \lesssim C(\|u(0)\|_{L^2}, \|v(0)\|_{H^{-1/2}}). \tag{41}$$

On the other hand, we deduce by using (14), (15), (16), (18), (33) and (34) with $b = b_1 = c = c_1 = \frac{1}{3}$, that

$$\|u\|_{X^{s,\frac{1}{3}}_{[0,\delta_0]}} \lesssim \|u(0)\|_{H^s} + \delta_0^{\frac{1}{3}}\left(\|u\|_{X^{0,\frac{1}{3}}_{[0,\delta_0]}}\|v\|_{Y^{s-\frac{1}{2},\frac{1}{3}}_{\gamma,[0,\delta_0]}} + \|u\|_{X^{s,\frac{1}{3}}_{[0,\delta_0]}}\|v\|_{Y^{-\frac{1}{2},\frac{1}{3}}_{\gamma,[0,\delta_0]}}\right), \tag{42}$$

and

$$\|v\|_{Y^{s-\frac12,\frac13}_{\gamma,[0,\delta_0]}} \lesssim \|v(0)\|_{H^{s-\frac12}} + \delta_0^{\frac13}\|u\|_{X^{0,\frac13}_{[0,\delta_0]}}\|u\|_{X^{s,\frac13}_{[0,\delta_0]}}, \tag{43}$$

where $0<\delta_0\le T_0$. Inserting the estimate (43) into (42) and recalling (40), (41), we get that

$$\|u\|_{X^{s,\frac13}_{[0,\delta_0]}} \lesssim \|u(0)\|_{H^s} + C(\|u(0)\|_{L^2},\|v(0)\|_{H^{-1/2}})\|v(0)\|_{H^{s-\frac12}}$$
$$+\delta_0^{\frac13}C(\|u(0)\|_{L^2},\|v(0)\|_{H^{-\frac12}},\tilde T)\|u\|_{X^{s,\frac13}_{[0,\delta_0]}}.$$

Thus, if we choose

$$\delta_0 \sim \frac{1}{1+C(\|u(0)\|_{L^2},\|v(0)\|_{H^{-\frac12}},\tilde T)^3}, \tag{44}$$

we deduce the *a priori* estimates

$$\|u\|_{X^{s,\frac13}_{[0,\delta_0]}} \lesssim \|u(0)\|_{H^s} + C(\|u(0)\|_{L^2},\|v(0)\|_{H^{-1/2}})\|v(0)\|_{H^{s-\frac12}},$$

and

$$\|v\|_{Y^{s-\frac12,\frac13}_{\gamma,[0,\delta_0]}} \lesssim \|v(0)\|_{H^{s-\frac12}} + C(\|u(0)\|_{L^2},\|v(0)\|_{H^{-1/2}})(\|u(0)\|_{H^s}+\|v(0)\|_{H^{s-\frac12}}).$$

Therefore we conclude from Lemma 3.3 that

$$(u,v)\in C([0,\delta_0];H^s(\mathbb{R}))\times C([0,\delta_0];H^{s-\frac12}(\mathbb{R})).$$

Since δ_0 defined in (44) only depends on $\|u(0)\|_{L^2}$, $\|v(0)\|_{H^{-\frac12}}$ and $\tilde T$, we can iterate the above argument a finite number of times to deduce that

$$(u,v)\in C([0,\tilde T];H^s(\mathbb{R}))\times C([0,\tilde T];H^{s-\frac12}(\mathbb{R})).$$

This completes the proof of Theorem 3.1 if one remembers that $\widetilde T>0$ is arbitrary. □

3.2. **Well-posedness on $\mathbb{T}$.** This subsection contains sharp bilinear estimates for the coupling terms uv and $\partial_x(|u|^2)$ of the SBO system in the periodic setting and the global well-posedness result in the energy space $H^1(\mathbb{T})\times H^{1/2}(\mathbb{T})$ (which is necessary for our subsequent stability theory).

Theorem 3.9 (Local well-posedness in $\mathbb{T}$). *Let $\gamma\in\mathbb{R}$ such that $\gamma\neq 0$, $|\gamma|\neq 1$ and $s\ge 1/2$. Then, the SBO system (1) is locally well-posed in $H^s(\mathbb{T})\times H^{s-1/2}(\mathbb{T})$, i.e. for all $(u(0),v(0))\in H^s(\mathbb{T})\times H^{s-1/2}(\mathbb{T})$, there exists $T=T(\|u(0)\|_{H^s},\|v(0)\|_{H^{s-1/2}})$ and a unique solution of the Cauchy problem (1) of the form $(\psi_T u,\psi_T v)$ such that $(u,v)\in X^{s,1/2+}_{per}\times Y^{s-1/2,1/2+}_{\gamma,per}$. Moreover, (u,v) satisfies the additional regularity*

$$(u,v)\in C([0,T];H^s(\mathbb{T}))\times C([0,T];H^{s-1/2}(\mathbb{T})) \tag{45}$$

and the map solution $S:(u(0),v(0))\mapsto(u,v)$ is smooth.

Using the conservation laws (2) as in [35], our local existence result implies

Theorem 3.10 (Global well-posedness in $\mathbb{T}$). *Let $\alpha,\beta,\gamma\in\mathbb{R}$ such that $\gamma\neq 0$, $|\gamma|\neq 1$ and $\frac{\alpha\gamma}{\beta}<0$. Then, the SBO system (1) is globally well-posed in $H^s(\mathbb{T})\times H^{s-1/2}(\mathbb{T})$, when $s\ge 1$.*

The fundamental technical points in the proof of Theorem 3.9 are the following bilinear estimates. The rest of the proof follows by standard arguments, as in [20].

Proposition 3.11. *Let $\gamma \in \mathbb{R}$ such that $\gamma \neq 0$, $|\gamma| \neq 1$ and $s \geq 1/2$. Then*

$$\|uv\|_{X_{per}^{s,-1/2+}} \lesssim \|u\|_{Y_{\gamma,per}^{s-1/2,1/2}} \|v\|_{X_{per}^{s,1/2}}, \tag{46}$$

$$\|\partial_x(u\bar{v})\|_{Y_{\gamma,per}^{s-1/2,-1/2+}} \lesssim \|u\|_{X_{per}^{s,1/2}} \|v\|_{X_{per}^{s,1/2}}, \tag{47}$$

where the implicit constants depend on γ.

These estimates are sharp in the following sense

Proposition 3.12. *Let $\gamma \neq 0$, $|\gamma| \neq 1$. Then*
- *(i) The estimate (46) fails for any $s < 1/2$.*
- *(ii) The estimate (47) fails for any $s < 1/2$.*

Proposition 3.13. *Let $\gamma \in \mathbb{R}$ such that $|\gamma| = 1$. Then*
- *(i) The estimate (46) fails for any $s \in \mathbb{R}$.*
- *(ii) The estimate (47) fails for any $s \in \mathbb{R}$.*

The following Bourgain-Strichartz estimates will be used in the proof of Proposition 3.11.

Proposition 3.14. *We have*

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,3/8}}, \tag{48}$$

and

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{Y_\gamma^{0,3/8}}, \tag{49}$$

for $u : \mathbb{R} \times \mathbb{T} \to \mathbb{C}$ and $\gamma \in \mathbb{R}$, $\gamma \neq 0$.

Proof. The first estimate of (48) was proved by Bourgain in [16] and the second one is a simple consequence of the first one (see for example [33]). □

Proof of Proposition 3.11. Fix $s \geq 1/2$. Without loss of generality we can suppose that $0 < |\gamma| < 1$ in the rest of the proof.

In order to prove (46), it is sufficient to prove that

$$\|uv\|_{X_{per}^{s,-3/8}} = \|\frac{\langle n\rangle^s}{\langle \tau + n^2\rangle^{3/8}} (uv)^\wedge(\tau, n)\|_{l^2_n L^2_\tau} \lesssim \|u\|_{Y_{\gamma,per}^{s-1/2,1/2}} \|v\|_{X_{per}^{s,1/2}}. \tag{50}$$

Letting $f(\tau, n) = \langle n\rangle^{s-1/2}\langle \tau + \gamma n|n|\rangle^{1/2}\widehat{u}(\tau, n)$, $g(\tau, n) = \langle n\rangle^s \langle \tau + n^2\rangle^{1/2}\widehat{v}(\tau, n)$ and using duality, we deduce that the estimate (50) is equivalent to

$$I \lesssim \|f\|_{L^2_\tau l^2_n} \|g\|_{L^2_\tau l^2_n} \|h\|_{L^2_\tau l^2_n}, \tag{51}$$

where

$$\begin{aligned} I := \sum_{n, n_1 \in \mathbb{Z}} \int_{\mathbb{R}^2} & \frac{\langle n\rangle^s h(\tau, n) f(\tau_1, n_1)}{\langle \tau + n^2\rangle^{3/8} \langle n_1\rangle^{s-1/2} \langle \tau_1 + \gamma |n_1| n_1\rangle^{1/2}} \\ & \times \frac{g(\tau_2, n_2)}{\langle n_2\rangle^s \langle \tau_2 + n_2^2\rangle^{1/2}} d\tau d\tau_1, \end{aligned} \tag{52}$$

where $\tau_2 = \tau - \tau_1$ and $n_2 = n - n_1$. In order to bound the integral in (52), we split the integration domain $\mathbb{R}^2 \times \mathbb{Z}^2$ in the following regions,

$$\begin{aligned} \mathcal{M} &= \{(\tau, \tau_1, n, n_1) \in \mathbb{R}^2 \times \mathbb{Z}^2 \ : \ n_1 = 0 \text{ or } |n| \leq c(\gamma)^{-1}|n_2|\}, \\ \mathcal{N} &= \{(\tau, \tau_1, n, n_1) \in \mathbb{R}^2 \times \mathbb{Z}^2 \ : \ n_1 \neq 0 \text{ and } |n_2| \leq c(\gamma)|n|\}, \end{aligned}$$

where $c(\gamma)$ is a positive constant depending on γ to be fixed later. We also denote by $I_{\mathcal{M}}$ and $I_{\mathcal{N}}$ the integral I restricted to the regions $\mathcal{M}$ and $\mathcal{N}$, respectively.

