SOLVING THE TEN MARTINI PROBLEM

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ABSTRACT. We discuss the recent proof of Cantor spectrum for the almost Mathieu operator for all conjectured values of the parameters.

1. INTRODUCTION

The almost Mathieu operator (a.k.a. the Harper operator or the Hofstadter model) is a Schrödinger operator on $\ell^2(\mathbb{Z})$,

$$(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos 2\pi(\theta + n\alpha)u_n,$$

where $\lambda, \alpha, \theta \in \mathbb{R}$ are parameters (the *coupling*, the *frequency* and the *phase*). This model first appeared in the work of Peierls [21]. It arises in physics literature as related, in two different ways, to a two-dimensional electron subject to a perpendicular magnetic field [15, 23]. It plays a central role in the Thouless et al theory of the integer quantum Hall effect [27]. The value of λ of most interest from the physics point of view is $\lambda = 1$. It is called the critical value as it separates two different behaviors as far as the nature of the spectrum is concerned.

If $\alpha = \frac{p}{q}$ is rational, it is well known that the spectrum consists of the union of q intervals possibly touching at endpoints. In the case of irrational α the spectrum (which then does not depend on θ) has been conjectured for a long time to be a Cantor set for all $\lambda \neq 0$ [7]. To prove this conjecture has been dubbed the *Ten Martini problem* by Barry Simon, after an offer of Kac in 1981, see Problem 4 in [25].

In 1984 Bellissard and Simon [8] proved the conjecture for generic pairs of (λ, α). In 1987 Sinai [26] proved Cantor spectrum for a.e. α in the perturbative regime: for $\lambda = \lambda(\alpha)$ sufficiently large or small. In 1989 Helffer-Sjöstrand proved Cantor spectrum for the critical value $\lambda = 1$ and an explicitly defined generic set of α [16]. Most developments in the 90s were related to the following observation. For $\alpha = \frac{p}{q}$ the spectrum of $H_{\lambda,\alpha,\theta}$ can have at most $q - 1$ gaps. It turns out that all these gaps are open, except for the middle one for even q [20, 11]. Choi, Eliott, and Yui obtained in fact an exponential lower bound on the size of the individual gaps from which they deduced Cantor spectrum for Liouville (exponentially well approximated by the rationals) α [11]. In 1994 Last, using certain estimates of Avron, van Mouche and Simon [6], proved zero measure Cantor spectrum for a.e. α (for an explicit set that intersects with but does not contain the set in [16]) and

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$\lambda = 1$ [18]. Just extending this result to the case of all (rather than a.e.) α was considered a big challenge (see Problem 5 in[25]).

A major breakthrough came recently with an influx of ideas coming from dynamical systems. Puig, using Aubry duality [1] and localization for $\theta = 0$ and $\lambda > 1$ [13], proved Cantor spectrum for Diophantine α and any non-critical λ [22]. At about the same time, Avila and Krikorian proved zero measure Cantor spectrum for $\lambda = 1$ and α satisfying a certain Diophantine condition, therefore extending the result of Last to all irrational α [3]. The solution of the Ten Martini problem as originally stated was finally given in [2]:

Main Theorem ([2]). *The spectrum of the almost Mathieu operator is a Cantor set for all irrational α and for all $\lambda \neq 0$.*

Here we present the broad lines of the argument of [2]. For a much more detailed account of the history as well as of the physics background and related developments see a recent review [19].

While the ten martini problem was solved, a stronger version of it, dubbed by B. Simon the *Dry Ten Martini problem* is still open. The problem is to prove that all the gaps prescribed by the gap labelling theorem are open. This fact would be quite meaningful for the QHE related applications [4]. Dry ten martini was only established for Liouville α [11, 2] and for Diophantine α in the perturbative regime [22], using a theorem of Eliasson [12].

1.1. **Rough strategy.** The history of the Ten Martini problem we described shows the existence of a number of different approaches, applicable on different parameter ranges.

Denote by $\Sigma_{\lambda,\alpha}$ the union over $\theta \in \mathbb{R}$ of the spectrum of $H_{\lambda,\alpha,\theta}$ (recall that the spectrum is actually θ-independent if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$). Due to the obvious symmetry $\Sigma_{\lambda,\alpha} = -\Sigma_{-\lambda,\alpha}$, we may assume that $\lambda > 0$. Aubry duality gives a much more interesting symmetry, which implies that $\Sigma_{\lambda,\alpha} = \lambda \Sigma_{\lambda^{-1},\alpha}$. The critical coupling $\lambda = 1$ separates two very distinct regimes. The transition at $\lambda = 1$ can be clearly seen by consideration of the Lyapunov exponent $L(E) = L_{\lambda,\alpha}(E)$, for which we have the following statement.

Theorem 1.1 ([9]). *Let $\lambda > 0$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. For every $E \in \Sigma_{\lambda,\alpha}$, $L_{\lambda,\alpha}(E) = \max\{\ln \lambda, 0\}$.*

With respect to the frequency α, one can broadly distinguish two approaches, applicable depending on whether α is well approximated by rationals or not (the Liouville and the Diophantine cases):

1. In the Liouvillian region, one can try to proceed by rational approximation, exploiting the fact that a significant part of the behavior at rational frequencies is accessible by calculation (this is a very special property of the cosine potential).
2. In the Diophantine region, one can attemp to solve two small divisor problems that have been linked with Cantor spectrum.

