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SQUARE-TILED CYCLIC COVERS

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ABSTRACT. A cyclic cover of the complex projective line branched at four appropriate points has a natural structure of a square-tiled surface. We describe the combinatorics of such a square-tiled surface, the geometry of the corresponding Teichmüller curve, and compute the Lyapunov exponents of the determinant bundle over the Teichmüller curve with respect to the geodesic flow. This paper includes a new example (announced by G. Forni and C. Matheus in [17]) of a Teichmüller curve of a square-tiled cyclic cover in a stratum of Abelian differentials in genus four with a maximally degenerate Kontsevich–Zorich spectrum (the only known example in genus three found previously by Forni also corresponds to a square-tiled cyclic cover [15]). We present several new examples of Teichmüller curves in strata of holomorphic and meromorphic quadratic differentials with a maximally degenerate Kontsevich–Zorich spectrum. Presumably, these examples cover all possible Teichmüller curves with maximally degenerate spectra. We prove that this is indeed the case within the class of square-tiled cyclic covers.

1. INTRODUCTION

The Kontsevich–Zorich cocycle is a dynamical system on the total space of the Hodge bundle over the moduli space of Abelian or quadratic differentials. It is a continuous-time cocycle in the standard sense: it is a flow which acts linearly on the fibers of the bundle. Its projection to the moduli space is given by the Teichmüller geodesic flow. Since the Kontsevich–Zorich cocycle is closely related to the tangent cocycle of the Teichmüller flow, its Lyapunov structure determines that of the Teichmüller flow and has implications for its dynamics. The Kontsevich–Zorich spectrum also plays a crucial role in applications of Teichmüller theory to the dynamics of translation flows and interval-exchange transformations— in particular, to results on the deviation of ergodic averages (see [38, 39, 40, 22, 14]), on the existence and nature of the limit distributions (see [7, 8]) and on the weak mixing property (see [2]).

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M. Kontsevich and A. Zorich conjectured that the Lyapunov spectrum of the cocycle is simple (in particular, that all the exponents are nonzero) for $SL(2,\mathbb{R})$-invariant canonical absolutely continuous measures on all connected components of strata of the moduli spaces of Abelian (and quadratic) holomorphic differentials. G. Forni [14] proved that for the canonical measures on strata of *Abelian differentials*, the exponents are all nonzero. A. Avila and M. Viana [3] later completed the proof of the Kontsevich–Zorich conjecture in this case. Recently, G. Forni developed his approach to give a general criterion for the nonvanishing property of the Kontsevich–Zorich exponents for $SL(2,\mathbb{R})$-invariant measures on the moduli space of Abelian differentials [16]. Based on this criterion (and on a standard construction of an orienting double-cover), R. Treviño [33] proved the nonvanishing of the exponents for all canonical measures on strata of quadratic differentials. The full Kontsevich–Zorich conjecture is still open for strata of nonorientable quadratic differentials.

The aforementioned results lead to the natural questions as to whether or not it is possible for the Kontsevich–Zorich cocycle to have zero exponents with respect to other invariant measures. It is well known to experts (*cf.* [34]) that it is possible to construct invariant measures *for the Teichmüller geodesic flow* (for instance, supported on periodic orbits) with maximally degenerate Kontsevich–Zorich spectra, that is, with all exponents equal to zero with the exception of the "trivial" ones. Answering a question of W. Veech, G. Forni found in [15] the first example of a $SL(2,\mathbb{R})$-invariant measure with a maximally degenerate spectrum. The example is given by the measure supported on the $SL(2,\mathbb{R})$-orbit of a genus three square-tiled cyclic cover; that is, a branched cover of the four-punctured Riemann sphere endowed with a quadratic differential with four simple poles at the punctures. Any cover of this type is "parallelogram-tiled" in the sense that it is also a branched cover of the torus with a single branching point. The flat surface in Forni's example, found independently by F. Herrlich, M. Möller and G. Schmithuesen, is very peculiar, very symmetric, and has so many remarkable properties that it has been aptly named *Eierlegende Wollmilchsau* [20]. Later G. Forni and C. Matheus announced in the preprint [17] a second example of the same kind in genus four (see also [27]). The present article has, in fact, grown out of that announcement.

M. Möller conjectured in [30] that these two examples are the only Teichmüller curves with maximally degenerate Kontsevich–Zorich spectra. He was able to prove his conjecture up to a few strata in genus five where the arithmetic conditions he derives to rule out maximally degenerate spectra could not be verified [30]. Möller result naturally led to the more general conjecture on whether the Teichmüller curves of the Eierlegende Wollmilchsau and the Forni–Matheus curve indeed give the only examples of $SL(2,\mathbb{R})$-invariant measures (or even of $SL(2,\mathbb{R})$-orbits) with maximally degenerate Kontsevich–Zorich spectra on strata of Abelian differentials. For sufficiently high genus the conjecture is proved in [11] for $SL(2,\mathbb{R})$-invariant suborbifolds in the moduli space of holomorphic Abelian and quadratic differentials, as a corollary of the key formula for the sum of the exponents. Other results in this direction for moduli spaces of holomorphic (Abelian or quadratic) differentials in all genera have been announced by A. Avila and M. Möller and, independently, by D. Aulicino. Similar conjectures for strata of *meromorphic* quadratic differentials are at the moment wide open, to the authors' best knowledge.

In this paper, we systematically investigate these questions within the class of all square-tiled cyclic covers. We remark that the idea of the construction of a square-tiled cyclic cover already appeared in [15] (and in [17]) but only in a particular case. Here we generalize the construction and derive the main topological, geometric, and combinatorial properties of the resulting translation and half-translation surfaces. We then classify all the examples of Teichmüller curves derived from square-tiled cyclic covers with maximally degenerate spectra in strata of Abelian holomorphic differentials and of quadratic holomorphic and meromorphic differentials. The main tool in our investigation of the spectrum of Lyapunov exponents is the formula from [11] for the sum of the nonnegative Lyapunov exponents (that is, for the exponent of the determinant bundle). The formula takes a particularly simple, *explicit* form in the case of square-tiled cyclic covers. In fact, our paper can be considered as a companion to the paper by A. Eskin, M. Kontsevich and A. Zorich [12] in which the authors derive a completely explicit formula for each individual Lyapunov exponent of a square-tiled cyclic cover. In a related paper D. Chen [9] computes the Lyapunov exponent of the determinant bundle for square-tiled cyclic covers by different methods and relates it to the slope of the corresponding Teichmüller curve.

Finally, we remark that the Eierlegende Wollmilchsau and the Forni–Matheus example were originally found in [15] and [17], respectively, by a completely different method based on the analysis of the action of the cyclic group of deck transformations on the second fundamental form of the Hodge bundle (related to the Kontsevich–Zorich spectrum by the variational formulas of [14, 15]). This symmetry approach has led us to conduct a systematic investigation of the spectrum of the Kontsevich–Zorich cocycle on equivariant subbundles of the Hodge bundle, which will appear in forthcoming paper [18].

Additional bibliographic remarks. Cyclic covers were already studied by I. Bouw and M. Möller in [6] in a similar context, but with respect to completely different (not square-tiled) flat structures. The papers [5] of I. Bouw and [28] of C. McMullen investigate more general cyclic covers but without any relation to flat metrics. The paper [37] of A. Wright studies general square-tiled *Abelian* (versus *cyclic*) covers.

1.1. **Reader's guide.** Cyclic covers are defined in Section 2.1. In Section 2.2 we introduce a square-tiled flat structure on any appropriate cyclic cover. A reader interested in the main results can then choose to pass directly to Section 3.2. In Section 2.3 we characterize square-tiled cyclic covers defined by holomorphic one-forms. We determine the corresponding ambient strata of holomorphic 1-forms (of quadratic differentials in the general situation). In Section

2.4 we describe in detail how to explicitly construct the square-tiled surface $(M_N(a_1, a_2, a_3, a_4), p^*q_0)$, and in Section 2.6 we characterize its Veech group and the corresponding arithmetic Teichmüller curve. In Section 2.5 we describe the automorphism group of a cyclic cover. In Section 3.1 we recall a general formula from [11] for the sum of positive Lyapunov exponents of the Hodge bundle over an arithmetic Teichmüller curve. From this formula we derive in Section 3.2 an explicit expression for the sum of exponents in the case of an arbitrary square-tiled cyclic cover. We apply these results in Section 3.3 to determine square-tiled cyclic covers giving rise to arithmetic Teichmüller curves with a maximally degenerate Kontsevich–Zorich spectrum.

In Appendix A we present an analytic computation of the spin-structure (different from the original computation in [27]), which allows us to determine the connected component of the ambient stratum corresponding to the exceptional square-tiled cyclic cover in genus four. Appendix B is provided for the sake of completeness: it is a technical exercise related to the proof of one of the main Theorems (namely Theorem 18).

2. SQUARE-TILED FLAT STRUCTURE ON A CYCLIC COVER

2.1. **Cyclic covers.** Consider an integer N such that $N > 1$ and a 4-tuple of integers $(a_1, \dots, a_4)$ satisfying the following conditions:

$$0 < a_i \le N; \quad \gcd(N, a_1, \dots, a_4) = 1; \quad \sum_{i=1}^{4} a_i \equiv 0 \pmod N\,. \tag{1}$$

Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be four distinct points. Conditions (1) imply that, possibly after a desingularization, a Riemann surface $M_N(a_1, a_2, a_3, a_4)$ defined by equation

$$w^N = (z - z_1)^{a_1}(z - z_2)^{a_2}(z - z_3)^{a_3}(z - z_4)^{a_4} \tag{2}$$

is closed, connected and nonsingular. By construction, $M_N(a_1, a_2, a_3, a_4)$ is a ramified cover over the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ branched over the points $z_1, \dots, z_4$. By puncturing the ramification points we obtain a regular N-fold cover over $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{z_1, z_2, z_3, z_4\}$.

REMARK 1. It is easy to see that quadruples (a_1, a_2, a_3, a_4) and $(\tilde a_1, \tilde a_2, \tilde a_3, \tilde a_4)$ with $a_i = \tilde a_i \pmod N$ for $i = 1, 2, 3, 4$, define isomorphic cyclic covers, which explains the first condition in formula (1). The condition on gcd in (1) is a necessary and sufficient condition of connectedness of the resulting cyclic cover. The third condition in formula (1) implies that there is no branching at infinity.

A group of deck transformations of this cover is the cyclic group $\mathbb{Z}/N\mathbb{Z}$ with a generator $T \colon M \to M$ given by

$$T(z, w) = (z, \zeta w)\,, \tag{3}$$

where ζ is a primitive Nth root of unity, $\zeta^N = 1$. Throughout this paper we will use the term *cyclic cover* when referring to a Riemann surface $M_N(a_1, \dots, a_4)$, with parameters $N, a_1, \dots, a_4$ satisfying relations (1).

2.2. **Square-tiled surface associated to a cyclic cover.** Any meromorphic quadratic differential $q(z)(dz)^2$ with at most simple poles on a Riemann surface defines a flat metric $g(z) = |q(z)|$ with conical singularities at zeroes and poles of q. Let us consider a meromorphic quadratic differential q_0 on $\mathbb{P}^1(\mathbb{C})$ of the form

$$q_0 := \frac{c_0(dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}\,, \quad \text{where } c_0 \in \mathbb{C} \smallsetminus \{0\}\,. \tag{4}$$

It has simple poles at z_1, z_2, z_3, z_4 and no other zeroes or poles. The quadratic differential q_0 defines a flat metric on a sphere obtained by the following construction. Consider an appropriate flat cylinder. On each boundary component of the cylinder mark a pair of opposite points and glue the resulting pairs of cords by isometries. The four marked points become conical points of the flat metric. For a convenient choice of parameters c_0, $z_1,\ldots,z_4$ the resulting flat sphere can be obtained by identifying the boundaries of two copies of a unit square. Metrically, we get a square pillow with four corners corresponding to the four poles of q_0, see Figure 1.

Now consider some cyclic cover $M_N(a_1,a_2,a_3,a_4)$ and the canonical projection $p\colon M_N(a_1,a_2,a_3,a_4) \to \mathbb{P}^1(\mathbb{C})$. Consider an induced quadratic differential $q = p^* q_0$ on $M_N(a_1,a_2,a_3,a_4)$ and the corresponding flat metric. By construction, the resulting flat surface is tiled with unit squares. In other words, we get a *square-tiled surface*, see [10, 41] (also called an *origami*, [25, 32]; also called an *arithmetic translation surface*, see [19]).

In this paper we mostly focus on *square-tiled cyclic covers*, which are pairs

$$(M_N(a_1,a_2,a_3,a_4)\,,\ p^* q_0)\,,$$

where the meromorphic quadratic differential q_0 on the underlying $\mathbb{P}^1(\mathbb{C})$ defines a “unit square pillow with vertical and horizontal sides” as in Figure 1.

2.3. **Singularity pattern of a square-tiled cyclic cover.** The Riemann surface $M_N(a_1,a_2,a_3,a_4)$ has $\gcd(N,a_i)$ ramification points over each branching point $z_i \in \mathbb{P}^1(\mathbb{C})$, where $i = 1,2,3,4$, on the base sphere. Each ramification point has degree $N/\gcd(N,a_i)$. The flat metric has four conical points $z = z_i$, $i = 1,\ldots,4$, on the base sphere with a cone angle π at each conical point. Hence, the induced flat metric on $M_N(a_1,a_2,a_3,a_4)$ has $\gcd(N,a_i)$ conical points over z_i; each conical point has cone angle $\big(N/\gcd(N,a_i)\big)\pi$.