Estimate on the region $\mathcal{M}$. We observe that, since $s \geq 1/2$, it holds $\frac{\langle n\rangle^s}{\langle n_1\rangle^{s-1/2}\langle n_2\rangle^s} \lesssim 1$ (where the implicit constant depends on γ) in the region $\mathcal{M}$. Thus, we deduce that, using the Plancherel identity and the $L^2_{t,x}L^4_{t,x}L^4_{t,x}$-Hölder inequality,

$$\begin{aligned} I_{\mathcal{M}} &\lesssim \sum_{n,n_1\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{h(\tau,n)f(\tau_1,n_1)g(\tau_2,n_2)d\tau d\tau_1}{\langle \tau+n^2\rangle^{3/8}\langle \tau_1+\gamma|n_1|n_1\rangle^{1/2}\langle \tau_2+n_2^2\rangle^{1/2}} \\ &\lesssim \int_{\mathbb{R}\times\mathbb{T}} \left(\frac{h(\tau,n)}{\langle \tau+n^2\rangle^{3/8}}\right)^{\vee} \left(\frac{f(\tau,n)}{\langle \tau+\gamma|n|n\rangle^{1/2}}\right)^{\vee} \left(\frac{g(\tau,n)}{\langle \tau+n^2\rangle^{1/2}}\right)^{\vee} dtdx \\ &\lesssim \| \left(\frac{h(\tau,n)}{\langle \tau+n^2\rangle^{3/8}}\right)^{\vee} \|_{L^2_{t,x}} \| \left(\frac{f(\tau,n)}{\langle \tau+\gamma|n|n\rangle^{1/2}}\right)^{\vee} \|_{L^4_{t,x}} \| \left(\frac{g(\tau,n)}{\langle \tau+n^2\rangle^{1/2}}\right)^{\vee} \|_{L^4_{t,x}}. \end{aligned}$$

This implies, together with Estimates (48) and (49), that

$$I_{\mathcal{M}} \lesssim \|f\|_{L^2_\tau l^2_n} \|g\|_{L^2_\tau l^2_n} \|h\|_{L^2_\tau l^2_n}. \tag{53}$$

Estimate on the region $\mathcal{N}$. The dispersive smoothing effect associated to the SBO system (1) can be translated by the following algebraic relation

$$-(\tau+n^2) + (\tau_1+\gamma n_1|n_1|) + (\tau_2+n_2^2) = Q_\gamma(n,n_1), \tag{54}$$

where

$$Q_\gamma(n,n_1) = n_2^2 + \gamma|n_1|n_1 - n^2. \tag{55}$$

We have in the region $\mathcal{N}$, $|n_1| \leq (1+c(\gamma))|n|$, so that

$$|Q_\gamma(n,n_1)| \geq \left(1-|\gamma|(1+c(\gamma))^2 - c(\gamma)^2\right)(1+c(\gamma))^{-2}|n_1|^2.$$

Now, we choose $c(\gamma)$ positive, small enough such that

$$\left(1-|\gamma|(1+c(\gamma))^2 - c(\gamma)^2\right) = \frac{1-|\gamma|}{2},$$

which is possible since $|\gamma| < 1$. Therefore, we divide the region $\mathcal{N}$ in three parts accordingly to which term of the left-hand side of (54) is dominant:

$$\begin{aligned} \mathcal{N}_1 &= \{(\tau,\tau_1,n,n_1)\in\mathcal{N} \;:\; |\tau+n^2| \geq |\tau_1+\gamma|n_1|n_1|, |\tau_2+n_2^2|\}, \\ \mathcal{N}_2 &= \{(\tau,\tau_1,n,n_1)\in\mathcal{N} \;:\; |\tau_1+\gamma|n_1|n_1| \geq |\tau+n^2|, |\tau_2+n_2^2|\}, \\ \mathcal{N}_3 &= \{(\tau,\tau_1,n,n_1)\in\mathcal{N} \;:\; |\tau_2+n_2^2| \geq |\tau+n^2|, |\tau_1+\gamma|n_1|n_1|\}. \end{aligned}$$

We denote by $I_{\mathcal{N}_1}$, $I_{\mathcal{N}_2}$ and $I_{\mathcal{N}_3}$ the restriction of the integral I to the regions $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$, respectively.

In the region $\mathcal{N}_1$, we have $\frac{\langle n\rangle^s}{\langle n_1\rangle^{s-1/2}\langle n_2\rangle^s} \times \frac{1}{\langle \tau+n^2\rangle^{3/8}} \lesssim 1$, so that we can conclude

$$I_{\mathcal{N}_1} \lesssim \|f\|_{L^2_\tau l^2_n} \|g\|_{L^2_\tau l^2_n} \|h\|_{L^2_\tau l^2_n}, \tag{56}$$

exactly as for $I_{\mathcal{M}}$. We note that $\frac{\langle n\rangle^s}{\langle n_1\rangle^{s-1/2}\langle n_2\rangle^s} \times \frac{1}{\langle \tau_1+|n_1|n_1\rangle^{1/2}} \lesssim 1$ in the region $\mathcal{N}_2$. Then, using the $L^4_{t,x}L^2_{t,x}L^4_{t,x}$-Hölder inequality, that

$$I_{\mathcal{N}_2} \lesssim \| \left(\frac{h(\tau,n)}{\langle \tau+n^2\rangle^{3/8}}\right)^{\vee} \|_{L^4_{t,x}} \|f\|_{L^2_\tau l^2_n} \| \left(\frac{g(\tau,n)}{\langle \tau+n^2\rangle^{1/2}}\right)^{\vee} \|_{L^4_{t,x}}.$$

Combining this with (48), we obtain that

$$I_{\mathcal{N}_2} \lesssim \|f\|_{L^2_\tau l^2_n} \|g\|_{L^2_\tau l^2_n} \|h\|_{L^2_\tau l^2_n}. \tag{57}$$

Similarly, $\frac{\langle n\rangle^s}{\langle n_1\rangle^{s-1/2}\langle n_2\rangle^s}\times\frac{1}{\langle \tau_2+(n_2)^2\rangle^{1/2}}\lesssim 1$ in $\mathcal{N}_3$ so that

$$\begin{aligned} I_{\mathcal{N}_3} &\lesssim \|\left(\frac{h(\tau,n)}{\langle\tau+n^2\rangle^{3/8}}\right)^{\vee}\|_{L^4_{t,x}}\|\left(\frac{f(\tau,n)}{\langle\tau+\gamma|n|n\rangle^{1/2}}\right)^{\vee}\|_{L^4_{t,x}}\|g\|_{L^2_\tau l^2_n} \\ &\lesssim \|f\|_{L^2_\tau l^2_n}\|g\|_{L^2_\tau l^2_n}\|h\|_{L^2_\tau l^2_n}. \end{aligned} \tag{58}$$

Then, we gather (53), (56), (57) and (58) to deduce (51), which concludes the proof of the estimate (46).

Next, in order to prove (47), we argue as above so that it is sufficient to prove

$$\|\partial_x(u\bar v)\|_{Y^{s-1/2,-3/8}_{\gamma,per}}\lesssim\|u\|_{X^{s,1/2}_{per}}\|v\|_{X^{s,1/2}_{per}}, \tag{59}$$

which is equivalent by duality and after performing the change of variable $f(\tau,n)=\langle n\rangle^s\langle\tau+n^2\rangle^{1/2}\widehat{u}(\tau,n)$ and $g(\tau,n)=\langle n\rangle^s\langle\tau-n^2\rangle^{1/2}\widehat{\bar v}(\tau,n)$ to

$$J\lesssim\|f\|_{L^2_\tau l^2_n}\|g\|_{L^2_\tau l^2_n}\|h\|_{L^2_\tau l^2_n}, \tag{60}$$

where

$$J:=\sum_{n,n_1\in\mathbb{Z}}\int_{\mathbb{R}^2}\frac{|n|\langle n\rangle^{s-1/2}h(\tau,n)f(\tau_1,n_1)}{\langle\tau+\gamma|n|n\rangle^{3/8}\langle n_1\rangle^s\langle\tau_1+n_1^2\rangle^{1/2}}\times\frac{g(\tau_2,n_2)}{\langle n_2\rangle^s\langle\tau_2-n_2^2\rangle^{1/2}}d\tau d\tau_1. \tag{61}$$

The algebraic relation associated to (61) is given by

$$-(\tau+\gamma|n|n)+(\tau_1+n_1^2)+(\tau_2-n_2^2)=\tilde{Q}_\gamma(n,n_1),$$

where

$$\tilde{Q}_\gamma(n,n_1)=-n_2^2-\gamma|n|n+n_1^2.$$

Therefore we can prove the estimate (60) using exactly the same arguments as for the estimate (51). □

Remark 3.15. Observe that we obtain our bilinear estimates in the spaces $X^{s,1/2+}_{per}$ and $Y^{s-1/2,1/2+}_{\gamma,per}$ which control the $L^\infty_t H^s_x$ and $L^\infty_t H^{s-1/2}_x$ norms respectively. Therefore, we do not need to use other norms as in the case of the periodic KdV equation [20].

