NON-UNIFORMLY HYPERBOLIC HORSESHOES IN THE STANDARD FAMILY

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ABSTRACT. We show that the non-uniformly hyperbolic horseshoes of Palis and Yoccoz occur in the standard family of area-preserving diffeomorphisms of the two-torus.

1. INTRODUCTION

In their *tour-de-force* work about the dynamics of surface diffeomorphisms, Palis and Yoccoz [2] proved that the so-called *non-uniformly hyperbolic horseshoes* are very frequent in the generic unfolding of a first heteroclinic tangency associated to periodic orbits in a horseshoe with Hausdorff dimension slightly bigger than one.

In the same article, Palis and Yoccoz gave an *ad-hoc* example of 1-parameter family of diffeomorphisms of the two-sphere fitting the setting of their main results, and, thus, exhibiting non-uniformly hyperbolic horseshoes: see page 3 (and, in particular, Figure 1) of [2].

In this note, we show that the *standard family* $f_k : \mathbb{T}^2 \to \mathbb{T}^2$, $k \in \mathbb{R}$,

$$f_k(x,y) := (-y + 2x + k\sin(2\pi x), x)$$

of area-preserving diffeomorphisms of the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ displays non-uniformly hyperbolic horseshoes.

More precisely, our main theorem is:

Theorem 1.1. *There exists $k_0 > 0$ such that, for all $|k| > k_0$, the subset of parameters $r \in \mathbb{R}$ such that $|r - k| < 4/k^{1/3}$ and f_r exhibits a non-uniformly hyperbolic horseshoe (in the sense of Palis–Yoccoz* [2]*) has positive Lebesgue measure.*

The remainder of this text is divided into three sections: in Section 2, we briefly recall the context of Palis–Yoccoz work [2]; in Section 3, we revisit some elements of Duarte's construction [1] of tangencies associated to certain (uniformly hyperbolic) horseshoes of f_k; finally, we establish Theorem 1.1 in Section 4 by modifying Duarte's constructions (from Section 3) in order to apply Palis-Yoccoz results (from Section 2).

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2. NON-UNIFORMLY HYPERBOLIC HORSESHOES

Suppose that F is a smooth diffeomorphism of a compact surface M displaying a first heteroclinic tangency associated to periodic points of a horseshoe K, that is:

- $p_s, p_u \in K$ belong to distinct periodic orbits of F;
- $W^s(p_s)$ and $W^u(p_u)$ have a quadratic tangency at a point $q \in M\backslash K$;
- for some neighborhoods U of K and V of the orbit $\mathcal{O}(q)$ such that the maximal invariant set of $U \cup V$ is precisely $K \cup \mathcal{O}(q)$.

Assume that K is *slightly thick* in the sense that its stable and unstable dimensions d^s and d^u satisfy $d_s + d_u > 1$ and

$$(d_s + d_u)^2 + \max(d_s + d_u)^2 < d_s + d_u + \max(d_s, d_u)$$

Remark 2.1. Since the stable and unstable dimensions of a horseshoe of an *area-preserving* diffeomorphism F always coincide, a slightly thick horseshoe K of an area-preserving diffeormorphism F has stable and unstable dimensions

$$0.5 < d_s = d_u < 0.6$$

In this setting, the results proved by Palis and Yoccoz [2] imply the following statement:

Theorem 2.2 (Palis–Yoccoz)**.** *Given a* 1*-parameter family* $(F_t)_{|t|<t_0}$ *with* $F_0 = F$ *and generically unfolding the heteroclinic tangency at q, the subset of parameters* $t \in (-t_0, t_0)$ *such that* F_t *has a non-uniformly hyperbolic horseshoe*[1] *has positive Lebesgue measure.*

3. HORSESHOES AND TANGENCIES IN THE STANDARD FAMILY

The standard family f_k generically unfolds tangencies associated to very thick horseshoes Λ_k: this phenomenon was studied in details by Duarte [1] during his proof of the almost denseness of elliptic islands of f_k for large generic parameters k.