If one of the cone angles is an odd multiple of π, then the flat metric has nontrivial holonomy; in other words, the quadratic differential $q = p^* q_0$ *is not* a global square of a holomorphic 1-form. Note, however, that although the condition that all cone angles be even multiples of π is necessary, it is not a sufficient condition for triviality of the holonomy of the flat metric.

Denote by $\mathcal{H}(m_1,\ldots,m_n)$ the stratum of Abelian differentials with zeroes of degrees $m_1,\ldots,m_n$ and by $\mathcal{Q}(d_1,\ldots,d_n)$ the stratum of meromorphic quadratic differentials with singularities of degrees $d_1,\ldots,d_n$. Here $m_i \in \mathbb{N}$, for $i = 1,\ldots,n$, and $\sum_{i=1}^n m_i = 2g-2$. We do not allow poles of orders higher than one for meromorphic quadratic differentials, so $d_i \in \{-1,1,2,3,\dots\}$, for $i=1,\dots,n$, and the sum of degrees of singularities satisfies the equality $\sum_{i=1}^n d_i = 4g-4$.

LEMMA 2. *If N is even and all a_i, $i=1,2,3,4$, are odd, the quadratic differential $q=p^*q_0$ is a global square of a holomorphic 1-form ω on $M_N(a_1,a_2,a_3,a_4)$, where ω belongs to the stratum*

$$\omega \in \mathcal{H}\Bigg(\underbrace{\frac{N}{2\gcd(N,a_1)}-1,\dots}_{\gcd(N,a_1)},\dots,\underbrace{\dots,\frac{N}{2\gcd(N,a_4)}-1}_{\gcd(N,a_4)}\Bigg). \tag{5}$$

The associated flat metric on such a square-tiled cyclic cover has a trivial linear holonomy.

If N is odd, or if N is even but at least one of a_i, $i=1,2,3,4$, is also even, the quadratic differential $q=p^*q_0$ is not a global square of a holomorphic 1-form on $M_N(a_1,a_2,a_3,a_4)$. In this case q belongs to the stratum

$$q \in \mathcal{Q}\Bigg(\underbrace{\frac{N}{\gcd(N,a_1)}-2,\dots}_{\gcd(N,a_1)},\dots,\underbrace{\dots,\frac{N}{\gcd(N,a_4)}-2}_{\gcd(N,a_4)}\Bigg), \tag{6}$$

and the flat metric on such a square-tiled cyclic cover $M_N(a_1,a_2,a_3,a_4)$ has nontrivial linear holonomy.

REMARK 3. In the case, when $\gcd(N,a_i)=\frac{N}{2}$, the corresponding "conical points" have cone angles 2π and so are actually regular points of the metric. Depending on the context we either consider such points as marked points or simply ignore them.

REMARK 4. Lemma 2 implies, in particular, that the quadratic differential $q = p^*q_0$ is *holomorphic* if and only if inequalities $a_i \le N$ are strict for all $i=1,2,3,4$. If $a_i = N$ for at least one index i, then the quadratic differential $q=p^*q_0$ is *meromorphic*; that is, it has simple poles.

Proof of the Lemma. Let σ_i be a small contour around z_i in the positive direction on the sphere (see Figure 1). The paths σ_i, $i=1,2,3$ generate the fundamental group of the sphere punctured at the four ramification points.

Since the cone angle at each cone singularity of the underlying "flat sphere" is π (whether it is glued from squares or not), the parallel transport along each loop σ_i brings a tangent vector $\vec{v}$ to $-\vec{v}$. Let $z\in\mathbb{P}^1(\mathbb{C})$ be a point of the loop σ_i, and let (w,z) be one of its preimages in the cover $M_N(a_1,a_2,a_3,a_4)$. By lifting the loop σ_i to a path on the cover which starts at the point (w,z), we land at the point $(\zeta^{a_i}w,z)$, where ζ is the primitive Nth root of unity. Thus we get the following representation of the fundamental group of the punctured sphere in the cyclic group $\mathbb{Z}/N\mathbb{Z}$ of deck transformations (3) and the holonomy group $\mathbb{Z}/2\mathbb{Z}$ of the flat metric on the sphere:

$$\text{Deck: } \sigma_i \mapsto a_i \in \mathbb{Z}/N\mathbb{Z} \qquad \text{Hol: } \sigma_i \mapsto 1 \in \mathbb{Z}/2\mathbb{Z} \tag{7}$$

Since the metric on $M_N(a_1,a_2,a_3,a_4)$ is induced from the metric on the sphere, the holonomy representation hol of the fundamental group of the covering surface $M_N(a_1,a_2,a_3,a_4)$ factors through the one of the sphere, *i.e.,* $\mathrm{hol} = \mathrm{Hol}\circ p_*$.

Let us suppose that N is odd. Let $\alpha_i = N/\gcd(N,a_i)$. Then

$$\mathrm{Deck}(\sigma_i^{\alpha_i}) = a_i\cdot\frac{N}{\gcd(N,a_i)} = \frac{a_i}{\gcd(N,a_i)}\,N = 0 \pmod N\ .$$

Hence $\sigma_i^{\alpha_i}$ can be lifted to a closed path on $M_N(a_1,a_2,a_3,a_4)$. On the other hand, $\mathrm{Hol}(\sigma_i^{\alpha_i}) = \alpha_i = N/\gcd(N,a_i) = 1 \pmod 2$. Thus, the flat metric on the cover $M_N(a_1,a_2,a_3,a_4)$ has nontrivial holonomy, and our quadratic differential q is *not* a global square of a holomorphic 1-form.

Let us suppose that N is even, but some a_i is also even. Since

$$\gcd(N,a_1,\dots,a_4) = 1\,,$$

at least one of a_j is odd. Since $a_1+\dots+a_4$ is divisible by N, this sum is even, and hence we have exactly two even entries. By relabeling the ramification points on the sphere, if necessary, we may assume that a_1,a_3 are odd while a_2,a_4 are even. Let us consider a path

$$\rho := \sigma_1^{(a_2+a_3)}\sigma_2^{-a_1}\sigma_3^{-a_1}$$

For the induced deck transformation we get

$$\mathrm{Deck}(\rho) = ((a_2+a_3)a_1 - a_1a_2 - a_1a_3) = 0\ .$$

Hence, a lift of ρ is closed on $M_N(a_1,a_2,a_3,a_4)$. On the other hand, this path acts as an element $\mathrm{Hol}(\rho) = (a_2+a_3) - a_1 - a_1 = 1 \pmod 2$ in the holonomy group $\mathbb{Z}/2\mathbb{Z}$. Hence, the holonomy of the metric along this closed path on the cover $M_N(a_1,a_2,a_3,a_4)$ is nontrivial, and our quadratic differential q is *not* a global square of a holomorphic 1-form.

Finally, suppose that N is even and all a_i, $i=1,2,3,4$ are odd. Any element of the fundamental group of the underlying four-punctured sphere can be represented by a product

$$\tau = \prod_{j=1}^{n}\sigma_{i_j}^{p_j}\,,\quad \text{where } p_j = \pm1 \text{ and } i_j\in\{1,2,3,4\}\,.$$

Let k_i be the algebraic number of entries of the "letter" σ_i in the word as above, where σ_i^{-1} is counted with the sign minus, and $i=1,2,3,4$. The loop τ as above can be lifted to a closed loop on the cover if and only if $\mathrm{Deck}(\tau)=0$, that is, if and only if $k_1a_1+\dots+k_4a_4 = 0 \pmod N$. In this case $\mathrm{Hol}(\tau) = k_1+\dots+k_4$. Since all a_i are odd we have $k_1+\dots+k_4 = k_1a_1+\dots+k_4a_4 \pmod 2$. Finally, since N is even and $k_1a_1+\dots+k_4a_4 = 0 \pmod N$, we also have $k_1a_1+\dots+k_4a_4 = 0 \pmod 2$. Thus, in this case, the flat metric on $M_N(a_1,a_2,a_3,a_4)$ has trivial linear holonomy.

Recall that a flat metric associated to a meromorphic quadratic differential with at most simple poles has cone angle $(d+2)\pi$ at a zero of order d (we consider a simple pole as a "zero of degree -1"). A flat metric associated to a holomorphic 1-form has cone angle $2(k+1)\pi$ at a zero of degree k. We have already evaluated the number of ramification points and their ramification degrees which define cone angles at all conical singularities, and hence the degrees of the corresponding quadratic (Abelian) differential. This completes the proof of the Lemma. □

LEMMA 5. *In the case when the quadratic differential $q = p^* q_0$ determining a square-tiled flat structure is the global square of a holomorphic* 1*-form, that is, $q = \omega^2$, the form ω is anti-invariant with respect to the action of a generator of the group of deck transformations,*

$$T^* \omega = -\omega\,. \tag{8}$$

Proof. By construction, the quadratic differential $q = p^* q_0$ is invariant under the action of deck transformations on $M_N(a_1, a_2, a_3, a_4)$. Hence, when N is even, all a_i, $i = 1,2,3,4$ are odd, and $q = p^* q_0 = \omega^2$, the holomorphic 1-form ω is either invariant or anti-invariant under the action of a generator of the group of deck transformations (3).

Invariance of ω under T^* would mean that ω can be pushed forward to $\mathbb{P}^1(\mathbb{C})$, which would imply in turn that q_0 is a global square of a holomorphic 1-form. This is not true. Hence, ω is anti-invariant. □

By the Riemann–Hurwitz formula, the genus g of $M_N(a_1, a_2, a_3, a_4)$ satisfies

$$\begin{aligned} 2-2g = 2N - &\sum_{\substack{\text{ramification}\\ \text{points}}} (\text{degree of ramification} - 1) \\ = \ & 2N - \sum_{i=1}^{4} \gcd(N, a_i) \cdot \big(N/\gcd(N, a_i) - 1\big) \\ = \ & \sum_{i=1}^{4} \gcd(N, a_i) - 2N \end{aligned}$$

and hence,

$$g = N + 1 - \frac{1}{2} \sum_{i=1}^{4} \gcd(a_i, N)\,. \tag{9}$$

The same result can be obtained by summing up the degrees of zeroes, which gives $2g-2$ for a holomorphic 1-form in (5), and $4g-4$ for a quadratic differential in (6).

2.4. **Combinatorics of a square-tiled cyclic cover.**

It is convenient to define a square-tiled surface corresponding to a holomorphic 1-form by a pair of permutations on the set of all squares, π_h and π_v, indicating for each square its neighbor to the right and its neighbor on top respectively. Let us evaluate these permutations for a square-tiled surface defined by a holomorphic 1-form ω on $M_N(a_1, a_2, a_3, a_4)$ (and in addition to the conditions in formula (1) we assume that N is even and all a_i are odd).

We start with an appropriate enumeration of the squares. By construction, our "square-tiled pillow" $(\mathbb{P}^1(\mathbb{C}), q_0)$ in the base of the cover

$$M_N(a_1, a_2, a_3, a_4) \to \mathbb{P}^1(\mathbb{C})$$

is tiled with two unit squares. Since the above cover has degree N, our square-tiled cyclic cover gets tiled with $2N$ squares.

Assume that the branch points z_1, z_2, z_3, z_4 are associated to the corners of the pillow as in Figure 1. We associate the letters A, B, C, D to the four corners respectively. We also associate the corresponding letters to the corners of each square on the surface $M_N(a_1, a_2, a_3, a_4)$.

Let us color one of the faces of our pillow white, and the other black. Let us lift this coloring to $M_N(a_1, a_2, a_3, a_4)$. Choose some white square, and associate the number 0 to it. Take a black square adjacent to the side $[CD]$ of the first one and associate the number 1 to it. Acting by deck transformations we associate to a white square $T^k(S_0)$ the number $2k$, and to a black square $T^k(S_1)$ the number $2k+1$. As usual, k is taken modulo N, so we may assume that $0 \le k < N$.

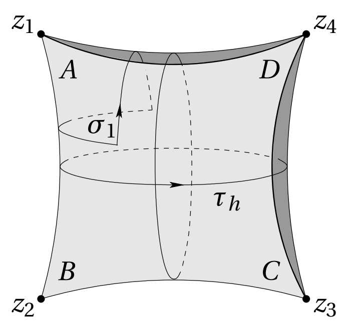

FIGURE 1. Flat sphere glued from two squares

Let us consider a small loop σ_i encircling z_i in a positive direction; we assume that σ_i does not have other ramification points inside an encircled domain, see Figure 1. Let us then consider a lift of σ_i to $M_N(a_1, a_2, a_3, a_4)$. The end-point of the lifted path is the image of the action of T^{a_i} on the starting point of the lifted path. Hence, starting at a square number j and "going around a corner" on $M_N(a_1, a_2, a_3, a_4)$ in the positive (counterclockwise) direction we get to a square number $j + 2a_j \pmod{2N}$ (see Figure 2).

Let us consider a horizontal path τ_h as in Figure 1 and a lift of τ_h to the surface $M_N(a_1, a_2, a_3, a_4)$. The end-point of the resulting path is the image of the action of $T^{a_1+a_4} = T^{-(a_2+a_3)}$ on the starting point of the lifted path. Hence, "moving two squares to the right" on $M_N(a_1, a_2, a_3, a_4)$ we move from a square number j to a square number $j + 2(a_1 + a_4) \pmod{2N}$ if vertices B and C are at the bottom of the squares, and to a square number $j - 2(a_1 + a_4) \pmod{2N}$ if vertices B and C are on top of the squares (see Figure 3).