In the proof of Proposition 3.12, we will use the following lemma which is a direct consequence of the Dirichlet theorem.

Lemma 3.16. *Let $\gamma\in\mathbb{R}$ such that $\gamma\neq 0$ and $|\gamma|<1$, and Q_γ defined as in (55). Then, there exists a sequence of positive integers $\{N_j\}_{j\in\mathbb{N}}$ such that*

$$N_j\to\infty\quad\text{and}\quad|Q_\gamma(N_j,N_j^0)|\lesssim 1, \tag{62}$$

where $N_j^0=[\frac{2N_j}{1+\gamma}]$ and $[x]$ denotes the closest integer to x.

Theorem 3.17 (Dirichlet)**.** *Let $\alpha\in\mathbb{R}\setminus\mathbb{Q}$. Then, the inequality*

$$0<\left|\alpha-\frac{p}{q}\right|<\frac{1}{q^2} \tag{63}$$

has infinitely many rational solutions $\frac{p}{q}$.

Proof of Lemma 3.16. Fix $\gamma \in \mathbb{R}$ such that $|\gamma| < 1$. Let N a positive integer, $N \geq 2$, $\alpha = \frac{2}{1+\gamma}$ and $N^0 = [\alpha N]$. Then, from the definition in (55), we deduce that

$$|Q_\gamma(N, N^0)| \lesssim 1 \qquad \Longleftrightarrow \qquad \left|\alpha - \frac{[\alpha N]}{N}\right| \leq \frac{1}{N^2}. \tag{64}$$

When $\alpha \in \mathbb{Q}$, $\alpha = \frac{p}{q}$, it is clear that we can find an infinity of positive integer N satisfying the right-hand side of (64) choosing $N_j = jq$, $j \in \mathbb{N}$. When $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, this is guaranteed by the Dirichlet theorem. □

Proof of Proposition 3.12. We will only show that the estimate (46) fails, since a counterexample for the estimate (47) can be constructed in a similar way. First observe that, letting $f(\tau, n) = \langle n\rangle^{s-1/2}\langle \tau + \gamma n|n|\rangle^{1/2}\widehat{u}(\tau, n)$ and $g(\tau, n) = \langle n\rangle^s\langle \tau + n^2\rangle^{1/2}\widehat{v}(\tau, n)$, the estimate (46) is equivalent to

$$\|B_\gamma(f, g; s)\|_{L^2_\tau l^2_n} \lesssim \|f\|_{L^2_\tau l^2_n}\|g\|_{L^2_\tau l^2_n}, \ \forall \ f, \ g \in L^2_\tau l^2_n, \tag{65}$$

where

$$B_\gamma(f, g; s)(\tau, n) := \frac{\langle n\rangle^s}{\langle \tau + n^2\rangle^{1/2-}} \sum_{n_1 \in \mathbb{Z}} \int_{\mathbb{R}} \frac{f(\tau_1, n_1)}{\langle n_1\rangle^{s-1/2}\langle \tau_1 + \gamma|n_1|n_1\rangle^{1/2}} \times \frac{g(\tau_2, n_2)}{\langle n_2\rangle^s\langle \tau_2 + n_2^2\rangle^{1/2}} d\tau_1, \tag{66}$$

and $\tau_2 = \tau - \tau_1$, $n_2 = n - n_1$, for all s and $\gamma \in \mathbb{R}$.

Fix $s < 1/2$ and γ such that $|\gamma| \neq 1$; without loss of generality, we can suppose that $|\gamma| < 1$. Consider the sequence of integer $\{N_j\}$ obtained in Lemma 3.16. Note that we can always assume $N_j \gg 1$. Define

$$f_j(\tau, n) = a_n \chi_{1/2}(\tau + \gamma|n|n) \quad \text{with} \quad a_n = \begin{cases} 1, & n = N_j^0, \\ 0, & \text{elsewhere}, \end{cases} \tag{67}$$

and

$$g_j(\tau, n) = b_n \chi_{1/2}(\tau + n^2) \quad \text{with} \quad b_n = \begin{cases} 1, & n = N_j - N_j^0, \\ 0, & \text{elsewhere}, \end{cases} \tag{68}$$

where χ_r is the characteristic function of the interval $[-r, r]$. Hence,

$$\|f_j\|_{L^2_\tau l^2_n} \sim \|g_j\|_{L^2_\tau l^2_n} \sim 1, \tag{69}$$

$$a_{n_1} b_{n-n_1} \neq 0 \quad \text{if and only if} \quad n_1 = N_j^0 \text{ and } n = N_j.$$

Using (54), we deduce that

$$\int_{\mathbb{R}} \chi_{1/2}(\tau_1 + \gamma|N_j^0|N_j^0)\chi_{1/2}(\tau - \tau_1 + (N_j - N_j^0)^2)d\tau_1 \sim \chi_1(\tau + N^2 + Q_\gamma(N_j, N_j^0)).$$

Therefore, we have from the definition in (66)

$$B_\gamma(f_j, g_j; s)(\tau, N_j) \gtrsim \frac{N_j^s \chi_1(\tau + N_j^2 + Q_\gamma(N_j, N_j^0))}{\langle \tau + N_j^2\rangle^{1/2-} N_j^{s-1/2} N_j^s}, \tag{70}$$

where the implicit constant depends on γ. Thus, we deduce using (62) that

$$\|B_\gamma(f_j, g_j; s)\|_{L^2_\tau l^2_n} \gtrsim N_j^{1/2-s}, \quad \forall \ j \in \mathbb{N} \tag{71}$$

which combined with (62) and (69) contradicts (65), since $s < 1/2$. □

Proof of Proposition 3.13. Let $s \in \mathbb{R}$, we fix $\gamma = 1$. As in the proof of Proposition 3.12, we will only show that the estimate (46) fails, since a counterexample for the estimate (47) can be constructed in a similar way. In this case, (46) is equivalent to

$$\|B_1(f,g;s)\|_{L^2_\tau l^2_n} \lesssim \|f\|_{L^2_\tau l^2_n} \|g\|_{L^2_\tau l^2_n}, \ \forall\ f,\ g \in L^2_\tau l^2_n, \tag{72}$$

where

$$B_1(f,g;s)(\tau,n) := \frac{\langle n\rangle^s}{\langle \tau+n^2\rangle^{1/2-}} \sum_{n_1\in\mathbb{Z}} \int_{\mathbb{R}} \frac{f(\tau_1,n_1)}{\langle n_1\rangle^{s-1/2}\langle \tau_1+|n_1|n_1\rangle^{1/2}} \times \frac{g(\tau_2,n_2)}{\langle n_2\rangle^s \langle \tau_2+n_2^2\rangle^{1/2}} d\tau_1, \tag{73}$$

and $\tau_2 = \tau - \tau_1$ and $n_2 = n - n_1$. Fix a positive integer N, such that $N \gg 1$, and define

$$f_N(\tau,n) = a_n \chi_{1/2}(\tau+|n|n) \quad \text{with} \quad a_n = \begin{cases} 1, & n = N, \\ 0, & \text{elsewhere}, \end{cases} \tag{74}$$

and

$$g_N(\tau,n) = b_n \chi_{1/2}(\tau+n^2) \quad \text{with} \quad b_n = \begin{cases} 1, & n = 0, \\ 0, & \text{elsewhere}, \end{cases} \tag{75}$$

where χ_r is the characteristic function of the interval $[-r,r]$. Hence,

$$\|f_N\|_{L^2_\tau l^2_n} \sim \|g_N\|_{L^2_\tau l^2_n} \sim 1, \tag{76}$$

$$a_{n_1} b_{n-n_1} \neq 0 \quad \text{if and only if} \quad n_1 = N \text{ and } n = N,$$

and

$$\int_{\mathbb{R}} \chi_{1/2}(\tau_1+N^2)\chi_{1/2}(\tau-\tau_1)d\tau_1 \sim \chi_1(\tau+N^2).$$

Therefore, we deduce from (73) that

$$\|B_1(f_N,g_N;s)\|_{L^2_\tau l^2_n} \gtrsim N^{1/2}, \quad \forall\ N \gg 1, \tag{77}$$

which combined with (76) contradicts (72). The case $\gamma = -1$ is similar. □

Remark 3.18. Observe that the same counterexamples would prove that the estimates (46) and (47) also fail whenever $|\gamma| \neq 1$, $\gamma \neq 0$ if $s < 1/2$ and whenever $|\gamma| = 1$ if $s \in \mathbb{R}$, if we substitute $-1/2+$ by b' and $1/2$ by b for any $b, b' \in \mathbb{R}$ (see also [31] for similar results).

4. **Existence of periodic traveling wave solutions.** The goal of this section is to show the existence of a smooth branch of periodic traveling wave solutions for (5). Based on the ideas of Angulo in [4], we first show the existence of a smooth branch of dnoidal waves solutions for (5) in the case $\gamma = 0$. Then by using the implicit function theorem we construct (in the case $\gamma \neq 0$ and close to 0) a curve of periodic solutions for (5) bifurcating from these dnoidal waves.