(a) Localization (for large coupling), whose relevance to Cantor spectrum was shown in [22].
(b) Floquet reducibility (for small coupling), which is connected to Cantor spectrum in [12], [22].

Although Aubry duality relates both problems for $\lambda \neq 1$, it is important to notice that the small divisor analysis is much more developed in the localization problem, where powerful non-perturbative methods are currently available.

To decide whether α should be considered Liouville or Diophantine for the Ten Martini problem, we introduce a parameter $\beta = \beta(\alpha) \in [0, \infty]$:

$$\beta = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}, \tag{1.1}$$

where $\frac{p_n}{q_n}$ are the rational approximations of α (obtained by the continued fraction algorithm). As β grows, the Diophantine approach becomes less and less efficient, until it ceases to work, while the opposite happens for the Liouville approach.

As discussed before, those lines of attack lead to the solution of the Ten Martini problem in a very large region of the parameters, which is both generic and of full Lebesgue measure. However there is no reason to expect that one could cover the whole parameter range by this Liouville/Diophantine dichotomy. Actually our analysis seems to indicate the existence of a critical range, $\beta \leq |\ln \lambda| \leq 2\beta$, where one is close enough to the rationals to make the small divisor problems intractable (so that, in particular, localization does not hold in the full range of phases for which it holds for larger λ), but not close enough so that one can borrow their gaps.

In order to go around the (seemingly) very real issues present in the critical range, we will use a somewhat convoluted argument which proceeds by contradiction. The contradiction argument allows us to exploit the following new idea: roughly, absence of Cantor spectrum is shown to imply much better, irrealistically good estimates. Still, those "fictitious" estimates are barely enough to cover the critical range of parameters, and we are forced to push the more direct approaches close to their technical limits.

We will need to apply this trick both in the Liouvillian side and in the Diophantine side. In the Liouvillian side, it implies improved continuity estimates for the dependence of the spectrum on the frequency. In the Diophantine side, it immediately solves the "non-commutative" part of Floquet reducibility: what remains to do is to solve the cohomological equation. Unfortunately, this can not be done directly. Instead, what we pick up from the ("soft") analysis of the cohomological equation is used to complement the ("hard") analysis of localization.

In the following sections we will succesively describe the analytic extension trick, the Liouville estimates, the two aspects of the Diophantine side (reducibility and localization), and we will conclude with some aspects of the proof of localization.

2. Analytic extension

In Kotani theory, the complex analytic properties of Weyl's m-functions are used to describe the absolutely continuous component of the spectrum of an ergodic Schrödinger operator. However, it can also be interpreted as a theory about certain dynamical systems, *cocycles*.

We restrict to the case of the almost Mathieu operator. A formal solution of $H_{\lambda,\alpha,\theta}u = Eu$, $u \in \mathbb{C}^{\mathbb{Z}}$, satisfies the equation

$$\begin{pmatrix} E - 2\lambda\cos 2\pi(\theta + n\alpha) & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}. \tag{2.1}$$

Defining $S_{\lambda,E}(x) = \begin{pmatrix} E - 2\lambda\cos 2\pi x & -1 \\ 1 & 0 \end{pmatrix}$, the importance of the products $S_{\lambda,E}(\theta + (n-1)\alpha) \cdots S_{\lambda,E}(\theta)$ becomes clear. Since $S_{\lambda,E}$ are matrices in $\mathrm{SL}(2,\mathbb{C})$, which has a natural action on $\overline{\mathbb{C}}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$, this leads to the consideration of the dynamical system

$$\begin{aligned} (\alpha, S_{\lambda,E}) :& \mathbb{R}/\mathbb{Z} \times \overline{\mathbb{C}} \to \mathbb{R}/\mathbb{Z} \times \overline{\mathbb{C}} \\ & (x, w) \mapsto (x + \alpha, S_{\lambda,E}(x) \cdot w), \end{aligned} \tag{2.2}$$

which is the projective presentation of the almost Mathieu cocycle.

An invariant section for the cocycle $(\alpha, S_{\lambda,E})$ is a function $m : \mathbb{R}/\mathbb{Z} \to \overline{\mathbb{C}}$ such that $S_{\lambda,E}(x) \cdot m(x) = m(x+\alpha)$. The existence of a (sufficiently regular) invariant section is of course a nice feature, as it in a sense means that the cocycle does not see the whole complexity of the group $\mathrm{SL}(2,\mathbb{C})$: the cocycle is conjugate to a cocycle in a simpler group (of triangular matrices). The existence of two distinct invariant sections means that the simpler group is isomorphic to an even simpler, abelian group (of diagonal matrices).