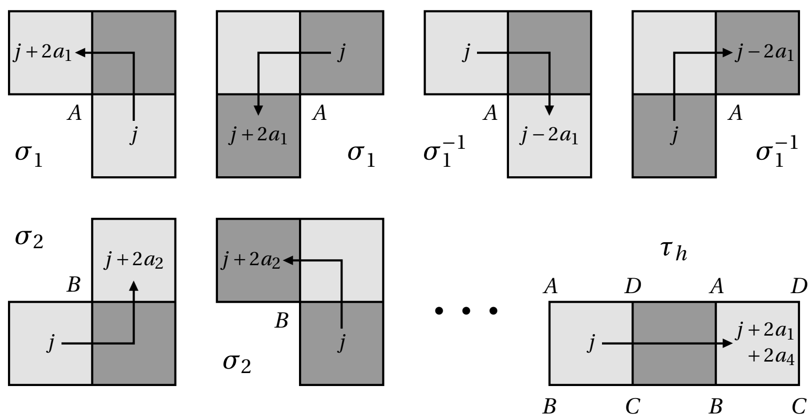

FIGURE 2. Local moves on a square-tiled cyclic cover

Using these rules, it is easy to determine the permutations π_h and π_v. Start with two neighboring squares numbered by 0 and 1. By iterating the operation τ_h, we can determine all the squares to the right of 0 and 1 till we close up and get a cylinder. Recall that we associate letters A, B, C, D to the corners of the squares. By applying appropriate operations σ_i we find the squares located atop those which are already constructed. By applying appropriate operations σ_i^{-1} we find a direct neighbor to the right for every square which does not belong to one of the previously constructed horizontal cylinders. Having two horizontally adjacent squares, we apply iteratively the operation τ_h to obtain all $2N/\gcd(N, a_1+a_4)$ squares in the corresponding cylinder (row), etc.

EXAMPLE 6. Figure 3 presents a construction of the enumeration for the square-tiling of $M_6(1,1,1,3)$, where the exponents $\{1,1,1,3\}$ are represented by vertices $\{A,B,C,D\}$ respectively. Note that by moving two squares to the right in the first row (say, $0 \longrightarrow 8$) we apply τ_h, while by moving two squares to the right in the second row (say, $10 \longrightarrow 2$) we apply τ_h^{-1}. In this example the permutations π_h and π_v have the following decompositions into cycles

$$\pi_h = (0,1,8,9,4,5)(11,10,3,2,7,6)$$
$$\pi_v = (0,7,4,11,8,3)(1,6,9,2,5,10)$$

When N is odd, or when N is even but at least one of a_i is also even, the quadratic differential $q = p^* q_0$ on $M_N(a_1,a_2,a_3,a_4)$ is *not* a global square of a holomorphic 1-form. The holonomy of the flat structure defined by q is no longer trivial: a parallel transport of a tangent vector v along certain closed paths brings it to $-v$. In particular the notions of "up–down" or "left–right" are no longer globally defined. However, the notions of horizontal direction and of vertical direction are still globally well-defined. Our flat surface is square-tiled and its combinatorial geometry can still be encoded by a pair of permutations π_h and π_v.

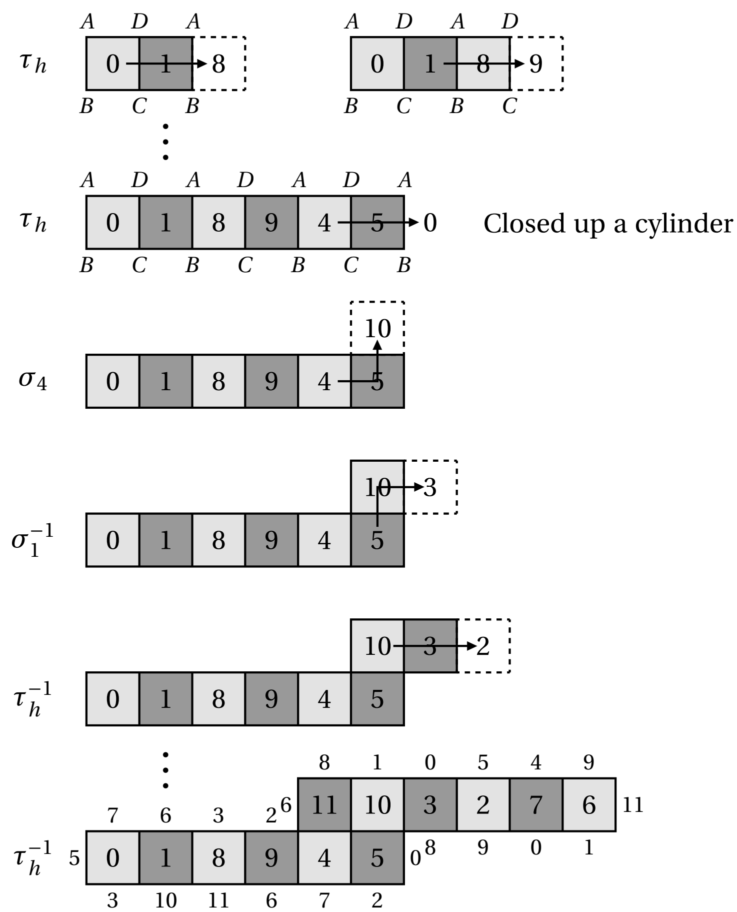

FIGURE 3. Cartoon movie construction of $M_6(1,1,1,3)$

Note that the vertices of every square of our tiling are naturally labeled by indices A, B, C, D according to the label of their projections on the Riemann sphere. By convention, $\pi_h(2k)$ indicates the number of a black square adjacent to the side $[CD]$ of the white square $2k$, and $\pi_h(2k+1)$ indicates the number of a white square adjacent to the side $[AB]$ of the black square $2k+1$. Similarly, $\pi_v(2k)$ indicates the number of a black square adjacent to the side $[AD]$ of the white square $2k$, and $\pi_h(2k+1)$ indicates the number of a white square adjacent to the side $[BC]$ of the black square $2k+1$.

With this definition the remaining part of the construction literally coincides with the one for an Abelian differential, which was described above.

EXAMPLE 7. Figure 4 illustrates a square-tiling of $M_4(1,3,2,2)$. The flat metric has nontrivial linear holonomy; the corresponding quadratic differential belongs to the stratum $\mathcal{Q}(2,2)$. In this example the permutations π_h and π_v are

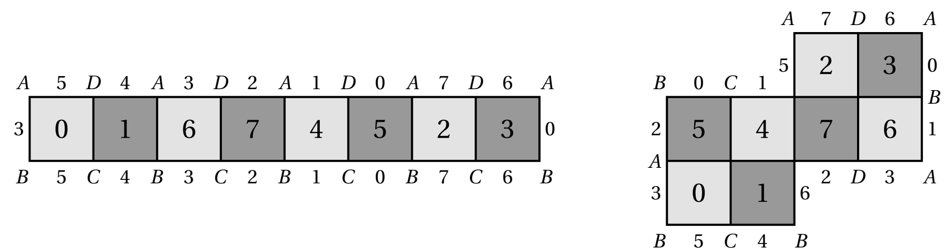

FIGURE 4. These two ways to unfold the square-tiled cyclic cover $M_4(1,3,2,2)$ represent decompositions into horizontal and into vertical cylinders

decomposed into cycles as

$$\pi_h = (0,1,6,7,4,5,2,3)$$
$$\pi_v = (0,5)(1,4)(2,7)(3,6)$$

REMARK 8. Note that when pairing sides of boundary squares of an abstract square-tiled surface one has to respect the orientation of the surface.

We proceed below with an elementary lemma which will, however, be important later.

LEMMA 9. *Consider a decomposition* $M_N(a_1,a_2,a_3,a_4) = \sqcup \mathrm{cyl}_i$ *of a square-tiled cyclic cover into cylinders* cyl_i *filled by closed horizontal trajectories. For every cylinder* cyl_i *we denote by* w_i *its width (the length of each closed horizontal trajectory) and by* h_i *its height (the length of each vertical segment).*

Assuming that the branch points z_1, z_2, z_3, z_4 *are numbered as indicated in Figure 1, the widths of the corresponding cylinders and the sum of the heights of all cylinders are given by the formulas:*

$$w_i = \frac{2N}{\gcd(N, a_1+a_4)}, \quad \textit{for all } i\,, \textit{ and } \sum_i h_i = \gcd(N, a_1+a_4)\,.$$

Proof. Clearly, the operation $\tau_h = T^{(a_1+a_4)}$ has order $\frac{N}{\gcd(N,a_1+a_4)}$. Hence, the length of each horizontal trajectory is equal to $\frac{2N}{\gcd(N,a_1+a_4)}$ which, in turn, is equal to the width of any cylinder. Since the area of the surface is $2N$, the total height of all cylinders is equal to $\gcd(N, a_1+a_4)$. □

REMARK 10. It is irrelevant whether or not the quadratic differential $q = p^* q_0$ defining the square-tiled flat structure in Lemma 9 is a global square of a holomorphic 1-form.

2.5. **Symmetries of cyclic covers.** We continue with a description of the isomorphisms of cyclic covers (certainly known to all who worked with them). Note that in the Lemma below we do not use any flat structure.

LEMMA 11. *Two cyclic covers* $M_N(a_1,\dots,a_4)$ *and* $M_N(\tilde a_1,\dots,\tilde a_4)$ *admit an isomorphism compatible with the projection to* $\mathbb{P}^1(\mathbb{C})$

$$\begin{tikzcd}[column sep=small] M_N(a_1,\dots,a_4) \arrow[dr] & \simeq & M_N(\tilde a_1,\dots,\tilde a_4) \arrow[dl] \\ & \mathbb{P}^1(\mathbb{C}) & \end{tikzcd}$$

if and only if for some primitive element $k \in \mathbb{Z}/N\mathbb{Z}$ one has

$$\tilde a_i = k a_i \pmod N, \quad \textit{for} \quad i = 1,2,3,4\,. \tag{10}$$

In particular,

$$M_N(a_1,a_2,a_3,a_4) \simeq M_N(N-a_1,N-a_2,N-a_3,N-a_4)\,. \tag{11}$$

Proof. Any isomorphism g of cyclic covers as above induces an isomorphism g_* of their groups of deck transformations such that,

$$\widetilde{\mathrm{Deck}}(\sigma_i) = g_*(\mathrm{Deck}(\sigma_i))\,, \quad \text{for } i = 1,\dots,4\,.$$

Since by construction $\widetilde{\mathrm{Deck}}(\sigma_i) = \tilde a_i$ and $\mathrm{Deck}(\sigma_i) = a_i$ in $\mathbb{Z}/N\mathbb{Z}$, for $i = 1,\dots,4$, and any automorphism of a cyclic group $\mathbb{Z}/N\mathbb{Z}$ is given by the multiplication by a primitive element $k \in \mathbb{Z}/N\mathbb{Z}$, the above relation yields formula (10).

On the other hand, when condition (10) is satisfied, one can find integers $m_1,\dots,m_4$ such that $\tilde a_i = k a_i + m_i N$, where $0 < \tilde a_i \le N$, and, hence, we have an obvious isomorphism

$$\tilde w = w^k (z-z_1)^{m_1}(z-z_2)^{m_2}(z-z_3)^{m_3}(z-z_4)^{m_4}$$

between the two cyclic covers $w^N = (z-z_1)^{a_1}(z-z_2)^{a_2}(z-z_3)^{a_3}(z-z_4)^{a_4}$ and $\tilde w^N = (z-z_1)^{\tilde a_1}(z-z_2)^{\tilde a_2}(z-z_3)^{\tilde a_3}(z-z_4)^{\tilde a_4}$. $\square$

Consider the particular case when $\{a_1,\dots,a_4\}$ and $\{\tilde a_1,\dots,\tilde a_4\}$ coincide as unordered sets (possibly with multiplicities).

DEFINITION 12. A permutation π in $\mathfrak{S}_4$ is called a *symmetry* of a cyclic cover $M_N(a_1,a_2,a_3,a_4)$ if there exists an integer k such that

$$k\cdot a_i \pmod N = a_{\pi(i)} \quad \text{for } i = 1,2,3,4\,.$$

2.6. **The Veech group of a square-tiled cyclic cover.** The groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$ act naturally on any stratum of holomorphic 1-forms and, respectively, meromorphic quadratic differentials with at most simple poles. The *Veech group* $\Gamma(S)$ of a flat surface S is the stabilizer of the corresponding point of the stratum under this action. In this section we study the Veech groups of square-tiled cyclic covers. In particular, we prove the following statement.

THEOREM 13. *The Veech group $\Gamma(S)$ of any square-tiled cyclic cover S contains the group $\Gamma(2)$ (respectively $\Gamma(2)/(\pm\mathrm{Id})$) as a subgroup. The Veech group $\Gamma(S)$ has one of the indices 1, 2, 3 or 6 in $SL(2,\mathbb{Z})$ (respectively in $PSL(2,\mathbb{Z})$).*

If the Veech groups $\Gamma(S_1)$ and $\Gamma(S_2)$ of two square-tiled cyclic covers S_1, S_2 have the same index in $SL(2,\mathbb{Z})$ (respectively in $PSL(2,\mathbb{Z})$), then $\Gamma(S_1)$ and $\Gamma(S_2)$ are conjugate.

REMARK 14. A proper subgroup of $SL(2,\mathbb{Z})$ containing $\Gamma(2)$ is said to be a *congruence subgroup of level two*. It is not hard to check that the conjugation class of a congruence subgroup of level two is uniquely determined by its index. An example of a congruence subgroup of $SL(2,\mathbb{Z})$ of level two with index 3 is

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \equiv d \equiv 1 (\mathrm{mod}\ 2), c \equiv 0 (\mathrm{mod}\ 2) \right\}.$$

In the literature, the congruence subgroups of level two of $SL(2,\mathbb{Z})$ are called *theta groups* (sometimes denoted Θ).

Theorem 13 will be derived from Lemma 15 and from a more precise Theorem 18.