4.1. **Dnoidal wave solutions.** We start by finding solutions for the case $\gamma = 0$ and $\sigma > 0$ in (5). Henceforth, without loss of generality, we will assume that $\alpha = 1$ and $\beta = \frac{1}{2}$. Hence, we will solve the system

$$\begin{cases} \phi_0'' - \sigma\phi_0 = \psi_0\phi_0 \\ \psi_0 = -\frac{1}{2c}\phi_0^2. \end{cases} \tag{78}$$

Then, by substituting the second equation of (78) into the first one, we obtain that

$$\phi_0'' - \sigma\phi_0 + \frac{1}{2c}\phi_0^3 = 0. \tag{79}$$

Equation (79) can be solved in a similar fashion to the method used by Angulo in [4] . For the sake of completeness, we provide here a sketch of the proof of this fact. Indeed, from (79), ϕ_0 must satisfy the first-order equation

$$[\phi_0']^2 = \frac{1}{4c}[-\phi_0^4 + 4c\sigma\phi_0^2 + 4cB_{\phi_0}] = \frac{1}{4c}(\eta_1^2 - \phi_0^2)(\phi_0^2 - \eta_2^2),$$

where B_{ϕ_0} is an integration constant and $-\eta_1, \eta_1, -\eta_2, \eta_2$ are the zeros of the polynomial $F(t) = -t^4 + 4c\sigma t^2 + 4cB_{\phi_0}$. Moreover,

$$\begin{cases} 4c\sigma = \eta_1^2 + \eta_2^2 \\ 4cB_{\phi_0} = -\eta_1^2\eta_2^2. \end{cases} \tag{80}$$

Now, if we are looking for by ϕ_0 to be a non-constant and smooth periodic solution, then $\eta_2 \leq \phi_0 \leq \eta_1$. So, by choosing $\eta_1 > \eta_2 > 0$ we obtain that ϕ_0 is a positive solution. Note that $-\phi_0$ is also a solution of (79). Next, define $\zeta = \phi_0/\eta_1$ and $k^2 = (\eta_1^2 - \eta_2^2)/\eta_1^2$. It follows from (80) that

$$[\zeta']^2 \quad = \tfrac{\eta_1^2}{4c}(1 - \zeta^2)(\zeta^2 + k^2 - 1).$$

Now define χ through $\zeta^2 = 1 - k^2 \sin^2\chi$. So we get that $4c(\chi')^2 = \eta_1^2(1 - k^2\sin^2\chi)$. Then for $l = \frac{\eta_1}{2\sqrt{c}}$, and assuming that $\zeta(0) = 1$, we have

$$\int_0^{\chi(\xi)} \frac{dt}{\sqrt{1 - k^2 sin^2 t}} \; = l\,\xi.$$

Then from the definition of the *Jacobian elliptic function* $sn(u; k)$, we have that $\sin\chi = sn(l\xi; k)$ and hence $\zeta(\xi) = \sqrt{1 - k^2 sn^2(l\xi; k)} \equiv dn(l\xi; k)$. Returning to the variable ϕ_0, we obtain the **dnoidal wave** solutions associated to equation (78),

$$\begin{cases} \phi_0(\xi) \equiv \phi_0(\xi; \eta_1, \eta_2) = \eta_1\; dn\Big(\dfrac{\eta_1}{2\sqrt{c}}\,\xi; k\Big) \\ \psi_0(\xi) \equiv \psi_0(\xi; \eta_1, \eta_2) = -\dfrac{\eta_1^2}{2c}\; dn^2\Big(\dfrac{\eta_1}{2\sqrt{c}}\,\xi; k\Big), \end{cases} \tag{81}$$

where

$$0 < \eta_2 < \eta_1, \qquad k^2 = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2}, \qquad \eta_1^2 + \eta_2^2 = 4c\sigma. \tag{82}$$

Next, since dn has the fundamental period $2K(k)$, it follows that ϕ_0 in (81) has the fundamental wavelength (i.e., period) T_{ϕ_0} given by

$$T_{\phi_0} \equiv \frac{4\sqrt{c}}{\eta_1}\,K(k).$$

Given $c > 0$, $\sigma > 0$, it follows from (82) that $0 < \eta_2 < \sqrt{2c\sigma} < \eta_1 < 2\sqrt{c\sigma}$. Moreover we can write

$$T_{\phi_0}(\eta_2) = \frac{4\sqrt{c}}{\sqrt{4c\sigma - \eta_2^2}}\, K(k(\eta_2)) \qquad \text{with} \qquad k^2(\eta_2) = \frac{4c\sigma - 2\eta_2^2}{4c\sigma - \eta_2^2}. \tag{83}$$

Then, using these formulas and the properties of the function K, we see that $T_{\phi_0} \in (\sqrt{\frac{2}{\sigma}}\,\pi, +\infty)$ for $\eta_2 \in (0, \sqrt{2c\sigma})$. Moreover, we will see in Theorem 4.1 below that $\eta_2 \mapsto T_{\phi_0}(\eta_2)$ is a strictly decreasing mapping and so we obtain the basic inequality

$$T_{\phi_0} > \sqrt{\frac{2}{\sigma}}\,\pi. \tag{84}$$

Two relevant solutions of (78) are hidden in (81). Namely, the constant and solitary wave solutions. Indeed, when $\eta_2 \to \sqrt{2c\sigma}$, i.e. $\eta_2 \to \eta_1$, it follows that $k \to 0^+$. Then since $d(u; 0^+) \to 1$ we obtain the constant solutions

$$\phi_0(\xi) = \sqrt{2c\sigma} \qquad \text{and} \qquad \psi_0(\xi) = -\sigma. \tag{85}$$

Next, for $\eta_2 \to 0$ we have $\eta_1 \to 4c\sigma^-$ and so $k \to 1^-$. Then since $dn(u; 1^-) \to sech(u)$ we obtain the classical solitary wave solutions

$$\phi_{0,s}(\xi) = 2\sqrt{c\sigma} sech(\sqrt{\sigma}\xi) \qquad \text{and} \qquad \psi_{0,s}(\xi) = -2\sigma sech^2(\sqrt{\sigma}\xi). \tag{86}$$

Our next theorem is the main result of this subsection and it proves that for a fixed period $L > 0$ there exists a smooth branch of dnoidal wave solutions with the same minimal period L to (78). The construction is an immediate consequence of the implicit function theorem. Indeed, fix $c > 0$ and $\omega \in \mathbb{R}$ such that $\sigma \equiv \omega - c^2/4 > 2\pi^2/L^2$. Since the function $\eta_2 \in (0, \sqrt{2c\sigma}) \to T_{\phi_0}(\eta_2)$ is strictly decreasing (see proof of Theorem 4.1 below), there is a unique $\eta_2 = \eta_2(\sigma) \in (0, \sqrt{2c\sigma})$ such that $\phi_0(\cdot; \eta_1(\sigma), \eta_2(\sigma))$ has the fundamental period $T_{\phi_0}(\eta_2(\sigma)) = L$. Next, we claim that the choice of $\eta_2(\sigma)$ depends smoothly of σ:

Theorem 4.1. *Let L and c be arbitrarily fixed positive numbers. Let $\sigma_0 > 2\pi^2/L^2$ and $\eta_{2,0} = \eta_2(\sigma_0)$ be the unique number in the interval $(0, \sqrt{2c\sigma})$ such that $T_{\phi_0}(\eta_{2,0}) = L$. Then,*

(1) *there are intervals $I(\sigma_0)$ and $B(\eta_{2,0})$ around of σ_0 and $\eta_2(\sigma_0)$ respectively, and an unique smooth function $\Lambda : I(\sigma_0) \to B(\eta_{2,0})$, such that $\Lambda(\sigma_0) = \eta_{2,0}$ and*

$$\frac{4\sqrt{c}}{\sqrt{4c\sigma - \eta_2^2}}\, K(k(\sigma)) = L, \tag{87}$$

where $\sigma \in I(\sigma_0)$, $\eta_2 = \Lambda(\sigma)$, and

$$k^2 \equiv k^2(\sigma) = \frac{4c\sigma - 2\eta_2^2}{4c\sigma - \eta_2^2} \in (0, 1). \tag{88}$$

(2) *Solutions $(\phi_0(\cdot; \eta_1, \eta_2), \psi_0(\cdot; \eta_1, \eta_2))$ given by (81) and determined by $\eta_1 = \eta_1(\sigma)$, $\eta_2 = \eta_2(\sigma) = \Lambda(\sigma)$, with $\eta_1^2 + \eta_2^2 = 4c\sigma$, have the fundamental period L and satisfy (78). Moreover, the mapping*

$$\sigma \in I(\sigma_0) \to \phi_0(\cdot; \eta_1(\sigma), \eta_2(\sigma)) \in H^n_{per}([0, L])$$

is a smooth function (for all $n \geq 1$ integer).