In the sequel, we review some facts from Duarte's article about Λ_k and its tangencies (for later use in the proof of our Theorem 1.1).

For technical reasons, it is convenient to work with the standard family f_k and their *singular* perturbations

$$g_k(x, y) = (-y + 2x + k\sin(2\pi x) + \rho_k(x), x),$$

where ρ_k is defined in Section 4 of [1]. Here, it is worth to recall that the key features of ρ_k are:

[1] We are not going to recall the definition of non-uniformly hyperbolic horseshoes here: instead, we refer to the original article [2] for the details.

- ρ_k has *poles* at the critical points $\nu_\pm = \pm 1/4 + O(1/k)$ of the function $2x + k\sin(2\pi x)$;
- ρ_k vanishes outside $|x \pm \frac{1}{4}| \leq \frac{2}{k^{1/3}}$.

In Section 2 of [1], Duarte constructs the stable and unstable foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ for g_k. As it turns out, $\mathcal{F}^s$, resp. $\mathcal{F}^u$, is an almost vertical, resp. horizontal, foliation in the sense that it is generated by a vector field $(\alpha^s(x,y),1)$, resp. $(1,\alpha^u(x,y))$, satisfying all properties described in Section 2 of Duarte's paper [1]. In particular, $\mathcal{F}^s$, resp. $\mathcal{F}^u$, describe the local stable, resp. unstable, manifolds for the standard map f_k at points whose future, resp. past, orbits stay in the region $\{f_k = g_k\}$, resp. $\{f_k^{-1} = g_k^{-1}\}$.

In Section 3 of [1], Duarte analyses the projections π^s and π^u obtained by thinking the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ as fibrations over the singular circles $C_s = \{(x,\nu_+) \in \mathbb{T}^2\}$ and $C_u = \{(\nu_+, y) \in \mathbb{T}^2\}$. Among many things, Duarte shows that the circle map $\Psi_k : \mathbb{S}^1 \to \mathbb{S}^1$ defined by

$$(\Psi_k(x), \nu_+) := \pi^s(g_k(x,\nu_+)) \text{ or, equivalently, } (\nu_+, \Psi_k(y)) = \pi^u(g_k^{-1}(\nu_+, y))$$

is *singular* expansive with small distortion.

In Section 4 of [1], Duarte considers a Cantor set

$$K_k = \bigcap_{n\in\mathbb{N}} \Psi_k^{-1}(J_0 \cup J_1)$$

of the circle map Ψ_k associated to a Markov partition $J_0 \cup J_1 \subset [-1/4, 3/4]$ with the following properties:

- the extremities of the intervals $J_0 = [a,b]$ and $J_1 = [b', a'+1]$ satisfy $a + \frac{1}{4}, \frac{1}{4} - b, -\frac{1}{4} - a', b' - \frac{1}{4}, \in (\frac{3}{k^{1/3}}, \frac{4}{k^{1/3}})$, so that J_0 and J_1 are contained in the region $\{\rho_k = 0\}$;
- $\Psi_k(a) = a = \Psi_k(a')$, $\Psi_k(b) = a' = \Psi_k(b')$.

In particular, Duarte uses these features of K_k to prove that

$$\Lambda_k = (\pi^s)^{-1}(K_k) \cap (\pi^u)^{-1}(K_k)$$

is a horseshoe of *both* g_k and f_k.