Sometimes it is convenient to "mark" (in other words "label" or "give names to") the zeroes (and the simple poles) of the corresponding holomorphic 1-form (respectively, quadratic differential). In this way, one gets the strata of marked (in other words "labeled", "named"), flat surfaces. The action of the group $SL(2,\mathbb{R})$ (respectively $PSL(2,\mathbb{R})$) on the strata of marked flat surfaces, and the Veech group $\Gamma(S_{marked})$ of a marked flat surface S_{marked} are defined analogously.

Let the flat surface S be a "unit square pillow", that is, let S be $\mathbb{P}^1(\mathbb{C})$ endowed with the quadratic differential (4), where the parameters are chosen in such way that the flat surface S is glued from two unit squares, and their sides are vertical and horizontal. The Veech group $\Gamma(S)$ of such a flat surface S coincides with $PSL(2,\mathbb{Z})$. We will need the following elementary Lemma for the marked version of the latter surface.

LEMMA 15. *The action of the group $PSL(2,\mathbb{Z})$ on the "unit square pillow with marked corners" factors through the free action of the group $\mathfrak{S}_3$ of permutations of three elements by means of the surjective homomorphism*

$$PSL(2,\mathbb{Z}) \to PSL(2,\mathbb{Z}/2\mathbb{Z}) \simeq \mathfrak{S}_3\,. \tag{12}$$

In particular, the Veech group $\Gamma(S_{marked})$ of the "unit square pillow with marked corners" S_{marked} coincides with the kernel $\Gamma(2)/(\pm \mathrm{Id})$ of this homomorphism.

Proof. Let the labels of the corners of the pillow be A, B, C, D, say, as in Figure 1. An element g of $PSL(2,\mathbb{Z})$ acts on the flat surface $(\mathbb{P}^1(\mathbb{C}), q_0)$, giving a new flat surface isomorphic to the original one. It can still be obtained by gluing two squares, but the labels A, B, C, D have moved around. Using an appropri-

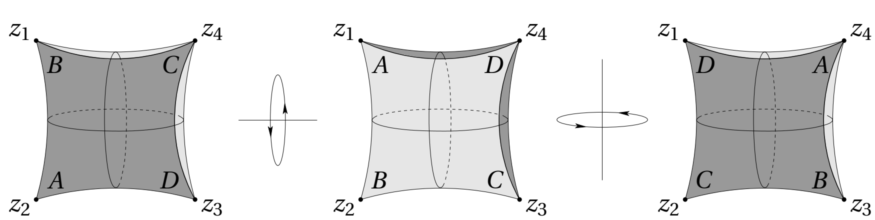

FIGURE 5. Symmetries of the flat surface $\left(\mathbb{P}^1(\mathbb{C}), q_0\right)$

ate "pillow symmetry" as in Figure 5, one can always move back one chosen label to the original position, but the other labels B, C, D are not fixed. Speaking more formally, the "pillow symmetries" define the normal subgroup

$$\mathfrak{K} = \big\{() \,;\, (1,2)(3,4) \,;\, (1,3)(2,4) \,;\, (1,4)(2,3)\big\} \tag{13}$$

of the symmetric group $\mathfrak{S}_4$. The subgroup $\mathfrak{K}$ is isomorphic to the Klein group. The quotient of $\mathfrak{S}_4$ over $\mathfrak{K}$ is isomorphic to the symmetric group $\mathfrak{S}_3$. The projection

$$\mathfrak{S}_4 \to \mathfrak{S}_3 \simeq \mathfrak{S}_4/\mathfrak{K} \tag{14}$$

is not canonical, since it depends on the choice of the fixed label. However, the *conjugacy class* of the image of any element is defined canonically.

Note that the "pillow symmetries" are diffeomorphisms of our flat sphere with differential $\pm\mathrm{Id}$ in flat coordinates. Thus, all "unit square pillows with marked corners" related by "pillow symmetries" define one and the same point of the stratum $\mathcal{Q}_{marked}(-1,-1,-1,-1)$.

It is easy to see that under the convention that $\mathrm{PSL}(2,\mathbb{Z})$ keeps one of the labels fixed, the elements $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ of $\mathrm{PSL}(2,\mathbb{Z})$ fix all the labels, and, thus, belong to the Veech group of S_{marked}. It is a well-known fact that the above two elements generate the kernel $\Gamma(2)/(\pm\mathrm{Id})$ of the homomorphism of formula (12), and that the group $\mathrm{PSL}(2,\mathbb{Z}/2\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathfrak{S}_3$.

It is also easy to check that under this identification the elements of $\mathfrak{S}_3$ act by the corresponding permutations of the three "free" labels; in particular, the action of $\mathfrak{S}_3$ on the six distinct "unit square pillows with marked corners" is free. □

Let S be a cyclic cover of a type $M_N(a_1,\dots,a_4)$ endowed with the flat structure induced from the flat structure (4) on $\mathbb{P}^1(\mathbb{C})$. As always, we assume that when N is even and all a_i are odd, the flat structure on S is defined by the holomorphic 1-form ω, such that $\omega^2 = p^* q_0$; otherwise it is defined by the quadratic differential $q = p^* q_0$.

It is easy to see, that for any g in $\mathrm{SL}(2,\mathbb{R})$ (respectively, for any g in $\mathrm{PSL}(2,\mathbb{R})$), the image $\tilde S := gS$ is also represented by a cyclic cover. Moreover, any affine diffeomorphism $A_g\colon S \to \tilde S$ of the corresponding flat surfaces intertwines the action of the group of the deck transformations on S and $\tilde S$, that is, $A_g \circ T = \tilde T \circ A_g$. Similarly, for any $g \in \mathrm{SL}(2,\mathbb{Z})$ (respectively, for any $g \in \mathrm{PSL}(2,\mathbb{Z})$) the image of a *square-tiled* cyclic cover under the action of g is again a *square-tiled* cyclic cover.

In the rest of this section (and, basically, in the remaining part of the paper) we consider only *square-tiled* cyclic covers, in particular, to avoid cumbersome notations, we denote by $M_N(a_1,a_2,a_3,a_4)$ a cyclic cover endowed with the flat structure induced from the "square pillow" as in Figure 1. Under this convention, the order of the entries $a_1,\dots,a_4$ matters in the definition of a square-tiled cyclic cover $M_N(a_1,a_2,a_3,a_4)$. However, the square-tiled cyclic covers

$$M_N(a_1,a_2,a_3,a_4), M_N(a_2,a_1,a_4,a_3), M_N(a_4,a_3,a_2,a_1), \text{ and } M_N(a_3,a_4,a_1,a_2),$$

related by "pillow symmetries" (see formula (13)) define the same flat surface.

The fact that the square-tiled cyclic cover $M_N(a_3,a_4,a_1,a_2)$ defines the same flat surface as $M_N(a_1,a_2,a_3,a_4)$ implies, in particular, that in the case when the flat structure on a square-tiled cyclic cover is defined by a holomorphic 1-form, the element $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ of $\mathrm{SL}(2,\mathbb{Z})$ belongs to the Veech group of the corresponding flat surface. Hence, the action of $\mathrm{SL}(2,\mathbb{Z})$ on square-tiled cyclic covers factors through the action of $\mathrm{PSL}(2,\mathbb{Z})$. We shall sometimes consider the latter action without specifying it explicitly.

It is easy to see, that if an element $g \in \mathrm{PSL}(2,\mathbb{Z})$ permutes the marking of the initial "unit square pillow with marked corners" by a permutation $\pi \in \mathfrak{S}_4$, then the square-tiled cyclic cover $M_N(a_1,a_2,a_3,a_4)$ is mapped by g to the square-tiled cyclic cover $M_N(a_{\pi(1)},a_{\pi(2)},a_{\pi(3)},a_{\pi(4)})$. Since the elements of the subgroup $\mathfrak{K}$ of $\mathfrak{S}_4$ correspond to isomorphic square-tiled cyclic covers, we conclude that the action of $\mathrm{SL}(2,\mathbb{Z})$ (respectively of $\mathrm{PSL}(2,\mathbb{Z})$) on square-tiled cyclic covers factors through the action on "unit square pillows with marked corners".

By combining the latter observation with Lemma 15 we conclude that the Veech group of $M_N(a_1,a_2,a_3,a_4)$ contains the group $\Gamma(2)$ (respectively the group $\Gamma(2)/(\pm\mathrm{Id})$), that it has index *at most* 6 in $\mathrm{SL}(2,\mathbb{Z})$ (respectively, in $\mathrm{PSL}(2,\mathbb{Z})$), and that it is determined by its index up to conjugation.

To complete the proof of Theorem 13 it remains to prove that all indices 1, 2, 3, 6 are realized.

In order to describe this Veech group more precisely we need the following remark. Let

$$\hat{f}\colon M_N(a_1,a_2,a_3,a_4)\to M_N(\tilde a_1,\tilde a_2,\tilde a_3,\tilde a_4)$$

be an isomorphism of square-tiled cyclic covers, that is, a diffeomorphism with differential equal to Id (respectively $\pm$Id) in flat coordinates. It is not hard to see that $\hat f$ is part of a commutative diagram

$$\begin{array}{ccc}
M_N(a_1,a_2,a_3,a_4) & \xrightarrow{\hat f} & M_N(\tilde a_1,\tilde a_2,\tilde a_3,\tilde a_4)\\
\Big\downarrow p & & \Big\downarrow p\\
(\mathbb{P}^1(\mathbb{C}),q_0) & \xrightarrow{f} & (\mathbb{P}^1(\mathbb{C}),q_0),
\end{array}$$

where p is the canonical projection and $f\colon \mathbb{P}^1(\mathbb{C})\to\mathbb{P}^1(\mathbb{C})$ is an automorphism of the underlying flat sphere. We have seen that the only automorphisms of the "square pillow" $(\mathbb{P}^1(\mathbb{C}),q_0)$ are the "pillow symmetries" (see Figure 5). It follows from formula (13) that all "pillow symmetries", that is, all elements of the Klein group $\mathfrak{K}$ are involutions. Let s be the the "pillow symmetry" corresponding to the automorphism $f\colon \mathbb{P}^1(\mathbb{C})\to\mathbb{P}^1(\mathbb{C})$ or, equivalently, to its inverse. Let it act on the canonical labeling of the corners of the pillow by a permutation $\kappa\in\mathfrak{K}$.

Let $\hat{s}$ be the induced automorphism of $M_N(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4)$. By definition of $\hat{s}$ the diagram

$$\begin{tikzcd} M_N(\tilde{a}_1,\tilde{a}_2,\tilde{a}_3,\tilde{a}_4) \arrow[r, "\hat{s}"] \arrow[d, "p"] & M_N(\tilde{a}_{\kappa(1)},\tilde{a}_{\kappa(2)},\tilde{a}_{\kappa(3)},\tilde{a}_{\kappa(4)}) \arrow[d, "\tilde{p}"] \\ (\mathbb{P}^1(\mathbb{C}), q_0) \arrow[r, "s"] & (\mathbb{P}^1(\mathbb{C}), q_0) \end{tikzcd}$$

commutes. Note that by construction the composition $f \circ s$ is the identity map, which allows us to merge the two commutative diagrams above into the commutative diagram

$$\begin{tikzcd} M_N(a_1,\ldots,a_4) \simeq M_N(\tilde{a}_{\kappa(1)},\tilde{a}_{\kappa(2)},\tilde{a}_{\kappa(3)},\tilde{a}_{\kappa(4)}) \arrow[d, shift right] \arrow[d, shift left] \\ \mathbb{P}^1(\mathbb{C}) . \end{tikzcd} \tag{15}$$

Let us consider the case when $\{a_1,\ldots,a_4\}$ and $\{\tilde{a}_1,\ldots,\tilde{a}_4\}$ coincide as unordered sets (possibly with multiplicities). Since all symmetries of cyclic covers such as those in formula (15) are described by Definition 12 and Lemma 11, our considerations imply the following statement.

LEMMA 16. *Consider $\pi \in \mathfrak{S}_4$. The square-tiled cyclic covers $M_N(a_1,a_2,a_3,a_4)$ and $M_N(a_{\pi(1)},a_{\pi(2)},a_{\pi(3)},a_{\pi(4)})$ are isomorphic (that is define the same point of the corresponding stratum) if and only if there exists a symmetry π' of the cyclic cover $M_N(a_1,a_2,a_3,a_4)$ such that the permutation $\pi' \cdot \pi^{-1}$ belongs to the Klein subgroup $\mathfrak{K}$ defined in formula* (13).

By passing to the quotient $\mathfrak{S}_3 \simeq \mathfrak{S}_4/\mathfrak{K}$ we get the following immediate Corollary of the Lemma above.

COROLLARY 17. *For any square-tiled cyclic cover $M_N(a_1,a_2,a_3,a_4)$ the index of its Veech group in $SL(2,\mathbb{Z})$ (in $PSL(2,\mathbb{Z})$ when the flat structure is defined by a quadratic differential) coincides with the index of the image of the subgroup of symmetries of $M_N(a_1,a_2,a_3,a_4)$ in $\mathfrak{S}_3$ under the projection of formula* (14).

In order to complete the proof of Theorem 13 it is sufficient to prove that all indices $1,2,3,6$ are realized. We prefer to prove a strengthened version of Theorem 13.

THEOREM 18. *The index of the Veech group of a square-tiled cyclic cover is described by the following list.*

- *If for some triple of pairwise distinct indices $i,j,k \in \{1,2,3,4\}$ one has $a_i = a_j = a_k$, the index of the Veech group of the corresponding square-tiled cyclic cover $M_N(a_1,a_2,a_3,a_4)$ is* 1.
- *If there is no such triple of pairwise distinct indices, but there is a pair of indices $i \neq j$, where $i,j \in \{1,2,3,4\}$, such that $a_i = a_j$, then the index of the Veech group is* 3.