(3) *$I(\sigma_0)$ can be chosen as $(\frac{2\pi^2}{L^2}, +\infty)$.*

(4) *The mapping $\Lambda : I(\sigma_0) \to B(\eta_{2,0})$ is a strictly decreasing function. Therefore, from (88), $\sigma \to k(\sigma)$ is a strictly increasing function.*

Proof. The key of the proof is to apply the implicit function theorem (Angulo [4]). Indeed, consider the open set $\Omega = \{(\eta,\sigma):\ \sigma > \frac{2\pi^2}{L^2},\ \eta \in (0,\sqrt{2c\sigma})\ \} \subseteq \mathbb{R}^2$ and define $\Psi : \Omega \to \mathbb{R}$ by

$$\Psi(\eta,\sigma) = \frac{4\sqrt{c}}{\sqrt{4c\sigma-\eta^2}}\ K(k(\eta,\sigma)) \tag{89}$$

where $k^2(\eta,\sigma) = \frac{4c\sigma-2\eta^2}{4c\sigma-\eta^2}$. By hypotheses $\Psi(\eta_{2,0},\sigma_0) = L$. Next, we show $\partial_\eta\Psi(\eta,\sigma) < 0$. In fact, it is immediate that

$$\partial_\eta\Psi(\eta,\sigma) = \frac{4\sqrt{c}\,\eta}{(4c\sigma-\eta^2)^{3/2}}\ K(k) + \frac{4\sqrt{c}}{\sqrt{4c\sigma-\eta^2}}\frac{dK}{dk}\frac{dk}{d\eta}.$$

Next, from

$$\frac{dk}{d\eta} = -\frac{4c\sigma\eta}{k(4c\sigma-\eta^2)^2},$$

and the relations (see [18])

$$\begin{cases} \frac{dE}{dk} = \frac{E-K}{k}, & \frac{d^2E}{dk^2} = -\frac{1}{k}\frac{dK}{dk}, \\ kk'^2\frac{d^2E}{dk^2} + k'^2\frac{dE}{dk} + kE = 0, & \end{cases} \tag{90}$$

with $k'^2 = 1-k^2$, and $E = E(k)$ being the complete elliptic integral of second kind defined as

$$E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\ dt,$$

we have the following formal equivalences

$$\begin{aligned} \partial_\eta\Psi(\eta,\sigma) < 0 &\Leftrightarrow k(4c\sigma-\eta^2)\Big(E - k\frac{dE}{dk}\Big) < -4c\sigma k\frac{d^2E}{dk^2} \\ &\Leftrightarrow k(4c\sigma-\eta^2)\Big(E - k\frac{dE}{dk}\Big) < \Big(\frac{dE}{dk} + \frac{k}{k'^2}E\Big)(4c\sigma-\eta^2)(2-k^2) \\ &\Leftrightarrow 2k'^2\frac{dE}{dk} + kE > 0 \Leftrightarrow \frac{dE}{dk} - k\frac{d^2E}{dk^2} > 0 \Leftrightarrow \frac{dE}{dk} + \frac{dK}{dk} > 0. \end{aligned}$$

So, since the condition $\frac{dE}{dk} + \frac{dK}{dk} > 0$ is equivalent to $2(1-k^2)K(k) < (2-k^2)E(k)$, which is easy to be obtained of the definitions of K and E.

Therefore, there is a unique smooth function, Λ, defined in a neighborhood $I(\sigma_0)$ of σ_0, such that $\Psi(\Lambda(\sigma),\sigma) = L$ for every $\sigma \in I(\sigma_0)$. So, we obtain (87). Moreover, since σ_0 was chosen arbitrarily in $\mathcal{I} = (\frac{2\pi^2}{L^2}, +\infty)$, it follows from the uniqueness of the function Λ that it can be extended to $\mathcal{I}$.

Next, we show that Λ is a strictly decreasing function. We know that $\Psi(\Lambda(\sigma),\sigma) = L$ for every $\sigma \in I(\sigma_0)$, then

$$\frac{d}{d\sigma}\Lambda(\sigma) = -\frac{\partial\Psi/\partial\sigma}{\partial\Psi/\partial\eta} < 0 \Leftrightarrow \partial\Psi/\partial\sigma < 0.$$

Thus, using the relation $\eta^2 = (4c\sigma-\eta^2)(1-k^2) \equiv (4c\sigma-\eta^2)k'^2$, we obtain the following formal equivalences

$$\frac{\partial\Psi}{\partial\sigma} < 0 \Leftrightarrow (4c\sigma-\eta^2)K > \frac{\eta^2}{k}\frac{dK}{dk} \Leftrightarrow K > \frac{k'^2}{k}\frac{dK}{dk}.$$

Then, since $\frac{dK}{dk} = (E - k'^2K)/(kk'^2)$, it follows that

$$\frac{\partial\Psi}{\partial\sigma} < 0 \Leftrightarrow k^2K > E - k'^2K \Leftrightarrow (k^2+k'^2)K > E \Leftrightarrow K > E.$$

This completes the proof of Theorem 4.1. □

The following result will be used in our stability theory.

Corollary 4.2. *Let L and c be arbitrarily fixed positive numbers. Consider the smooth curve of dnoidal waves $\sigma \in (\frac{2\pi^2}{L^2},\infty) \to \phi_0(\cdot;\eta_1(\sigma),\eta_2(\sigma))$ determined by Theorem 4.1. Then*

$$\frac{d}{d\sigma}\int_0^L \phi_0^2(\xi)\, d\xi > 0.$$

Proof. By (81), (87), and the formula $\int_0^{K(k)} dn^2(x;k)\, dx = E(k)$ (see page 194 in [18]) it follows that

$$\int_0^L \phi_0^2(\xi)d\xi = 2\eta_1\sqrt{c}\int_0^{2K(k)} dn^2(x;k)dx = \frac{16cK}{L}\int_0^{K(k)} dn^2(x;k)dx = \frac{16c}{L}E(k)K(k).$$

So, since $k \to K(k)E(k)$ and $\sigma \to k(\sigma)$ are strictly increasing functions we have that

$$\frac{d}{d\sigma}\int_0^L \phi_0^2(\xi)\, d\xi = \frac{16c}{L}\frac{d}{dk}[K(k)E(k)]\frac{dk}{d\sigma} > 0.$$

□

4.2. **Periodic traveling wave solutions for the equation (5).** In this subsection we show the existence of a branch of periodic traveling waves solutions of (5) for γ close to zero such that these solutions bifurcate the dnoidal waves solutions found in Theorem 78.

We start our analysis by studying the periodic eigenvalue problem considered on $[0,L]$,

$$\begin{cases} \mathcal{L}_0\chi \equiv (-\dfrac{d^2}{dx^2}+\sigma-\dfrac{3}{2c}\phi_0^2)\chi = \lambda\chi \\ \chi(0)=\chi(L), \quad \chi'(0)=\chi'(L), \end{cases} \tag{91}$$

where for $\sigma > 2\pi^2/L^2$, ϕ_0 is given by Theorem 78 and satisfies (79).

Theorem 4.3. *The linear operator $\mathcal{L}_0$ defined in (91) with domain $H^2_{per}([0,L]) \subseteq L^2_{per}([0,L])$, has its first three eigenvalues simple with zero being its second eigenvalue (with eigenfunction $\frac{d}{dx}\phi_0$). Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues which are double and converging to infinity.*

Proof. Theorem 4.3 is a consequence of the Floquet theory (see Angulo [4] and Magnus and Winkler [32]). Indeed, from the classical theory of compact symmetric linear operator, we have that (91) determines a countable infinity set of eigenvalues $\{\lambda_n | n = 0,1,2,...\}$ with $\lambda_0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4 \le ...$, where double eigenvalue is counted twice and $\lambda_n \to \infty$ as $n \to \infty$. We shall denote by χ_n the eigenfunction associated to the eigenvalue λ_n. From Sturm's Oscillation Theory we have that $\lambda_0 < \lambda_1 \le \lambda_2$. Since $\mathcal{L}_0\frac{d}{dx}\phi_0 = 0$ and $\frac{d}{dx}\phi_0$ has 2 zeros in $[0,L)$, it follows that 0 is either λ_1 or λ_2 (see Theorem 3.1 in [4]). We will show that $0=\lambda_1<\lambda_2$. So, we consider $\Psi(x) \equiv \chi(\gamma x)$ with $\gamma^2 = 4c/\eta_1^2$. Then from (91) and from the identity $k^2sn^2x + dn^2x = 1$, we obtain

$$\begin{cases} \dfrac{d^2}{dx^2}\Psi + [\rho - 6k^2sn^2(x;k)]\Psi = 0 \\ \Psi(0)=\Psi(2K(k)), \quad \Psi'(0)=\Psi'(2K(k)), \end{cases} \tag{92}$$

where

$$\rho = \frac{4c(\lambda-\sigma)}{\eta_1^2} + 6. \tag{93}$$

Now, from [32], it follows that $(-\infty,\rho_0),(\mu_0',\mu_1')$ and (ρ_1,ρ_2) are the instability intervals associated to this Lamé's equation, where for $i\geq 0$, μ_i' are the eigenvalues associated to the semi-periodic problem. Therefore, ρ_0,ρ_1,ρ_2 are simple eigenvalues for (92) and the other eigenvalues $\rho_3\leq\rho_4<\rho_5\leq\rho_6<\cdots$ satisfy $\rho_3=\rho_4,\rho_5=\rho_6,\cdots$, i.e., they are double eigenvalues.

It is easy to verify that the first three eigenvalues ρ_0,ρ_1,ρ_2 and its corresponding eigenfunctions Ψ_0,Ψ_1,Ψ_2 are given by the formulas (see Ince [28])

$$\begin{cases} \rho_0 = 2[1+k^2-\sqrt{1-k^2+k^4}], \;\; \Psi_0(x) = 1-(1+k^2-\sqrt{1-k^2+k^4}\,)sn^2(x), \\ \rho_1 = 4+k^2, \;\; \Psi_1(x) = snx\; cnx, \\ \rho_2 = 2[1+k^2+\sqrt{1-k^2+k^4}], \;\; \Psi_2(x) = 1-(1+k^2+\sqrt{1-k^2+k^4}\,)sn^2(x). \end{cases} \tag{94}$$

Next, Ψ_0 has no zeros in $[0,2K]$ and Ψ_2 has exactly 2 zeros in $[0,2K)$, then ρ_0 is the first eigenvalue to (92). Since $\rho_0<\rho_1$ for every $k^2\in(0,1)$, we obtain from (93) and (82) that

$$4c\lambda_0 = \eta_1^2(k^2-2-2\sqrt{1-k^2+k^4})<0 \Leftrightarrow \rho_0<\rho_1.$$

Therefore λ_0 is the first negative eigenvalue to $\mathcal{L}_0$ with eigenfunction $\chi_0(x) = \Psi_0(x/\gamma)$. Similarly, since $\rho_1<\rho_2$ for every $k^2\in(0,1)$, we obtain from (93) that

$$4c\lambda_2 = \eta_1^2(k^2-2+2\sqrt{1-k^2+k^4})>0 \Leftrightarrow \rho_1<\rho_2.$$

Hence λ_2 is the third eigenvalue to $\mathcal{L}_0$ with eigenfunction $\chi_2(x)=\Psi_2(x/\gamma)$. Finally, since $\chi_1(x)=\Psi_1(x/\gamma)=\mu\frac{d}{dx}\phi_0(x)$ we finish the proof. □

Next, we have our theorem of existence of solutions for (5). For $s\geq 0$, let $H^s_{per,e}([0,L])$ denote the closed subspace of all even functions in $H^s_{per}([0,L])$.