In Section 5 of [1], Duarte studies the tangencies associated to the invariant foliations of Λ_k. More concretely, denote by $\mathcal{G}^u = (f_k)_*(\mathcal{F}^u)$ the foliation obtained by pushing the almost horizontal foliation $\mathcal{F}^u$ by the standard map f_k. The vector fields $(\beta^u(x,y),1)$ defining $\mathcal{G}^u$ and $(\alpha^s(x,y),1)$ defining $\mathcal{F}^s$ coincide along two (almost horizontal) circles of tangencies $\{(x,\sigma_+(x) : x \in \mathbb{S}^1\} \cup \{(x,\sigma_-(x) : x \in \mathbb{S}^1\}$ (with $|\sigma_\pm(x) - \nu_\pm| \leq \frac{1}{270k^{5/3}}$ and $|\sigma_\pm'(x)| \leq \frac{1}{12k^{4/3}}$ for all $x \in \mathbb{S}^1$). The projections of Λ_k along $\mathcal{F}^s$ and $\mathcal{G}^u$ on the circle of tangencies $\{(x,\sigma_+(x)) : x \in \mathbb{S}^1\}$ define two Cantor sets

$$K_h^s = \{(x,\sigma_+(x)) : x \in \mathbb{S}^1\} \cap (\pi^s)^{-1}(K_k)$$

and

$$K_h^u = \{(x, \sigma_+(x)) : x \in \mathbb{S}^1\} \cap f_k((\pi^u)^{-1}(K_k))$$

whose intersection points $x \in K_h^s \cap K_h^u$ are points of tangencies between the invariant manifolds of Λ_k. Furthermore, it is shown in Propositions 18 and 20 of [1] that these tangencies are quadratic[2] and unfold generically[3].

4. Proof of Theorem 1.1

After these preliminaries on the works of Palis–Yoccoz and Duarte, we are ready to prove the main result of this note.

The standard map f_k has fixed points at $p_s = (0,0) \in \Lambda_k$ and $p_u = (-\frac{1}{12} + O(\frac{1}{k}), -\frac{1}{12} + O(\frac{1}{k})) \in \Lambda_k$.

The local stable leaf $\mathcal{F}^s(p_s)$ is tangent to some leaf of $\mathcal{G}^u$ at a point q. Since K_k is $\frac{2}{k^{1/3}}$-dense in $\mathbb{S}^1$ (cf. page 394 of [1]), and f_k sends the vertical circle $f_k^{-1}(\{(x, \sigma_+(x)) : x \in \mathbb{S}^1\}) := \{(\rho_+(x), x) : x \in \mathbb{S}^1\}$ into the horizontal circle $\{(x, \sigma_+(x)) : x \in \mathbb{S}^1\}$ as a C^1-perturbation of size $\frac{1}{81k^2}$ of a rigid rotation (cf. page 397 of [1]), we can find a point of K_h^u in the $\frac{7}{2k^{1/3}}$-neighborhood of the tangency point $q \in \{(x, \sigma_+(x)) : x \in \mathbb{S}^1\}$.

Therefore, the fact that the tangency at q unfolds generically (cf. footnote 3) permits to take a parameter $|\overline{k} - k| < \frac{4}{k^{1/3}}$ such that the local stable leaf $\mathcal{F}^s(p_s)$ is tangent to the unstable manifold of some point of $\Lambda_{\overline{k}}$.

Because the unstable manifold of the fixed point p_u is dense in $\Lambda_{\overline{k}}$ (and the tangencies unfold generically), we can replace $\overline{k}$ by a parameter $|r - k| < \frac{4}{k^{1/3}}$ such that the local stable manifold $\mathcal{F}^s(p_s)$ has a quadratic tangency with the unstable manifold of p_u at q which is unfolded generically.

Next, we observe that the right part of a small neighborhood of q in the circle of tangencies is transversal to leaves of $\mathcal{F}^s$ to the right of p_s, and the left part of a small neighborhood of q in the circle of tangencies is transversal to a certain (fixed) iterate of the leaves of $\mathcal{F}^u$ which are either all above or all below p_u. In the former, resp. latter, case, we consider a Markov partition $I_- \cup I_0 \cup I_1$ for the singular expansive map $\Psi_r : \mathbb{S}^1 \to \mathbb{S}^1$ where:

- I_0 has extremities $\pi^s(p_s)$ and $a \in [\frac{1}{8}, \frac{1}{8} + \frac{1}{k^{1/3}}]$;
- I_1 has extremities $b \in [\frac{15}{32} - \frac{1}{k^{1/3}}, \frac{15}{32}]$ and $c \in [\frac{19}{32}, \frac{19}{32} + \frac{1}{k^{1/3}}]$;
- I_- has extremities $\pi^u(p_u)$ and $d \in [-\frac{1}{48}, -\frac{1}{48} + \frac{1}{k^{1/3}}]$, resp. $d \in [-\frac{7}{48} - \frac{1}{k^{1/3}}, -\frac{7}{48}]$;
- $\Psi_r(c) = \pi^u(p_u)$, $\Psi_r(b) = c = \Psi_r(d)$ and $\Psi_r(a) = d$, resp. $\Psi_r(a) = \pi^u(p_u)$, $\Psi_r(d) = \pi^s(p_s)$, $\Psi_r(c) = d$ and $\Psi_r(b) = c$.

[2] The difference in curvatures at tangency points is $\geq 4\pi^2 k - \frac{3}{k^{1/3}}$

[3] The leaves of $\mathcal{F}^s$ move with speed $\leq \frac{3}{k^{2/3}}$ and the leaves of $\mathcal{G}^u$ move with speed $\geq 1 - \frac{3}{k^{2/3}}$.

This defines a Cantor set

$$L_r := \bigcap_{n\in\mathbb{N}} \Psi_r^{-n}(I_- \cup I_0 \cup I_1)$$

and a horseshoe

$$\Theta_r := (\pi^s)^{-1}(L_r) \cap (\pi^u)^{-1}(L_r)$$

containing p_s and p_u.

By definition, we can select neighborhoods U of Θ_r and V of the orbit $\mathcal{O}(q)$ of q such that the f_r-maximal invariant set of $U \cup V$ is exactly $\Theta_r \cup \mathcal{O}(q)$: this happens because our choices were made so that the local stable leafs of Θ_r approach q only from the right, while certain (fixed) iterates of the local unstable manifolds of Θ_r approach q only from the left.

Therefore, we can conclude Theorem 1.1 from Palis–Yoccoz work (cf. Theorem 2.2) once we verify that Θ_r is slightly thick.

In view of Remark 2.1, our task is reduced to check that the stable and unstable Hausdorff dimensions of Θ_r are comprised between 0.5 and 0.6. In this direction, note that these Hausdorff dimensions coincide with the Hausdorff dimension $d(r)$ of L_r. Moreover, the distortion constant $C_1(r)$ of Ψ_r is small (namely, $0 \leq C_1(k) \leq \frac{9}{k^{1/3}}$, cf. page 388 of [1]). Hence, $d(r)$ is close to the solution $\kappa(r)$ of "Bowen's equation"

$$(\text{length } I_-)^{\kappa(r)} + (\text{length } I_0)^{\kappa(r)} + (\text{length } I_1)^{\kappa(r)} = (\text{length } I)^{\kappa(r)}$$

where I is the convex hull of $I_- \cup I_0 \cup I_1$. Since length $I_- = \frac{1}{16} + O(\frac{1}{k^{1/3}})$, length $I_0 =$ length $I_1 = \frac{1}{8} + O(\frac{1}{k^{1/3}})$,

$$\text{length } I = \frac{19}{32} + \frac{1}{12} + O(\frac{1}{k^{1/3}}), \quad \text{resp. } \frac{19}{32} + \frac{7}{48} + O(\frac{1}{k^{1/3}})$$

and

$$(1/16)^{0.5809\ldots} + (1/8)^{0.5809\ldots} + (1/8)^{0.5809\ldots} = (65/96)^{0.5809\ldots}, \text{ resp.}$$

$$(1/16)^{0.5546\ldots} + (1/8)^{0.5546\ldots} + (1/8)^{0.5546\ldots} = (71/96)^{0.5546\ldots},$$

we derive that $0.554 < d(r) < 0.581$. This completes the argument.

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