If all a_i are pairwise distinct, then

- *If $M_N(a_1,\dots,a_4)$ does not have nontrivial symmetries, or if any nontrivial symmetry decomposes into two cycles of length* 2*, the index of the Veech group is* 6.
- *If $M_N(a_1,\dots,a_4)$ has a symmetry represented by a cycle of length* 4 *or by a single cycle of length* 2*, the index of the Veech group is* 3.
- *If $M_N(a_1,\dots,a_4)$ has a symmetry represented by a cycle of length* 3*, the index of the Veech group is* 2.

All the symmetries listed above are realized.

Proof. If for some triple of pairwise distinct indices $i,j,k \in \{1,2,3,4\}$ the numbers $a_i = a_j = a_k$ coincide, it is clear that the index of the Veech group of the square-tiled cyclic cover $M_N(a_1,a_2,a_3,a_4)$ is 1, say, as for $M_4(1,1,1,1)$ or for $M_6(3,1,1,1)$.

Let us suppose that there is no such triple of indices, but there is a pair of indices $i \neq j$, where $i,j \in \{1,2,3,4\}$, such that $a_i = a_j$, say, as for $M_{10}(1,1,3,5)$ or for $M_6(1,1,5,5)$. Then the transposition in the symmetric group $\mathfrak{S}_4$ which interchanges the two labels corresponding to a_i and a_j fixes the square-tiled cyclic cover $M_N(a_1,\dots,a_4)$. The image of a transposition under the projection (14) is again a transposition. Hence, the index of the Veech group in this case is either 3 or 1.

Let us show that index 1 is excluded. Let us suppose that the index of the Veech group is 1, that is, for all permutations $\pi \in \mathfrak{S}_4$ the flat surfaces represented by square-tiled cyclic covers $M_N(a_{\pi(1)},a_{\pi(2)},a_{\pi(3)},a_{\pi(4)})$ are isomorphic. Thus, without loss of generality we may assume that $a_2 = a_1$, and $a_3 \neq a_1$, $a_4 \neq a_1$. Then

$$\mathrm{Deck}(\sigma_1\sigma_2^{-1}) = a_1 - a_2 = 0\,.$$

This property can be formulated in a form invariant under “pillow symmetries”, namely: for at least one of the two vertical saddle connections of the “square pillow” $(\mathbb{P}^1(\mathbb{C}), q_0)$ the loop encircling first one of the corresponding singularities in the positive direction and then the other singularity in the negative direction lifts to a closed loop on $M_N(a_1,a_2,a_3,a_4)$. Clearly this property is not valid for the surface $M_N(a_1,a_3,a_2,a_4)$ and we get a contradiction.

Consider now the remaining case when all a_i are pairwise distinct.

If any nontrivial symmetry decomposes into two cycles of length 2, all the symmetries are reduced to “pillow symmetries” and by Corollary 17 the index of the Veech group of $M_N(a_1,\dots,a_4)$ is 6. This situation realizes, for example, for $M_8(1,3,5,7)$.

Let us suppose that $M_N(a_1,\dots,a_4)$ has a symmetry represented by a cycle of length 4. As an example, consider $M_{10}(1,3,9,7)$ and a symmetry corresponding to the multiplication by $k=3$. The image of such symmetry under projection (14) is a transposition. By Corollary 17 this implies that the index of the Veech group of $M_N(a_1,\dots,a_4)$ is either 3 or 1. It is left as an exercise to verify that index 1 is excluded (see Appendix B).

Let us suppose now that $M_N(a_1,\dots,a_4)$ has a symmetry represented by a cycle of length 2. As an example, consider $M_{40}(1,9,5,25)$ and a symmetry induced by multiplication by 9. A transposition is mapped by the projection (14) to a transposition. Hence, the index of the Veech group in this case is again either 3 or 1. It is left as an exercise to verify that index 1 is excluded (see Appendix B).

Finally, let us suppose that $M_N(a_1,\dots,a_4)$ has a symmetry represented by a cycle of length 3. As an example, consider $M_{14}(1,9,11,7)$ and a symmetry induced by multiplication by $k=9$. A cycle of length 3 is mapped by the projection (14) to a cycle of length 3. Hence, the index of the Veech group is either 2 or 1. If it were 1, one of the symmetries would be an odd permutation, *i.e.,* a single cycle of length 2 or 4. We have proved that the presence of such a symmetry excludes index 1. Theorem 18 and, thus, Theorem 13 are proved. □

To complete this section we note that the $\mathrm{SL}(2,\mathbb{R})$-orbit of any square-tiled surface (respectively, the $\mathrm{PSL}(2,\mathbb{R})$-orbit in the case when the flat structure is represented by a quadratic differential) inside the ambient moduli space of Abelian or quadratic differentials is closed. Its projection to the moduli space of curves is a Riemann surface with cusps, often called an *arithmetic Teichmüller curve*, see [35, 19]. Any arithmetic Teichmüller curve is a finite cover of the modular curve. Theorem 13 shows that for a square-tiled cyclic cover, the corresponding arithmetic Teichmüller curve is small: it is a 1, 2, 3, or 6-fold cover of the modular curve.

3. SUM OF LYAPUNOV EXPONENTS

3.1. **Sum of the Lyapunov exponents for a square-tiled surface.** Let us consider a Teichmüller curve $\mathscr{C}$. Each point x of $\mathscr{C}$ is represented by a Riemann surface S_x. We can consider the cohomology space of $H^1(S_x,\mathbb{R})$ as a fiber of a vector bundle H^1 over $\mathscr{C}$, called the *Hodge bundle*. Similarly one defines the bundles $H^{1,0}$ and $H^1_{\mathbb{C}}$. Note that each fiber is endowed with a natural integer lattice $H^1(S_x,\mathbb{Z})$, which enables us to identify the fibers at nearby points x_1, x_2. Hence, the bundle H^1 is endowed with a natural flat connection called the *Gauss–Manin connection*.

The Teichmüller curve $\mathscr{C}$ is endowed with a natural hyperbolic metric associated to the complex structure of $\mathscr{C}$; the total area of $\mathscr{C}$ with respect to this metric is finite. Consider a geodesic flow in this metric, and consider the monodromy of the Gauss–Manin connection in H^1 with respect to the geodesic flow on $\mathscr{C}$. We get a $2g$-dimensional symplectic cocycle. The geodesic flow on $\mathscr{C}$ is ergodic with respect to the natural finite Lebesgue measure. Let us denote by $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{2g}$ the Lyapunov exponents of the corresponding cocycle. Since the cocycle is symplectic, its Lyapunov spectrum is symmetric in the sense that $\lambda_k = -\lambda_{2g-k+1}$ for all $k = 1,\dots,2g$.

Note that $\mathscr{C}$ is isometrically immersed (usually embedded) into the corresponding moduli space of curves with respect to the hyperbolic metric on the Teichmüller curve $\mathscr{C}$ and Teichmüller metric in the moduli space. The cocycle described above is a particular case of a more general cocycle related to the Teichmüller geodesic flow on the moduli space (sometimes called the *Kontsevich–Zorich cocycle*).

The Lyapunov exponents of this cocycle are important in the study of the dynamics of flows on surfaces and of interval-exchange transformations. They were studied by many authors including A. Avila and M. Viana [3]; M. Bainbridge [4]; I. Bouw and M. Möller [6]; G. Forni [13]–[15]; A. Eskin, M. Kontsevich and A. Zorich [11, 22, 42]; W. Veech [35]; see surveys [15] and [43] for an overview. In particular, from elementary geometric arguments it follows that one always has $\lambda_1 = 1$.

We need two results from [11] concerning the sum $\lambda_1+\cdots+\lambda_g$ of all nonnegative Lyapunov exponents of the Hodge bundle H^1 along the geodesic flow on an arithmetic Teichmüller curve.

THEOREM 19 ([11]). *The sum of nonnegative Lyapunov exponents of the Hodge bundle H^1 along the geodesic flow on an arithmetic Teichmüller curve in a stratum $\mathcal{H}(m_1,\ldots,m_n)$, where $m_1+\cdots+m_n = 2g-2$, satisfies the following relation:*

$$
\begin{aligned}
(16)\quad & 1+\lambda_2+\cdots+\lambda_g \\
&= \frac{1}{12}\cdot\sum_{i=1}^{n}\frac{m_i(m_i+2)}{m_i+1} + \frac{1}{\operatorname{card}(SL(2,\mathbb{Z})\cdot S_0)}\sum_{S_i\in SL(2,\mathbb{Z})\cdot S_0}\ \sum_{\substack{\textit{horizontal}\\ \textit{cylinders } cyl_{ij}\\ \textit{such that}\\ S_i=\sqcup cyl_{ij}}}\frac{h_{ij}}{w_{ij}}.
\end{aligned}
$$

where S_0 is a square-tiled surface representing the corresponding arithmetic Teichmüller curve.

REMARK 20. Note that the sum of the top g Lyapunov exponents of the Hodge bundle coincides with a single positive Lyapunov exponent of the complex line bundle $\Lambda^g H^{1,0}$ often called *determinant* line bundle.

When an arithmetic Teichmüller curve belongs to a stratum $\mathcal{Q}(d_1,\ldots,d_n)$ of quadratic differentials, one can consider the same vector bundle H^1 as above and define the cocycle and the Lyapunov exponents exactly in the same way as for holomorphic 1-forms. By reasons which will become clear below, it is convenient to denote these Lyapunov exponents by $\lambda_1^+ \geq \cdots \geq \lambda_{2g}^+$. Since the cocycle is symplectic, we again have the symmetry $\lambda_k^+ = -\lambda_{2g-k+1}^+$ for all $k = 1,\ldots,2g$. However, for quadratic differentials one has $\lambda_1^+ < 1$ for any invariant suborbifold (in fact, for any invariant probability measure, see [14]).

Following M. Kontsevich [22], in the case of quadratic differentials one can define one more vector bundle, H^1_-, over $\mathcal{Q}(d_1,\ldots,d_n)$. Consider a pair (Riemann surface S, quadratic differential q) representing a point of a stratum of quadratic differentials $\mathcal{Q}(d_1,\ldots,d_n)$. By assumption, q is not a global square of a 1-form. There exists a canonical (possibly ramified) double cover $p_2\colon \hat{S}\to S$ such that $p_2^*q = \omega^2$, where ω is a holomorphic 1-form on $\hat{S}$. Following [22] it will be convenient to introduce the following notation. Let $\hat{g}$ be the genus of the cover $\hat{S}$. By *effective genus* we call the positive integer

$$g_{\mathit{eff}} := \hat{g} - g\ . \tag{17}$$

The cohomology space $H^1(\hat{S},\mathbb{R})$ splits into a direct sum $H^1(\hat{S},\mathbb{R}) = H^1_+(\hat{S},\mathbb{R}) \oplus H^1_-(\hat{S},\mathbb{R})$ of invariant and anti-invariant subspaces with respect to the action $H^1(\hat{S},\mathbb{R}) \to H^1(\hat{S},\mathbb{R})$ induced on cohomology by the canonical involution which commutes with the double covering map $p_2\colon \hat{S} \to S$. Note that the invariant part is canonically isomorphic to the cohomology of the underlying surface, that is, $H^1_+(\hat{S},\mathbb{R}) \simeq H^1(S,\mathbb{R})$.

We consider the subspaces $H^1_+(\hat{S},\mathbb{R})$ and $H^1_-(\hat{S},\mathbb{R})$ as fibers of natural vector bundles H^1_+ and H^1_- over $\mathcal{Q}(d_1,\dots,d_n)$. The bundle H^1_+ is canonically isomorphic to the bundle H^1. The splitting $H^1 = H^1_+ \oplus H^1_-$ is equivariant with respect to the Gauss–Manin connection. The symplectic form restricted to each summand is nondegenerate. Thus, the monodromy of the Gauss–Manin connection on H^1_+ and on H^1_- along Teichmüller geodesic flow defines two symplectic cocycles. Following the notations established above, we denote the Lyapunov exponents of the cocycle acting on H^1_+ by $\lambda^+_1 \geq \cdots \geq \lambda^+_{2g}$ and the ones of the cocycle acting on H^1_- by $\lambda^-_1 \geq \cdots \geq \lambda^-_{2g_{\mathit{eff}}}$. As always for symplectic cocycles we have the symmetries $\lambda^+_k = -\lambda^+_{2g-k+1}$ and $\lambda^-_k = -\lambda^-_{2g_{\mathit{eff}}-k+1}$. It follows from the analogous result for the case of Abelian differentials that one always has $\lambda^-_1 = 1$.

Unlike H^1_+, the vector bundle H^1_- on a stratum of quadratic differentials is not induced from a vector bundle on an underlying moduli space of curves. However, it can be descended to $\mathcal{Q}(d_1,\dots,d_n)/\mathbb{C}^*$, where $\mathbb{C}^*$ is identified with the subgroup of $\mathrm{GL}(2,\mathbb{R})$ acting on $\mathcal{Q}(d_1,\dots,d_n)$ by multiplying a quadratic differential by a nonzero constant. In particular, for any Veech surface (S,q) in $\mathcal{Q}(d_1,\dots,d_n)$ the bundle H^1_- can be descended from the $\mathrm{PSL}(2,\mathbb{R})$-orbit $\mathcal{O}$ of (S,q) in $\mathcal{Q}(d_1,\dots,d_n)$ to the corresponding Teichmüller curve $\mathcal{C} = \mathcal{O}/\mathbb{C}^*$.