Theorem 4.4. *Let $L,\alpha,\beta,c>0$ and $\sigma>2\pi^2/L^2$ be fixed numbers. Then there exist $\gamma_1>0$ and a smooth branch*

$$\gamma\in(-\gamma_1,\gamma_1)\to(\phi_\gamma,\psi_\gamma)\in H^2_{per,e}([0,L])\times H^1_{per,e}([0,L])$$

of solutions for Eq. (5). In particular, for $\gamma\to 0$, $(\phi_\gamma,\psi_\gamma)$ converges to (ϕ_0,ψ_0) uniformly for $x\in[0,L]$, where (ϕ_0,ψ_0) is given by Theorem 4.1 and it is defined by (81). Moreover, the mapping

$$\gamma\in(-\gamma_1,\gamma_1)\to(\frac{d}{d\sigma}\phi_\gamma,\frac{d}{d\sigma}\psi_\gamma)$$

is continuous.

Proof. Without loss of generality, we take $\alpha=1$ and $\beta=1/2$. Let $X_e=H^2_{per,e}([0,L])\times H^3_{per,e}([0,L])$ and define the map

$$G:\mathbb{R}\times(0,+\infty)\times X_e\to L^2_{per,e}([0,L])\times H^2_{per,e}([0,L])$$

by

$$G(\gamma,\lambda,\phi,\psi)=(-\phi''+\lambda\phi+\phi\psi,-\gamma D\psi+c\psi+\frac{1}{2}\phi^2).$$

A calculation shows that the Fréchet derivative $G_{(\phi,\psi)} = \partial G(\gamma,\lambda,\phi,\psi)/\partial(\phi,\psi)$ exists and it is defined as a map from $\mathbb{R}\times(0,+\infty)\times X_e$ to $B(X_e; L^2_{per,e}([0,L])\times H^2_{per,e}([0,L]))$ by

$$G_{(\phi,\psi)}(\gamma,\lambda,\phi,\psi) = \begin{pmatrix} -\dfrac{d^2}{dx^2}+\lambda+\psi & \phi \\ \phi & -\gamma D + c \end{pmatrix}.$$

¿From Theorem 4.1 it follows that for $\Phi_0 = (\phi_0,\psi_0)$, $G(0,\sigma,\Phi_0) = \vec{0}^t$. Moreover, from Theorem 4.3 we have that $G_{(\phi,\psi)}(0,\sigma,\Phi_0)$ has a kernel generated by ${\Phi_0'}^t$. Next, since $\Phi_0' \notin X_e$, the boundedness of $G_{(\phi,\psi)}(0,\sigma,\Phi_0)^{-1}$ follows trivially from the surjectivity of $G_{(\phi,\psi)}(0,\sigma,\Phi_0)$ and the Open Mapping Theorem. Hence, since G and $G_{(\phi,\psi)}$ are smooth maps on their domains, the Implicit Function Theorem implies that there are $\gamma_1>0$, $\lambda_1\in(0,\sigma)$, and a smooth curve

$$(\gamma,\lambda)\in(-\gamma_1,\gamma_1)\times(\sigma-\lambda_1,\sigma+\lambda_1)\to(\phi_{\gamma,\lambda},\psi_{\gamma,\lambda})\in X_e$$

such that $G(\gamma,\lambda,\phi_{\gamma,\lambda},\psi_{\gamma,\lambda}) = 0$. Then, for $\lambda=\sigma$ we obtain a smooth branch $\gamma\in(-\gamma_1,\gamma_1)\to(\phi_{\gamma,\sigma},\psi_{\gamma,\sigma})\equiv(\phi_\gamma,\psi_\gamma)$ of solutions of (5) such that $\gamma\in(-\gamma_1,\gamma_1)\to(\frac{d}{d\sigma}\phi_\gamma,\frac{d}{d\sigma}\psi_\gamma)$ is continuous. □

5. **Stability of the periodic traveling wave solutions.** We begin this section defining the type of stability of our interest. For any $c\in\mathbb{R}^+$ define the functions $\Phi(x) = e^{icx/2}\phi(x)$ and $\Psi(x)=\psi(x)$, where (ϕ,ψ) is a solution of (5). Then we say that the orbit generated by (Φ,Ψ), namely,

$$\Omega_{(\Phi,\Psi)} = \{(e^{i\theta}\Phi(\cdot+x_0),\Psi(\cdot+x_0))\ :(\theta,x_0)\in[0,2\pi)\times\mathbb{R}\},$$

is stable in $H^1_{per}([0,L])\times H^{\frac12}_{per}([0,L])$ by the flow generated by Eq. (1), if for every $\epsilon>0$ there exists $\delta(\epsilon)>0$ such that for (u_0,v_0) satisfying $\|u_0-\Phi\|_1<\delta$ and $\|v_0-\Psi\|_{\frac12}<\delta$, we have that the global solution (u,v) of (1) with $(u(0),v(0)) = (u_0,v_0)$ satisfies that $(u,v)\in C(\mathbb{R};H^1_{per}([0,L]))\times C(\mathbb{R};H^{\frac12}_{per}([0,L]))$ and

$$\inf_{x_0\in\mathbb{R},\theta\in[0,2\pi)}\{\|e^{i\theta}u(\cdot+x_0,t)-\Phi\|_1+\|v(\cdot+x_0,t)-\Psi\|_{\frac12}\}<\epsilon, \tag{95}$$

for all $t\in\mathbb{R}$.

The main result to be proved in this section is that the periodic traveling waves solutions of (1) determined by Theorem 4.4 are stable for $\sigma>2\pi^2/L^2$ and γ negative close to 0.

Theorem 5.1. *Let $L,\alpha,\beta,c>0$ and $\sigma>2\pi^2/L^2$ be fixed numbers. We consider the smooth curve of periodic traveling waves solutions for (5), $\gamma\to(\phi_\gamma,\psi_\gamma)$, determined by Theorem 4.4. Then there exists $\gamma_0>0$ such that for each $\gamma\in(-\gamma_0,0)$, the orbit generated by $(\Phi_\gamma(x),\Psi_\gamma(x))$ with*

$$\Phi_\gamma(x) = e^{icx/2}\phi_\gamma(x)\ \ \textit{and}\ \ \Psi_\gamma(x)=\psi_\gamma(x),$$

is orbitally stable in $H^1_{per}([0,L])\times H^{\frac12}_{per}([0,L])$.

The proof of Theorem 5.1 is based on the ideas developed by Angulo in [8] (see also Benjamin [12] and Weinstein [38]) which give us an easy form of manipulating with the required spectral information and the positivity property of the quantity $\frac{d}{d\sigma}\int\phi_\gamma^2(x)dx$, which are basic in our stability theory. We do not use the abstract stability theory of Grillakis *et al.* basically by these circumstances. So, we consider

$(\phi_\gamma, \psi_\gamma)$ a solution of (5) obtained in Theorem 4.4. For $(u_0, v_0) \in H^1_{per}([0, L]) \times H^{\frac{1}{2}}_{per}([0, L])$ and the global solution (u, v) to (1) corresponding to these initial data given by Theorem 3.10, we define for $t \geq 0$ and $\sigma > 2\pi^2/L^2$

$$\Omega_t(x_0, \theta) = \|e^{i\theta}(T_c u)'(\cdot + x_0, t) - \phi'_\gamma\|^2 + \sigma\|e^{i\theta}(T_c u)(\cdot + x_0, t) - \phi_\gamma\|^2 \tag{96}$$

where we denote by T_c the bounded linear operator defined by

$$(T_c u)(x, t) = e^{-ic(x-ct)/2}u(x, t).$$

Then, the deviation of the solution $u(t)$ from the orbit generated by Φ is measured by

$$\rho_\sigma(u(\cdot, t), \phi_\gamma)^2 \equiv \inf_{x_0 \in [0,L], \theta \in [0,2\pi]} \Omega_t(x_0, \theta). \tag{97}$$

Hence, since Ω_t is continuous on $[0, L] \times [0, 2\pi]$ we have that the infimum in (97) is attained in $(x_0, \theta) = (x_0(t), \theta(t))$.