THEOREM 21 ([11])**.** *The sums of all nonnegative Lyapunov exponents of the bundles H^1_+ and H^1_- along the geodesic flow on an arithmetic Teichmüller curve in a stratum $\mathcal{Q}(d_1,\dots,d_n)$, of meromorphic quadratic differentials with at most simple poles, satisfy the following relations:*

$$\lambda^+_1 + \lambda^+_2 + \cdots + \lambda^+_g = \frac{1}{24}\cdot\sum_{i=1}^{n} \frac{d_i(d_i+4)}{d_i+2} + \frac{1}{\mathit{card}(PSL(2,\mathbb{Z})\cdot S_0)} \sum_{S_i \in PSL(2,\mathbb{Z})\cdot S_0} \sum_{\substack{\text{horizontal}\\ \text{cylinders } cyl_{ij}\\ \text{such that}\\ S_i = \sqcup cyl_{ij}}} \frac{h_{ij}}{w_{ij}}. \tag{18}$$

for the Lyapunov exponents of the bundle $H^1 = H^1_+$ and

$$(19)\quad 1+\lambda_2^-+\cdots+\lambda_{g_{\mathit{eff}}}^- = \frac{1}{24}\cdot\sum_{i=1}^{n}\frac{d_i(d_i+4)}{d_i+2}+\frac{1}{4}\cdot\sum_{\substack{j \textit{ such that}\\ d_j \textit{ is odd}}}\frac{1}{d_j+2}$$
$$+\frac{1}{card(PSL(2,\mathbb{Z})\cdot S_0)}\sum_{S_i\in PSL(2,\mathbb{Z})\cdot S_0}\ \sum_{\substack{\textit{horizontal}\\ \textit{cylinders } cyl_{ij}\\ \textit{such that}\\ S_i=\sqcup cyl_{ij}}}\frac{h_{ij}}{w_{ij}}.$$

for the Lyapunov exponents of the bundle H^1_-. Here S_0 is a square-tiled surface representing the corresponding arithmetic Teichmüller curve, and g and g_{eff} are the genus and the effective genus (17) of S_0.

3.2. **Sum of the Lyapunov exponents for a square-tiled cyclic cover.** Now everything is ready to apply the results of the previous section to square-tiled cyclic covers.

THEOREM 22. *Let us consider an even integer N and a collection of odd integers a_1, a_2, a_3, a_4 satisfying the relations in formula (1).*

The sum of all nonnegative Lyapunov exponents of the Hodge bundle H^1 along the geodesic flow on the arithmetic Teichmüller curve of the square-tiled cyclic cover $M_N(a_1, a_2, a_3, a_4)$ is expressed by the formula below:

$$(20)\quad 1+\lambda_2+\cdots+\lambda_g = \frac{N}{6}-\frac{1}{6N}\sum_{i=1}^{4}\gcd^2(N,a_i)$$
$$+\frac{1}{6N}\Big(\gcd^2(N,a_1+a_2)+\gcd^2(N,a_1+a_3)+\gcd^2(N,a_1+a_4)\Big)$$

Proof. We apply formula (16) taking into account the following data. The singularity pattern $(m_1,\ldots,m_n)$ of the holomorphic 1-form corresponding to the square-tiled cyclic cover $M_N(a_1,a_2,a_3,a_4)$ is computed in Lemma 2, see formula (5). The $\mathrm{SL}(2,\mathbb{Z})$-orbit of the square-tiled surface $M_N(a_1,a_2,a_3,a_4)$ is described by Theorem 18 and the cylinder decomposition for each square-tiled surface in the orbit is given in Lemma 9. By plugging the above data in formula (16) we obtain formula (20). □

THEOREM 23. *Let us consider integers N and $a_1,\ldots,a_4$ satisfying the relations in formula (1). Let us suppose, in addition, that either N is odd, or N is even and at least one of a_i, $i=1,2,3,4$, is also even.*

The sum of all nonnegative Lyapunov exponents of the bundle H^1_+ along the geodesic flow on the arithmetic Teichmüller curve of the square-tiled cyclic cover

$M_N(a_1,a_2,a_3,a_4)$ is expressed by the following formula:

$$\begin{aligned}(21)\quad \lambda_1^+ + \lambda_2^+ + \cdots + \lambda_g^+ &= \frac{N}{6} - \frac{1}{6N}\sum_{i=1}^{4}\gcd^2(N,a_i)\\ &+ \frac{1}{6N}\big(\gcd^2(N,a_1+a_2)+\gcd^2(N,a_1+a_3)+\gcd^2(N,a_1+a_4)\big)\end{aligned}$$

The sum of all nonnegative Lyapunov exponents of the bundle H^1_- along the geodesic flow on the arithmetic Teichmüller curve of the square-tiled cyclic cover $M_N(a_1,a_2,a_3,a_4)$ is expressed by the following formula:

$$\begin{aligned}(22)\quad 1+\lambda_2^- + \cdots + \lambda_{g_{\mathit{eff}}}^- &= \frac{N}{6}\\ &+ \frac{1}{12N}\sum_{\substack{i \text{ such that}\\ \frac{N}{\gcd(N,a_i)} \text{ is odd}}}\gcd^2(N,a_i) - \frac{1}{6N}\sum_{\substack{i \text{ such that}\\ \frac{N}{\gcd(N,a_i)} \text{ is even}}}\gcd^2(N,a_i)\\ &+ \frac{1}{6N}\big(\gcd^2(N,a_1+a_2)+\gcd^2(N,a_1+a_3)+\gcd^2(N,a_1+a_4)\big)\end{aligned}$$

Proof. We apply formulae (18) and (19) taking into account the following data. The singularity pattern $(d_1,\ldots,d_n)$ of the quadratic differential corresponding to the square-tiled cyclic cover $M_N(a_1,a_2,a_3,a_4)$ is computed in Lemma 2, see (6). The $\mathrm{PSL}(2,\mathbb{Z})$-orbit of the square-tiled cyclic surface $M_N(a_1,a_2,a_3,a_4)$ is described by Theorem 13 and a cylinder decomposition for each square-tiled surface in the orbit is given in Lemma 9. By plugging the above data in (18) and (19) we obtain (21) and (22) respectively. □

REMARK 24. Actually, the Hodge bundles $H^{1,0}$ and $H^1_{\mathbb{C}}$ over the Teichmüller curve of a square-tiled cyclic cover have an explicit decomposition into a direct sum of one- and two-dimensional vector subbundles, see [5]. A similar decomposition, used also in [6, 28] and [18], enables, in particular, to compute explicitly all *individual* Lyapunov exponents for any square-tiled cyclic cover, see [12].

3.3. **Degenerate Lyapunov spectrum.** In this section we list all examples of arithmetic Teichmüller curves coming from square-tiled cyclic covers with *maximally degenerate* Lyapunov spectra. We recall that by elementary geometric reasons in strata of Abelian differentials λ_1 is equal to one, while in strata of quadratic differentials (which are not squares) λ_1^- is equal to one, for any ergodic invariant measure. Thus, for strata of Abelian differentials we speak of "maximally degenerate spectrum" whenever $\lambda_2 = \cdots = \lambda_g = 0$, while for strata of quadratic differentials (which are not squares) we speak of "maximally degenerate spectrum" of λ^--exponents, whenever $\lambda_2^- = \cdots = \lambda_{g_{\mathit{eff}}}^- = 0$ and "maximally degenerate spectrum" of λ^+-exponents, whenever $\lambda_1^+ = \cdots = \lambda_g^+ = 0$.

3.3.1. *Abelian Differentials.* We start with square-tiled cyclic covers $M_N(a_1,a_2,a_3,a_4)$, which give rise to holomorphic 1-forms. By Lemma 2 this corresponds to even N and odd a_i, $i=1,2,3,4$.

THEOREM 25. *The cyclic cover $M_2(1,1,1,1)$ has genus one, so there is a single nonnegative Lyapunov exponent λ_1 of the Hodge bundle H^1 along the geodesic flow on the arithmetic Teichmüller curve of this cyclic cover; as always $\lambda_1=1$.*

For the arithmetic Teichmüller curves corresponding to the square-tiled cyclic covers

$$M_4(1,1,1,1) \simeq M_4(3,3,3,3) \quad \textit{and} \quad M_6(1,1,1,3) \simeq M_6(5,5,5,3)$$

the Lyapunov spectrum is maximally degenerate, that is $\lambda_2=\cdots=\lambda_g=0$.

For all other cyclic covers of the form $M_N(a_1,a_2,a_3,a_4)$ with even N and odd a_i, $i=1,2,3,4$, one has $\lambda_2>0$.

REMARK 26. The fact that the Lyapunov spectrum of $M_4(1,1,1,1)$ is maximally degenerate was discovered by G. Forni in [15] using a symmetry argument. Later G. Forni and C. Matheus discovered by the same approach that the Lyapunov spectrum of $M_6(1,1,1,3)$ is also maximally degenerate, see [17] (and also [18]).

REMARK 27. M. Möller [30] has an independent and by far stronger result showing that the two aforementioned examples of arithmetic Teichmüller curves with a maximally degenerate Lyapunov spectrum are really exceptional; see Conjecture 29 and Remarks 30 and 31 below.

Proof. Applying formula (20) to $M_4(1,1,1,1)$ and $M_6(1,1,1,3)$ we get a relation $1+\lambda_2+\cdots+\lambda_g=1$. Since $\lambda_2\ge\cdots\ge\lambda_g\ge 0$, this implies that, actually, $\lambda_2=\cdots=\lambda_g=0$.

It remains to prove that for all other collections $N,a_1,\dots,a_4$ the right-hand side of formula (20) is strictly greater than 1. Applying formula (20) we see that this statement is valid for the remaining two collections for $N=4$. Now we can assume that $N\ge 6$.

Since $\gcd(N,a_i)$ is a divisor of N, and $1\le a_i<N$, we conclude that $\gcd(N,a_i)$ might be $N/2$, $N/3$ or less. Hence, we always have

$$\sum_{i=1}^{4}\gcd{}^2(N,a_i)\le N^2\ .$$

This implies that if we have $\gcd(N,a_i)=\gcd(N,a_j)=N/2$ for two distinct indices $i\ne j$, then $a_i=a_j=N/2$ and at least one of the summands in

$$\gcd{}^2(N,a_1+a_2)+\gcd{}^2(N,a_1+a_3)+\gcd{}^2(N,a_1+a_4)$$

is equal to N^2. This means that the expression on the right of (20) is strictly greater than 1.

Thus, we can assume that there is at most one a_i such that $\gcd(N, a_i) = N/2$, while for the other indices $j \neq i$ we have $\gcd(N, a_j) \leq N/3$. Then

$$\frac{N}{6} - \frac{1}{6N}\sum_{i=1}^{4} \gcd^2(N, a_i) \geq \frac{5N}{72}\ .$$

For $N \geq 16$ we have $5N/72 > 1$, and hence, for $N \geq 16$ the expression on the right of (20) is strictly greater than 1.

It remains to consider finite number of arrangements $N, a_1, \ldots, a_4$, with $6 \leq N \leq 14$. This can be done either by a straightforward check, or by considerations similar to the ones as above. □

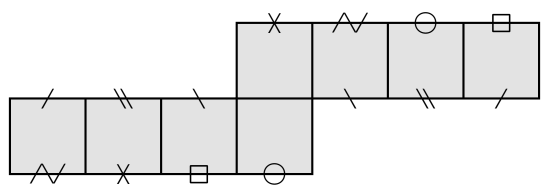

FIGURE 6. Eierlegende Wollmilchsau

REMARK 28. Figure 6 presents the square-tiled cyclic cover $M_4(1,1,1,1)$. This surface is also a 2-fold cover over a torus branched at four points. It has plenty of unusual properties. In particular, T. Monteil noticed that for every saddle connection there is always a twin saddle connection with the same length and the same direction, and there are no simple saddle connections joining a singularity to itself. M. Möller proved that the Teichmüller curves of the square-tiled cyclic covers $M_4(1,1,1,1)$, presented in Figure 6, and $M_6(1,1,1,3)$, presented in Figure 3, are also Shimura curves [30].

CONJECTURE 29. *The only Teichmüller curves in the strata of Abelian differentials in genus $g \geq 2$ with maximally degenerate Lyapunov spectra, that is, such that $\lambda_2 = \cdots = \lambda_g = 0$, are the arithmetic Teichmüller curves of the cyclic coverings $M_4(1,1,1,1)$ and $M_6(1,1,1,3)$.*

REMARK 30. According to M. Möller [30], the conjecture holds in all genera different from five and for some strata in genus five. For the remaining strata in genus five the statement requires some extra verification.

PROBLEM. *Are there any closed invariant suborbifolds (of any dimension) in strata of Abelian differentials with maximally degenerate Lyapunov spectra ($\lambda_2 = \cdots = \lambda_g = 0$) different from the two Teichmüller curves presented above?*

REMARK 31. It is proved in [11] that there are no such regular $SL(2,\mathbb{R})$-invariant suborbifolds in *any* stratum of Abelian differentials of genus 7 and higher and in *some* strata in genera 5 and 6.

REMARK 32. In genus $g = 2$, M. Bainbridge [4] has proved that the second exponent λ_2 equals $1/2$ for all ergodic $SL(2,\mathbb{R})$-invariant measures supported in the stratum $\mathcal{H}(1,1)$, corresponding to two simple zeros of the holomorphic differential, and λ_2 equals $1/3$ for all ergodic $SL(2,\mathbb{R})$-invariant measures supported in the stratum $\mathcal{H}(2)$, corresponding to a double zero. In particular, in genus 2 the Lyapunov spectrum is always nondegenerate and simple. Bainbridge's result was already known conjecturally since [22] as a consequence of a formula for the sum of exponents for an $SL(2,\mathbb{R})$-invariant submanifold of the hyperelliptic locus in any stratum. Such a formula has now been proved in [11].

Thus, any regular $SL(2,\mathbb{R})$-invariant suborbifold with maximally degenerate Lyapunov spectrum might live in genera 3 and 4 and in some strata in genera 5 and 6 only.

3.3.2. *Holomorphic quadratic differentials.* Let us consider next square-tiled cyclic covers $M_N(a_1,a_2,a_3,a_4)$, which give rise to holomorphic quadratic differentials. In particular, we assume through Section 3.3.2 that inequalities in formula (1) are strict.

THEOREM 33. *A square-tiled cyclic cover* $M_4(3,2,2,1)$ *in the stratum* $\mathcal{Q}(2,2)$ *has effective genus one, so there is a single nonnegative Lyapunov exponent* λ_1^- *of the vector bundle* H^1_- *along the geodesic flow on the arithmetic Teichmüller curve of this cyclic cover; as always* $\lambda_1^- = 1$.

The Lyapunov spectrum of the vector bundle H^1_- *along the geodesic flow on the arithmetic Teichmüller curves of the following cyclic covers*

- $M_5(2,1,1,1) \simeq M_5(4,2,2,2) \simeq M_5(1,3,3,3) \simeq M_5(3,4,4,4)$ *in* $\mathcal{Q}(3,3,3,3)$*;*
- $M_6(5,3,2,2) \simeq M_6(1,3,4,4)$ *in the stratum* $\mathcal{Q}(4,1,1,1,1)$*;*
- $M_8(4,2,1,1) \simeq M_8(4,6,3,3) \simeq M_8(4,2,5,5) \simeq M_8(4,6,7,7)$ *and* $M_8(7,4,3,2) \simeq M_8(1,4,5,6)$ *in the stratum* $\mathcal{Q}(6,6,2,2)$

is maximally degenerate, that is, $\lambda_2^- = \cdots = \lambda_{g_{\mathit{eff}}}^- = 0$.

For all other cyclic covers of the form $M_N(a_1,a_2,a_3,a_4)$ *with odd* N*, or with even* N *and at least one even* a_i, $i = 1,2,3,4$*, one has* $g_{\mathit{eff}} \geq 2$ *and* $\lambda_2^- > 0$*. Here we assume that* $0 < a_i < N$ *for all* $i = 1,2,3,4$.

Proof. Applying formula (22) to the cyclic covers from the list given in the Theorem we check that $1 + \lambda_2^- + \cdots + \lambda_{g_{\mathit{eff}}}^- = 1$. Since $\lambda_2^- \geq \cdots \geq \lambda_{g_{\mathit{eff}}}^- \geq 0$ this proves the equalities $\lambda_2^- = \cdots = \lambda_{g_{\mathit{eff}}}^- = 0$ in the cases mentioned above. The remaining part of the proof is completely analogous to the one of Theorem 25. □

PROBLEM. *Are there any other Teichmüller curves in a stratum of holomorphic quadratic differentials with effective genus at least two and with maximally degenerate Lyapunov spectrum of the bundle* H^1_-*?*

Are there closed invariant submanifolds (suborbifolds) of dimension greater than one satisfying this property?

REMARK 34. The only holomorphic quadratic differentials in genus one are the squares of holomorphic 1-forms. In genus two the strata $\mathcal{Q}(4)$ and $\mathcal{Q}(3,1)$ are empty, see [26]. The stratum $\mathcal{Q}(2,2)$ in genus two has effective genus one. The remaining two strata, namely $\mathcal{Q}(2,1,1)$ and $\mathcal{Q}(1,1,1,1)$, have effective genera two and three respectively; both of them are hyperelliptic, see [24]. It is proved in [11] that one has $\lambda_2^- = 1/3$ for any regular PSL(2,$\mathbb{R}$)-invariant suborbifold in $\mathcal{Q}_1(2,1,1)$ and one has $\lambda_2^- + \lambda_3^- = 2/3$ for any regular PSL(2,$\mathbb{R}$)-invariant suborbifold in $\mathcal{Q}_1(1,1,1,1)$.

It is proved in [11] that there are no regular PSL(2,$\mathbb{R}$)-invariant suborbifolds with maximally degenerate Lyapunov spectra on the bundle H^1_- in *any* stratum of holomorphic quadratic differentials of genus 7 and higher and in *some* strata in genera 5 and 6. Thus, if the answer to Problem 3.3.2 is affirmative, the corresponding invariant suborbifold should correspond to genera 3 or 4 or to some particular strata in genera 5 or 6.

A formula in [11], which generalizes formula (18), shows that one always has $\lambda_1^+ > 0$, hence the spectrum of Lyapunov exponents of the subbundle H^1_+ over an PSL(2,$\mathbb{R}$)-invariant submanifold in a stratum of *holomorphic* quadratic differentials can never be maximally degenerate. Theorem 35 in the section below shows that the situation is different for strata of *meromorphic* quadratic differentials with at most simple poles.

3.3.3. *Meromorphic quadratic differentials.* Let us consider square-tiled cyclic covers $M_N(a_1,a_2,a_3,a_4)$ which give rise to meromorphic quadratic differentials with simple poles. This case corresponds to collections $(N;a_1,a_2,a_3,a_4)$ of parameters where at least one of the a_i's is equal to N.

THEOREM 35. *If $a_i = N$ for at least one of $i = 1,2,3,4$, then the Lyapunov spectrum of the vector bundle H^1_+ along the geodesic flow on the arithmetic Teichmüller curve of the cyclic cover $M_N(a_1,a_2,a_3,a_4)$ is maximally degenerate: all Lyapunov exponents λ_i^+ are equal to zero.*

Proof. Without loss of generality we may assume that $a_1 = N$. Applying formula (21) we get:

$$\begin{aligned}\lambda_1^+ + \lambda_2^+ + \cdots + \lambda_g^+ &= \frac{N}{6} - \frac{\gcd^2(N,N)}{6N} - \frac{1}{6N}\sum_{i=2}^{4}\gcd^2(N,a_i)\\ &+ \frac{1}{6N}\Big(\gcd^2(N,N+a_2)+\gcd^2(N,N+a_3)+\gcd^2(N,N+a_4)\Big)\\ &= -\frac{1}{6N}\sum_{i=2}^{4}\gcd^2(N,a_i)+\frac{1}{6N}\sum_{i=2}^{4}\gcd^2(N,a_i) = 0\end{aligned}$$

□

THEOREM 36. *The square-tiled cyclic cover $M_2(2,2,1,1)$ belongs to the stratum $\mathcal{Q}(-1^4)$. There is a single nonnegative Lyapunov exponent $\lambda_1^- = 1$ of the vector bundle H^1_- along the geodesic flow on the arithmetic Teichmüller curve of this cyclic cover and no other Lyapunov exponents.*

The Lyapunov spectrum of the vector bundle $H^1_- \oplus H^1_+$ along the geodesic flow on the arithmetic Teichmüller curve of the following cyclic covers in genus one

- $M_3(3,1,1,1) \simeq M_3(3,2,2,2)$ *in the stratum* $\mathcal{Q}(1^3,-1^3)$*;*
- $M_4(4,2,1,1) \simeq M_4(4,2,3,3)$ *in the stratum* $\mathcal{Q}(2^2,-1^4)$

is maximally degenerate, that is, $\lambda_2^- = \cdots = \lambda_{g_{\mathit{eff}}}^- = 0$ *and* $\lambda_1^+ = 0$.

For all other cyclic covers of the form $M_N(N,a_2,a_3,a_4)$ *one has* $g_{\mathit{eff}} \geq 2$ *and* $\lambda_2^- > 0$.

Proof. Without loss of generality we may assume that $a_1 = N \geq 3$. Formula (22) applied to this case can be rewritten as follows:

$$\begin{aligned} 1+\lambda_2^- + \cdots + \lambda_{g_{\mathit{eff}}}^- &= \frac{N}{4} \\ &+ \frac{1}{12N} \sum_{\substack{i\geq 2 \text{ such that} \\ \frac{N}{\gcd(N,a_i)} \text{ is odd}}} \gcd^2(N,a_i) - \frac{1}{6N} \sum_{\substack{i \text{ such that} \\ \frac{N}{\gcd(N,a_i)} \text{ is even}}} \gcd^2(N,a_i) \\ &+ \frac{1}{6N}\Big(\gcd^2(N,a_2)+\gcd^2(N,a_3)+\gcd^2(N,a_4)\Big) \geq \frac{N}{4} \end{aligned}$$

Hence, for $N > 4$ we get $\lambda_2^- > 0$. Applying the above formula to remaining data for $N = 2,3,4$ we complete the proof of the Theorem. □

REMARK 37. The maximally degenerate examples in Theorem 36 have the following origin. For each of the cyclic covers as above consider a canonical ramified double cover such that the induced quadratic differential is a global square of a holomorphic 1-form. The resulting square-tiled surface is isomorphic to the square-tiled cyclic cover $M_6(3,1,1,1)$ for the double cover over $M_3(3,1,1,1)$ and also to the square-tiled cyclic cover $M_4(1,1,1,1)$ for the double cover over $M_4(4,2,1,1)$. Thus the statement of Theorem 36 for such examples follows immediately from Theorem 25.

PROBLEM. *Are there other Teichmüller curves in strata of meromorphic quadratic differentials with at most simple poles with effective genus at least two and with maximally degenerate Lyapunov spectrum of the bundle* H^1_+*? Same question for the bundle* H^1_-*? For both bundles* H^1_- *and* H^1_+ *simultaneously?*

Are there closed invariant submanifolds (suborbifolds) of dimension greater than one satisfying this property?

APPENDIX A. PARITY OF THE SPIN STRUCTURE

The ambient stratum $\mathcal{H}(1,1,1,1)$ for the Eierlegende Wollmilchsau representing the square-tiled $M_4(1,1,1,1)$ is connected. The ambient stratum $\mathcal{H}(2,2,2)$ for the square-tiled surface corresponding to the cyclic cover $M_6(1,1,1,3)$ contains two connected components representing even and odd parity of the spin-structure of the corresponding holomorphic 1-form, see [23]. The corresponding parity of the spin-structure of the square-tiled cyclic cover $M_6(1,1,1,3)$ was computed in [27] by combinatorial methods. We present another calculation to illustrate an alternative analytic technique, which is less known in the dynamical community.

PROPOSITION 38. *A holomorphic* 1*-form* ω *defining a square-tiled surface associated to the cyclic cover* $M_6(1,1,1,3)$ *has even parity of the spin structure, hence*

$$(M_6(1,1,1,3),\omega) \in \mathcal{H}^{even}(2,2,2)\ .$$

Proof. Let us consider a holomorphic 1-form ω with zeroes of even degrees only, that is, with a pattern of zeroes of the form $(2d_1,\ldots,2d_n)$ and let $K(\omega) = 2d_1P_1+\cdots+2d_nP_n$ be its zero divisor. An equivalent definition of the *parity of the spin structure* $\phi(\omega)$ associated to ω is the dimension of the space of holomorphic 1-forms with zeroes of degrees at least $d_1,\ldots,d_n$ at $P_1,\ldots,P_n$ respectively, computed modulo 2, that is,

$$\phi(\omega) := \dim\left|\frac{1}{2}K(\omega)\right| + 1 \pmod 2$$

(see [1, 21, 29, 31] for more information on the spin-structure). Let us compute the above dimension for the holomorphic 1-form ω defining the square-tiled flat structure $\omega^2 = p^*q_0$ on $M_6(1,1,1,3)$.

We recall that $M_6(1,1,1,3)$ is defined by the equation

$$w^6 = (z-z_1)(z-z_2)(z-z_3)(z-z_4)^3\ .$$

Let us consider the following 1-forms on $M_6(1,1,1,3)$,

$$\begin{aligned} \alpha(c_1,c_2) &= (c_1z+c_2)(z-z_4)^2\frac{dz}{w^5} \qquad c_1,c_2 = const\\ \beta &= (z-z_4)\frac{dz}{w^4}\\ \gamma &= (z-z_4)\frac{dz}{w^3} \end{aligned} \tag{23}$$

We claim that all these forms are holomorphic (in the case of $\alpha(c_1,c_2)$, it is holomorphic for all values of parameters $c_1,c_2 \in \mathbb{C}$). For example, let us check this for γ. When $w \neq 0$ and $z \neq \infty$, the form γ is clearly holomorphic. In a neighborhood of any of the ramification points P_i, where $i = 1,2,3$, we have $w^6 \sim (z-z_i)$, so $dz \sim w^5\,dw$, and with respect to a local coordinate w in a neighborhood of P_4, we get $\gamma \sim w^2\,dw$. This shows that γ has a double zero at each of the points P_1,P_2,P_3. In a neighborhood of z_4 we have $w^2 \sim (z-z_4)$, so with respect to a local coordinate w we get $dz = w\,dw$, and hence $\gamma \sim w^2\dfrac{w\,dw}{w^3} = dw$. Hence, each of the three preimages of z_4 on our Riemann surface is a regular point of γ. Finally, choosing a local coordinate $t = 1/z$ in a neighborhood of $z = \infty$ we see that $(z-z_i) \sim t^{-1}$, $dz \sim t^{-2}\,dt$, and $w \sim t^{-1}$, so $\gamma \sim dt$. Hence, any preimage of $z = \infty$ is a regular point of γ. In a similar way we can check that $\alpha(c_1,c_2)$ and β are holomorphic (see [5, 6, 12] for more details). Clearly,

$$\alpha(c_1,c_2) + c_3\beta + c_4\gamma \tag{24}$$

is identically zero if and only if $c_1 = c_2 = c_3 = c_4 = 0$. By formula (9), we know that the genus of $M_6(1,1,1,3)$ is equal to 4. Hence, we have constructed a basis of the space of holomorphic 1-forms on $M_6(1,1,1,3)$: every holomorphic 1-form can be represented as a linear combination (24), and this representation is unique.