Proof of Theorem 5.1. Consider the perturbation of the periodic traveling wave $(\phi_\gamma, \psi_\gamma)$

$$\begin{cases} \xi(x, t) = e^{i\theta}(T_c u)(x + x_0, t) - \phi_\gamma(x) \\ \eta(x, t) = v(x + x_0, t) - \psi_\gamma(x). \end{cases} \tag{98}$$

Hence, by the property of minimum of $(x_0, \theta) = (x_0(t), \theta(t))$, we obtain from (98) that $p(x, t) = \mathrm{Re}(\xi(x, t))$ and $q(x, t) = \mathrm{Im}(\xi(x, t))$ satisfy the compatibility relations

$$\begin{cases} \displaystyle\int_0^L q(x, t)\phi_\gamma(x)\psi_\gamma(x)\, dx = 0 \\ \displaystyle\int_0^L p(x, t)(\phi_\gamma(x)\psi_\gamma(x))'\, dx = 0. \end{cases} \tag{99}$$

Now we take the continuous functional L defined on $H^1_{per}([0, L]) \times H^{\frac{1}{2}}_{per}([0, L])$ by

$$L(u, v) = E(u, v) + c\, G(u, v) + \omega\, H(u, v),$$

where E, G, H are defined by (2). Then, from (98) and (5), we have

$$\begin{aligned} \Delta L(t) &:= L(u(t), v(t)) - L(\Phi_\gamma, \Psi_\gamma) = L(\Phi_\gamma + e^{icx/2}\xi, \psi_\gamma + \eta) - L(\Phi_\gamma, \psi_\gamma) \\ &= < \mathcal{L}_\gamma p, p > + < \mathcal{L}^+_\gamma q, q > \\ &\quad + \frac{\alpha}{2\beta}\int_0^L \left[\mathcal{K}_\gamma^{1/2}\eta + 2\beta\mathcal{K}_\gamma^{-1/2}(\phi_\gamma p) + \beta\mathcal{K}_\gamma^{-1/2}(p^2 + q^2)\right]^2 dx \\ &\quad - \frac{\alpha\beta}{2}\int_0^L \left[|\mathcal{K}_\gamma^{-1/2}(p^2 + q^2)|^2 + 4\mathcal{K}_\gamma^{-1/2}(\phi_\gamma p)\mathcal{K}_\gamma^{-1/2}(p^2 + q^2)\right] dx, \end{aligned} \tag{100}$$

where, for $\gamma < 0$ we define $\mathcal{K}_\gamma^{-1}$ as

$$\widehat{\mathcal{K}_\gamma^{-1} f}(k) = \frac{1}{-\gamma|k| + c}\widehat{f}(k) \qquad \text{for } k \in \mathbb{Z},$$

which is the inverse operator of $\mathcal{K}_\gamma : H^s_{per}([0, L]) \to H^{s-1}_{per}([0, L])$ defined by $\mathcal{K}_\gamma = -\gamma D + c$. The operator $\mathcal{L}_\gamma$ is

$$\mathcal{L}_\gamma = -\frac{d^2}{dx^2} + \sigma + \alpha\psi_\gamma - 2\alpha\beta\phi_\gamma \circ \mathcal{K}_\gamma^{-1} \circ \phi_\gamma, \tag{101}$$

with $\phi_\gamma \circ \mathcal{K}_\gamma^{-1} \circ \phi_\gamma$ given by $[\phi_\gamma \circ \mathcal{K}_\gamma^{-1} \circ \phi_\gamma](f) = \phi_\gamma \mathcal{K}_\gamma^{-1}(\phi_\gamma f)$. Here $\mathcal{L}_\gamma^+$ is defined by

$$\mathcal{L}_\gamma^+ = -\frac{d^2}{dx^2} + \sigma + \alpha\psi_\gamma \tag{102}$$

and $\mathcal{K}_\gamma^{1/2}$, $\mathcal{K}_\gamma^{-1/2}$ are the positive roots of $\mathcal{K}_\gamma$ and $\mathcal{K}_\gamma^{-1}$ respectively.

Now, we need to find a lower bound for $\Delta L(t)$. The first step will be to obtain a suitable lower bound of the last term on the right-hand side of (100). In fact, since $\mathcal{K}_\gamma^{-1/2}$ is a bounded operator on $L^2_{per}([0,L])$, ϕ_γ is uniformly bounded, and from the continuous embedding of $H^1_{per}([0,L])$ in $L^4_{per}([0,L])$ in $L^\infty([0,L])$, we have that

$$-\frac{\alpha\beta}{2}\int_0^L \Big[|\mathcal{K}_\gamma^{-1/2}(p^2+q^2)|^2 + 4\mathcal{K}_\gamma^{-1/2}(\phi_\gamma p)\mathcal{K}_\gamma^{-1/2}(p^2+q^2)\Big]dx \geq -C_1\|\xi\|_1^3 - C_2\|\xi\|_1^4 \tag{103}$$

where C_1 and C_2 are positive constants.

The estimates for $\langle \mathcal{L}_\gamma p, p\rangle$ and $\langle \mathcal{L}_\gamma^+ q, q\rangle$ will be obtained from the following theorem.

Lemma 5.2. *Let $L, \alpha, \beta, c > 0$ and $\sigma > 2\pi^2/L^2$ be fixed numbers. Then, there exists $\gamma_2 > 0$ such that, if $\gamma \in (-\gamma_2, 0)$, the self-adjoint operators $\mathcal{L}_\gamma$ and $\mathcal{L}_\gamma^+$ defined in (101) and (102), respectively, with domain $H^2_{per}([0,L])$ have the following properties:*

(1) *$\mathcal{L}_\gamma$ has a simple negative eigenvalue λ_γ with eigenfunction φ_γ and $\int_0^L \phi_\gamma\varphi_\gamma dx \neq 0$.*

(2) *$\mathcal{L}_\gamma$ has a simple eigenvalue at zero with eigenfunction $\frac{d}{dx}\phi_\gamma$.*

(3) *There is $\eta_\gamma > 0$ such that for $\beta_\gamma \in \Sigma(\mathcal{L}_\gamma) - \{\lambda_\gamma, 0\}$, we have that $\beta_\gamma > \eta_\gamma$.*

(4) *$\mathcal{L}_\gamma^+$ is a non-negative operator which has zero as its first eigenvalue with eigenfunction ϕ_γ. The remainder of the spectrum is constituted by a discrete set of eigenvalues.*

Proof. The idea of the proof is to use the min-max principle. From (5) it follows that $\mathcal{L}_\gamma\phi_\gamma = 2\phi_\gamma\psi_\gamma$ and so since $\psi_\gamma < 0$ we have that for $\gamma < 0$, $\langle \mathcal{L}_\gamma\phi_\gamma, \phi_\gamma\rangle = 2\int_{\mathbb{R}} \phi_\gamma^2\psi_\gamma dx < 0$. Therefore $\mathcal{L}_\gamma$ has a negative eigenvalue. Moreover, $\mathcal{L}_\gamma \frac{d}{dx}\phi_\gamma = 0$. Next, for $f \in H^1_{per}([0,L])$ and $\|f\| = 1$, we have

$$\begin{aligned}
&\langle \mathcal{L}_\gamma f, f\rangle = \langle \mathcal{L}_0 f, f\rangle - \frac{\gamma}{c}\alpha^2\langle \phi_0 f, D\mathcal{K}_\gamma^{-1}(\phi_0 f)\rangle + \alpha\int_0^L (\psi_\gamma - \psi_0)f^2 dx \\
&+\alpha^2\int_0^L [\phi_0 f\mathcal{K}_\gamma^{-1}(\phi_0 f) - \phi_\gamma f\mathcal{K}_\gamma^{-1}(\phi_\gamma f)]dx \\
&\geq \langle \mathcal{L}_0 f, f\rangle + \alpha\int_0^L (\psi_\gamma - \psi_0)f^2 dx + \alpha^2\int_0^L [\phi_0 f\mathcal{K}_\gamma^{-1}(\phi_0 f) - \phi_\gamma f\mathcal{K}_\gamma^{-1}(\phi_\gamma f)]dx,
\end{aligned} \tag{104}$$

where the last inequality is due to that $\gamma < 0$ and $D\mathcal{K}_\gamma^{-1}$ is a positive operator. So, since

$$\begin{aligned}
&\Big|\int_0^L (\psi_\gamma - \psi_0)f^2 dx\Big| \leq \|\psi_\gamma - \psi_0\|_\infty \\
&\Big|\int_0^L [\phi_0 f\mathcal{K}_\gamma^{-1}(\phi_0 f) - \phi_\gamma f\mathcal{K}_\gamma^{-1}(\phi_\gamma f)]dx\Big| \leq (\|\phi_\gamma\| + \|\phi_0\|)\|\phi_\gamma - \phi_0\|_\infty,
\end{aligned} \tag{105}$$

we have from Theorem 4.4 that for γ near 0^- and ϵ small, $\langle \mathcal{L}_\gamma f, f\rangle \geq \langle \mathcal{L}_0 f, f\rangle - \epsilon$. Hence, for $f \perp \chi_0$ and $f \perp \frac{d}{dx}\phi_0$, where $\mathcal{L}_0\chi_0 = \lambda_0\chi_0$ with $\lambda_0 < 0$, we have from the spectral structure of $\mathcal{L}_0$ (Theorem 4.3) that $\langle \mathcal{L}_\gamma f, f\rangle \geq \eta_\gamma > 0$. Therefore, from min-max principle ([36]) we obtain the desired spectral structure for $\mathcal{L}_\gamma$. Moreover, let φ_γ be such that $\mathcal{L}_\gamma\varphi_\gamma = \lambda_\gamma\varphi_\gamma$ with $\lambda_\gamma < 0$. Therefore, if $\phi_\gamma \perp \varphi_\gamma$ then from the spectral structure of $\mathcal{L}_\gamma$ we must have that $\langle \mathcal{L}_\gamma\phi_\gamma, \phi_\gamma\rangle \geq 0$. But we know that $\langle \mathcal{L}_\gamma\phi_\gamma, \phi_\gamma\rangle < 0$. Hence, $\langle \phi_\gamma, \varphi_\gamma\rangle \neq 0$. Finally, since $\mathcal{L}^+_\gamma\phi_\gamma = 0$ with $\phi_\gamma > 0$, it follows that zero is simple and it is the first eigenvalue. It follows from the theory of Hill's equation (see [32]) that the remainder of the spectrum is discrete. □

The next theorem follows the same ideas of Lemma 2.7 in [8].

Lemma 5.3. *Consider $\gamma < 0$ close to zero such that Theorem 5.2 is true. Then*

(a) $\inf\{\langle \mathcal{L}_\gamma f, f\rangle \;:\; \|f\| = 1, \langle f, \phi_\gamma\rangle = 0,\; \} \equiv \beta_0 = 0.$

(b) $\inf\{\langle \mathcal{L}_\gamma f, f\rangle \;:\; \|f\| = 1, \langle f, \phi_\gamma\rangle = 0, \langle f, (\phi_\gamma\psi_\gamma)'\rangle = 0\;\} \equiv \mu > 0.$

Proof. Part (a). Since $\mathcal{L}_\gamma \frac{d}{dx}\phi_\gamma = 0$ and $\langle \frac{d}{dx}\phi_\gamma, \phi_\gamma\rangle = 0$ then $\beta_0 \leq 0$. Next we show that $\beta_0 \geq 0$ by using Lemma E.1 in Weinstein [39]. So, we shall show initially that the infimum is attained. Let $\{\psi_j\} \subseteq H^1_{per}([0,L])$ with $\|\psi_j\| = 1$, $\langle \psi_j, \phi_\gamma\rangle = 0$ and $\lim_{j\to\infty}\langle \mathcal{L}_\gamma\psi_j, \psi_j\rangle = \beta_0$. Then there is a subsequence of $\{\psi_j\}$, which we denote again by $\{\psi_j\}$, such that $\psi_j \rightharpoonup \psi$ weakly in $H^1_{per}([0,L])$, so $\psi_j \to \psi$ in $L^2_{per}([0,L])$. Hence $\|\psi\| = 1$ and $\langle \psi, \phi_\gamma\rangle = 0$. Since $\|\psi'\|^2 \leq \liminf \|\psi'\|^2$ and $\mathcal{K}^{-1}_\gamma(\phi_\gamma\psi_j) \to \mathcal{K}^{-1}_\gamma(\phi_\gamma\psi)$ in $L^2_{per}([0,L])$, we have $\beta_0 \leq \langle \mathcal{L}_\gamma\psi, \psi\rangle \leq \liminf\langle \mathcal{L}_\gamma\psi_j, \psi_j\rangle = \beta_0$. Next we show that $\langle \mathcal{L}^{-1}_\gamma\phi_\gamma, \phi_\gamma\rangle \leq 0$. From (5) and Theorem 4.4 we obtain for $\chi_\gamma = -\frac{d}{d\sigma}\phi_\gamma$ that $\mathcal{L}_\gamma\chi_\gamma = \phi_\gamma$. Moreover, from Corollary 4.2 it follows that $\langle -\frac{d}{d\sigma}\phi_0, \phi_0\rangle < 0$ and so for γ small enough $\langle -\frac{d}{d\sigma}\phi_\gamma, \phi_\gamma\rangle < 0$. Hence from [39] we obtain that $\beta_0 \geq 0$. This shows part (a) of the theorem.

Part (b). From (a) we have that $\mu \geq 0$. Suppose $\mu = 0$. Then following a similar analysis to that used in part (a) above, we have that the infimum define in (b) is attained at an admissible function ζ. So, from Lagrange's multiplier theory, there are λ, θ, η such that

$$\mathcal{L}_\gamma\zeta = \lambda\zeta + \theta\phi_\gamma + \eta(\phi_\gamma\psi_\gamma)'. \tag{106}$$

Using (106) and $\langle \mathcal{L}_\gamma\zeta, \zeta\rangle = 0$ we obtain that $\lambda = 0$. Taking the inner product of (106) with ϕ'_γ, we have from $\mathcal{L}_\gamma\phi'_\gamma = 0$ that

$$0 = \eta\int_0^L \phi'_\gamma(\phi_\gamma\psi_\gamma)'dx, \tag{107}$$

but the integral in (107) converges to

$$\int_0^L \phi'_0(\phi_0\psi_0)'dx = \frac{-3\beta}{c}\int_0^L \phi_0^2(\phi'_0)^2dx < 0$$

as $\gamma \to 0$ (see (5) and (78)). Then, from (107), we obtain $\eta = 0$ and therefore $\mathcal{L}_\gamma\zeta = \theta\phi_\gamma$. So, since $\mathcal{L}_\gamma(-\frac{d}{d\sigma}\phi_\gamma) = \phi_\gamma$, we obtain $0 = \langle \zeta, \phi_\gamma\rangle = \theta\langle \phi_\gamma, -\frac{d}{d\sigma}\phi_\gamma\rangle$. Therefore $\theta = 0$ and $\mathcal{L}_\gamma\zeta = 0$. Then $\zeta = \nu\phi'_\gamma$ for some $\nu \neq 0$, which is a contradiction. Thus $\mu > 0$ and the proof of the lemma is complete. □

Theorem 5.4. *Consider $\gamma < 0$ close to zero such that Lemma 5.2 is true. If $\mathcal{L}^+_\gamma$ is defined as in (102) then*

$$\inf\{\langle \mathcal{L}^+_\gamma f, f\rangle \;:\; \|f\| = 1, \langle f, \phi_\gamma\psi_\gamma\rangle = 0\;\} \equiv \mu_0 > 0.$$

Proof. The proof follows from Lemma 5.2 and the ideas in Lemma 2.8 in [8]. □

Next we finish the proof of Theorem 5.1 by returning to (100). Our task is to estimate the terms $\langle \mathcal{L}_\gamma p, p\rangle$ and $\langle \mathcal{L}_\gamma^+ q, q\rangle$ where p and q satisfy (99). From Theorem 5.4 there is $C_1 > 0$ such that

$$\langle \mathcal{L}_\gamma^+ q, q\rangle \geq C_1 \|q\|_1^2. \tag{108}$$

Now we estimate $\langle \mathcal{L}_\gamma p, p\rangle$. Suppose without loss of generality that $\|\phi_\gamma\| = 1$. We write $p_\perp = p - p_{||}$, where $p_{||} = \langle p, \phi_\gamma\rangle \phi_\gamma$. Then, from (99), (5), and integration by parts it follows that $\langle p_\perp, (\phi_\gamma \psi_\gamma)'\rangle = 0$. Therefore from Lemma 5.3, it follows $\langle \mathcal{L}_\gamma p_\perp, p_\perp\rangle \geq \beta \|p_\perp\|^2$. Now we suppose that $\|u_0\| = \|\phi_\gamma\| = 1$. Since $\|u(t)\|^2 = 1$ for all t, we have that $\langle p, \phi_\gamma\rangle = -\|\xi\|^2/2$. So, $\langle \mathcal{L}_\gamma p_\perp, p_\perp\rangle \geq \beta_1 \|p\|^2 - \beta_2 \|\xi\|_{1,\sigma}^4$. Since $\langle \mathcal{L}_\gamma \phi_\gamma, \phi_\gamma\rangle < 0$ it follows that $\langle \mathcal{L}_\gamma p_{||}, p_{||}\rangle \geq -\beta_3 \|\xi\|_{1,\sigma}^4$. Moreover, Cauchy-Schwarz inequality implies $\langle \mathcal{L}_\gamma p_{||}, p_\perp\rangle \geq -\beta_4 \|\xi\|_1^3$. Therefore we conclude from the specific form of $\mathcal{L}_\gamma$ that

$$\langle \mathcal{L}_\gamma p, p\rangle \geq D_1 \|p\|_{1,\sigma}^2 - D_2 \|\xi\|_{1,\sigma}^3 - D_3 \|\xi\|_{1,\sigma}^4, \tag{109}$$

with $D_i > 0$ and $\|f\|_{1,\sigma}^2 = \|f'\|^2 + \sigma \|f\|^2$.

Next, by collecting the results in (103), (108) and (109) and substituting them in (100), we obtain

$$\Delta L(t) \geq d_1 \|\xi\|_{1,\sigma}^2 - d_2 \|\xi\|_{1,\sigma}^3 - d_3 \|\xi\|_{1,\sigma}^4, \tag{110}$$

where $d_i > 0$. Therefore, from standard arguments ([15]), for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that, if $\|u_0 - \Phi_\gamma\|_{1,\sigma} < \delta(\epsilon)$ and $\|v_0 - \Psi_\gamma\|_{\frac{1}{2}} < \delta(\epsilon)$, then for $t \in [0, \infty)$

$$\rho_\sigma(u(t), \phi_\gamma)^2 = \|\xi(t)\|_{1,\sigma}^2 < \epsilon. \tag{111}$$

Now, it follows from (100) and from the above analysis of ξ that

$$\epsilon \geq \frac{\alpha}{2\beta} \int_0^L \left[\mathcal{K}_\gamma^{1/2} \eta + 2\beta \mathcal{K}_\gamma^{-1/2} (\phi_\gamma p) + \beta \mathcal{K}_\gamma^{-1/2} (p^2 + q^2) \right]^2 dx.$$

Thus, from (111) and the equivalence of the norms $\|\mathcal{K}_\gamma^{1/2} \eta\|$ and $\|\eta\|_{\frac{1}{2}}$, we obtain (95). This proves that $(\Phi_\gamma, \Psi_\gamma)$ is stable relative to small perturbation which preserves the $L^2_{per}([0, L])$ norm of Φ_γ. The general case follows from that $\gamma \in (-\gamma_1, \gamma_1) \to \phi_\gamma$ is a continuous curve and $\frac{d}{d\sigma} \|\phi_\gamma\|^2 > 0$. □

Acknowledgements. The authors are grateful to the anonymous referee for his valuable comments and suggestions. Also, the authors would like to thank James Colliander, Justin Holmer and Nikolaos Tzirakis for the interesting discussions concerning the Schrödinger-Benjamin-Ono system.

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