Note that the forms α,β,γ are eigenforms of the deck transformation $(z,w)\mapsto(z,\zeta w)$, where ζ is a sixth primitive root of unity, $\zeta^6 = 1$. The corresponding eigenvalues are ζ,ζ^2,ζ^3. In particular, since $\zeta^3 = -1$, the form γ is anti-invariant with respect to a generator of the group of deck transformations. Our calculation shows that the eigenspace corresponding to the eigenvalue -1 is one-dimensional. Hence, the form γ differs from the holomorphic 1-form ω defining our square-tiled flat structure on $M_6(1,1,1,3)$ only by a (nonzero) multiplicative constant.

Assume that c_1, c_2 are not simultaneously equal to zero. When $-c_2/c_1$ satisfies $-c_2/c_1 \notin \{z_1,z_2,z_3,z_4,\infty\}$, the holomorphic 1-form $\alpha(c_1,c_2)$ has six simple zeroes: one at each of the six preimages of the root of the polynomial (c_1z+c_2). When $c_1 = 0$, the holomorphic 1-form $\alpha(c_1,c_2)$ also has six simple zeroes: one at each of the six preimages of $z=\infty$. When $-c_2/c_1 = z_i$, where $i = 1,2,3$, the form $\alpha(c_1,c_2)$ has a single zero of degree 6 at P_i. Finally, when $-c_2/c_1 = z_4$, the form $\alpha(c_1,c_2)$ has three zeroes of degree two: one at each of the three preimages of $z = z_4$.

The holomorphic form β has 6 simple zeroes: one at each of the six ramification points $z = z_i$, where $i = 1,2,3,4$.

Our consideration implies that a linear combination (24) has a zero at *each* ramification point P_1,P_2,P_3 if and only if $\alpha(c_1,c_2)$ is not present in our linear combination (*i.e.,* if and only if $c_1 = c_2 = 0$). This means that

$$\phi(\omega) = \phi(\gamma) = \dim \mathrm{Vect}(\beta,\gamma) = 2 \equiv 0 \pmod 2$$

and our Lemma is proved. □

APPENDIX B. AN EXERCISE IN ARITHMETICS

In this appendix we present an exercise mentioned in the proof of Theorem 18. We show that if a square-tiled cyclic cover $M_N(a_1,a_2,a_3,a_4)$ has a symmetry represented by a single cycle of length 4 or 2, then the index of the Veech group is different from 1.

We start with a symmetry represented by a cycle of length 4. If the index of the Veech group is 1, then for all permutations $\pi\in\mathfrak{S}_4$ the flat surfaces represented by square-tiled cyclic covers $M_N(a_{\pi(1)},a_{\pi(2)},a_{\pi(3)},a_{\pi(4)})$ are isomorphic. By Lemma 9 this implies that for any $i \neq j$ and any $m \neq l$, where $i,j,m,l \in \{1,2,3,4\}$ one has

$$\gcd(N,a_i+a_j) = \gcd(N,a_m+a_l)\,. \tag{25}$$

Since the symmetry group is the entire $\mathfrak{S}_4$, we may assume that the ramification points are numbered in such way that the cycle π acts as

$$a_1 \to a_2 \to a_3 \to a_4 \to a_1\,,$$

that is

$$a_i = a_1 \cdot k^{i-1} \pmod N, \text{ where } i = 1,2,3,4\,.$$

Conditions (1) imply that $\gcd(a_1, N) = 1$. Hence,

$$\begin{aligned}
&\gcd(a_1+a_2, N) = \gcd(a_1\cdot(1+k), N) = \gcd(1+k, N)\\
&\gcd(a_1+a_3, N) = \gcd(a_1\cdot(1+k^2), N) = \gcd(1+k^2, N) \qquad (26)\\
&\gcd(a_2+a_3, N) = \gcd(a_1\cdot(k+k^2), N) = \gcd(k+k^2, N)\,.
\end{aligned}$$

By (25) we have

$$\gcd(a_1+a_2, N) = \gcd(a_1+a_3, N) = \gcd(a_2+a_3, N) := d\,.$$

Clearly

$$\gcd((a_2+a_3)-(a_1+a_3)), N) = \gcd(a_1\cdot(k-1), N) = \gcd(k-1, N)$$

is divisible by d. Since $\gcd(k+1, N) = d$ we conclude that $2 = (k+1)-(k-1)$ is divisible by d, and, hence, d is either 1 or 2.

By (1) $a_1+a_2+a_3+a_4$ is divisible by N. On the other hand,

$$\begin{aligned}
\gcd(a_1+a_2+a_3+a_4, N) = \gcd\big((a_1(1+k)(1+k^2), N\big) =&\\
= \gcd\big((1+k)(1+k^2), N\big)&
\end{aligned}$$

divides the product $\gcd(1+k, N)\cdot\gcd(1+k^2, N)$ which is equal to d^2 by (26). Hence, N is one of the integers $1,2,4$, which contradicts the assumption that all a_i, a_j are pairwise distinct.

Let us suppose now that $M_N(a_1,\dots,a_4)$ has a symmetry represented by a cycle of length 2 and let us show that the Veech group cannot have index 1. Since the symmetry group is the entire $\mathfrak{S}_4$, without loss of generality we may assume that

$$\begin{aligned}
k\cdot a_1 &\pmod N = a_1\\
k\cdot a_2 &\pmod N = a_2\\
k\cdot a_3 &\pmod N = a_4\\
k\cdot a_4 &\pmod N = a_3
\end{aligned}$$

Let $\ell := N/\gcd(k-1, N)$. Since $(k-1)\cdot a_1 \pmod N = 0$ and $a_1 \le N$, $1 < k < N$, we have $\ell > 1$, and ℓ divides a_1. Similarly, ℓ divides a_2. Hence ℓ divides a_1+a_2 and thus $\gcd(a_1+a_2, N)$. We have seen that, when the index of the Veech group is 1, the relations in formula (25) hold, in particular

$$\gcd(a_1+a_2, N) = \gcd(a_1+a_3, N)\,.$$

Since ℓ is a divisor of both $\gcd(a_1+a_3, N)$ and a_1, it divides a_3, hence it also divides $a_4 = k\cdot a_3 \pmod N$. Thus, ℓ divides $\gcd(N, a_1, a_2, a_3, a_4)$. Since $\ell > 1$, this contradicts the second condition in formula (1).

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REFERENCES

[1] M. Atiyah, *Riemann surfaces and spin structures*, Ann. scient. ÉNS (4), **4** (1971), 47–62.

[2] A. Avila and G. Forni, *Weak mixing for interval exchange transformations and translation flows*, Ann. of Math., **165** (2007), 637–664.

[3] A. Avila and M. Viana, *Simplicity of Lyapunov spectra: Proof of the Zorich-Kontsevich conjecture*, Acta Math., **198** (2007), 1–56.

[4] M. Bainbridge, *Euler characteristics of Teichmüller curves in genus two*, Geometry and Topology, **11** (2007), 1887–2073.

[5] I. Bouw, *The p-rank of ramified covers of curves*, Compositio Math., **126** (2001), 295–322.

[6] I. Bouw and M. Möller, *Teichmüller curves, triangle groups and Lyapunov exponents*, Ann. of Math. (2), **172** (2010), 139–185.

[7] A. Bufetov, *Hölder cocycles and ergodic integrals for translation flows on flat surfaces*, Electron. Res. Announc. Math. Sci., **17** (2010), 34–42.

[8] ______, *Limit theorems for translation flows*, (2010), 1–69, arXiv:0804.3970v3.

[9] D. Chen, *Covers of the projective line and the moduli space of quadratic differentials*, (2010), 1–19, arXiv:1005.3120v1.

[10] A. Eskin and A. Okounkov, *Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials*, Inventiones Mathematicae, **145** (2001), 59–103.

[11] A. Eskin, M. Kontsevich and A. Zorich, *Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow*, preprint, 2011.

[12] ______, *Lyapunov spectrum of square-tiled cyclic covers*, Journal of Modern Dynamics **5** (2011), 319–353.

[13] G. Forni, *Solutions of the cohomological equation for area-preserving flows on compact surfaces of higher genus*, Ann. of Math. (2), **146** (1997), 295–344.

[14] ______, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*, Annals of Math. (2), **155** (2002), 1–103.

[15] ______, *On the Lyapunov exponents of the Kontsevich-Zorich cocycle*, in "Handbook of Dynamical Systems" (eds. B. Hasselblatt and A. Katok), **1B**, Elsevier B. V., Amsterdam, 2006, 549–580.

[16] ______, *A geometric criterion for the non-uniform hyperbolicity of the Kontsevich-Zorich cocycle*, JMD, **5** (2011), 355–395.

[17] G. Forni and C. Matheus, *An example of a Teichmüller disk in genus 4 with degenerate Kontsevich-Zorich spectrum*, 2008, 1–8, arXiv:0810.0023v1.

[18] G. Forni, C. Matheus and A. Zorich, *Lyapunov spectrum of equivariant subbundles of the Hodge bundle*, in preparation.

[19] E. Gutkin and C. Judge, *Affine mappings of translation surfaces: Geometry and arithmetic*, Duke Math. J., **103** (2000), 191–213.

[20] F. Herrlich and G. Schmithüsen, *An extraordinary origami curve*, Math. Nachr., **281** (2008), 219–237.

[21] D. Johnson, *Spin structures and quadratic forms on surfaces*, J. London Math. Soc. (2), **22** (1980), 365–373.

[22] M. Kontsevich, *Lyapunov exponents and Hodge theory*, in "The Mathematical Beauty of Physics," (Saclay, 1996), Adv. Ser. Math. Phys., **24**, World Scientific Publ., River Edge, NJ, 1997, 318–332.

[23] M. Kontsevich and A. Zorich, *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*, Inventiones Mathematicae, **153** (2003), 631–678.

[24] E. Lanneau, *Hyperelliptic components of the moduli spaces of quadratic differentials with prescribed singularities*, Comment. Math. Helvetici, **79** (2004), 471–501.

[25] P. Lochak, *On arithmetic curves in the moduli spaces of curves*, J. Inst. Math. Jussieu, **4** (2005), 443–508.

[26] H. Masur and J. Smillie, *Quadratic differentials with prescribed singularities and pseudo-Anosov diffeomorphisms*, Comment. Math. Helvetici, **68** (1993), 289–307.

[27] C. Matheus and J.-C. Yoccoz, *The action of the affine diffeomorphisms on the relative homology group of certain exceptionally symmetric origamis*, Journal of Modern Dynamics, **4** (2010), 453–486.

[28] C. McMullen, *Braid groups and Hodge theory*, to appear in Math. Ann.

[29] J. Milnor, *Remarks concerning spin manifolds*, Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), Princeton Univ. Press, Princeton, NJ, 1965, 55–62.

[30] M. Möller, *Shimura and Teichmüller curves*, Journal of Modern Dynamics, **5** (2011), 1–32.

[31] D. Mumford, *Theta-characteristics of an algebraic curve*, Ann. scient. Éc. Norm. Sup. (4), **4** (1971), 181–191.

[32] G. Schmithüsen, *An algorithm for finding the Veech group of an origami*, Experimental Mathematics, **13** (2004), 459–472.

[33] R. Treviño, *On the non-uniform hyperbolicity of the Kontsevich-Zorich cocycle for quadratic differentials*, 2010, 1–25, `arXiv:1010.1038v2`.

[34] W. Veech, *Gauss measures for transformations on the space of interval exchange maps,* Ann. of Math. (2), **115** (1982), 201–242.

[35] ______, *The Teichmüller Geodesic Flow*, Annals of Mathematics (2), **124** (1986), 441–530.

[36] ______, *Moduli spaces of quadratic differentials*, J. Anal. Math., **55** (1990), 117–171.

[37] A. Wright, *Abelian square-tiled surfaces*, preprint, 2011.

[38] A. Zorich, *Asymptotic flag of an orientable measured foliation on a surface*, in "Geometric Study of Foliations," (Tokyo, 1993), World Scientific Publ., River Edge, NJ, 1994, 479–498.

[39] ______, *Finite Gauss measure on the space of interval-exchange transformations. Lyapunov exponents*, Ann. Inst. Fourier (Grenoble), **46** (1996), 325–370.

[40] ______, *Deviation for interval-exchange transformations*, Ergod. Th. & Dynam. Sys., **17** (1997), 1477–1499.

[41] ______, *Square-tiled surfaces and Teichmüller volumes of the moduli spaces of Abelian differentials*, "Rigidity in Dynamics and Geometry" (eds. M. Burger and A. Iozzi), Proceedings of the Programme on Ergodic Theory, Geometric Rigidity and Number Theory, Isaac Newton Institute for the Mathematical Sciences, Cambridge, UK, January 5–July 7, 2000, Springer-Verlag, Berlin, 2002, 459–471.

[42] ______, *How do the leaves of a closed* 1*-form wind around a surface*, in the collection: "Pseudoperiodic Topology" (eds. Vladimir Arnold, Maxim Kontsevich and Anton Zorich), AMS Translations, Ser. 2, **197**, Advances in the Mathematical Sciences, **46**, AMS, Providence, RI, 1999, 135–178.

[43] ______, *Flat surfaces*, in collection "Frontiers in Number Theory, Physics and Geometry. Vol. 1: On Random Matrices, Zeta Functions and Dynamical Systems" (eds. P. Cartier, B. Julia, P. Moussa and P. Vanhove), Ecole de physique des Houches, France, March 9–21, 2003, Springer-Verlag, Berlin, 2006, 439–586.

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