It turns out that the cocycle is well behaved when E belongs to the resolvent set $\mathbb{C} \setminus \Sigma_{\lambda,\alpha}$: it is hyperbolic, which in particular means the existence of two continuous invariant sections. Moreover, the dependence of the invariant sections on E is analytic. Kotani showed that the existence of an open interval J in the spectrum where the Lyapunov exponent is zero allows one to use the Schwarz reflection principle with respect to E, and to conclude that the invariant sections can be analytically continued through J. Thus for $E \in J$, there are still two continuous invariant sections.

A crucial new idea is that those invariant sections are actually analytic also in the other variable.

Theorem 2.1. *Let $0 < \lambda \le 1$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $J \in \Sigma_{\lambda,\alpha}$ be an open interval. For $E \in J$, there exists an analytic map $B_E : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ such that*

$$B_E(x + \alpha) \cdot S_{\lambda,E}(x) \cdot B_E(x)^{-1} \in \mathrm{SO}(2,\mathbb{R}), \tag{2.3}$$

that is, $(\alpha, S_{\lambda,E})$ is analytically conjugate to a cocycle of rotations. Moreover, $(x, E) \mapsto B_E(x)$ is analytic for $(x, E) \in \mathbb{R}/\mathbb{Z} \times J$.

The proof uses the analyticity of the almost Mathieu cocycle $(\alpha, S_{\lambda,E})$ coupled with an analytic extension (Hartogs) argument.

3. The Liouvillian side

The rational approximation argument centers around two estimates, on the size of gaps for rational frequencies, and on the modulus of continuity (in the Hausdorff topology) of the spectrum as a function of the frequency.

3.1. **Gaps for rational approximants.** The best effective estimate for the size of gaps had been given in [11], which established that all gaps of $\Sigma_{\lambda,\frac{p}{q}}$ (except the central collapsed gap for q even) have size at least $C(\lambda)^{-q}$, where $C(\lambda)$ is some explicit constant (for instance, $C(1) = 8$). Such effective constants are not good enough for our argument (for instance, it is important to have $C(\lambda)$ close to 1 when λ is close to 1). On the other hand, we only need asymptotic estimates, addressing rationals $\frac{p}{q}$ approximating some given irrational frequency for which we want to prove Cantor spectrum.

Theorem 3.1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\lambda > 0$. For every $\epsilon > 0$, if $\frac{p}{q}$ is close enough to α then all open gaps of $\Sigma_{\lambda,\frac{p}{q}}$ have size at least $e^{-(|\ln\lambda|+\epsilon)q/2}$.*

It was pointed out to us by Bernard Helffer (during the Qmath9 conference) that this asymptotic estimate does not hold under the sole assumption of $q \to \infty$, as is demonstrated by the analysis of Helffer and Sjostrand, so it is important to only consider approximations of a given irrational frequency.

The proof starts as in [11], which gives a global inequality relating all bands in the spectrum. We then use the integrated density of states to get a better (asymptotic) estimate on the position of bands in the spectrum. Using the Thouless formula, we get an asymptotic estimate for the size of gaps near a given frequency α and near a given energy $E \in \Sigma_{\lambda,\alpha}$ in terms of the Lyapunov exponent $L_{\lambda,\alpha}(E)$. Theorem 1.1 then leads to the precise estimate above.

3.2. **Continuity of the spectrum.** The best general result on continuity of the spectrum was obtained in [6], 1/2-Hölder continuity. Coupled with the gap estimate for rational approximants, we get the following contribution to the Dry Ten Martini problem.

Theorem 3.2. *If $e^{-\beta} < \lambda < e^{\beta}$ then all gaps of $\Sigma_{\lambda,\frac{p}{q}}$ are open.*

Unfortunately this cannot be complemented by any Diophantine method that in one way or another requires localization, as it would miss the parameters such that $|\ln\lambda| = \beta > 0$. Indeed, by the Gordon's argument enhanced by the Theorem 1.1, for any θ operator $H_{\lambda,\alpha,\theta}$ has no exponentially decaying eigenfunctions for $\lambda \leq e^{\beta}$.

Better estimates on continuity of the spectrum were obtained by [14] in the Diophantine range, but these estimates get worse in the critical range and can not be used. What we do instead is a "fictitious" improvement based on Theorem 2.1.

Theorem 3.3. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\lambda > 0$. If $J \subset \operatorname{int} \Sigma_{\lambda,\alpha}$ is a closed interval then there exists $C > 0$ such that for every $E \in J$, and for every $\alpha' \in \mathbb{R}$, there exists $E' \in \Sigma_{\lambda,\alpha'}$ with $|E - E'| < C|\alpha - \alpha'|$.*

This estimate, Lipschitz continuity, is obtained in the range $0 < \lambda \leq 1$ using Theorem 2.1 and a direct dynamical estimate on perturbations of cocycles of rotations.

This result can be applied in an argument by contradiction:

Theorem 3.4. *If $e^{-2\beta} < \lambda < e^{2\beta}$ then $\Sigma_{\lambda,\alpha}$ is a Cantor set.*

4. The Diophantine side

The Diophantine side is ruled by small divisor considerations. Two traditional small divisor problems are associated to quasiperiodic Schrödinger operators: localization for large coupling and Floquet reducibility for small coupling. Those two problems are largely related by Aubry duality.

While originally both problems were attacked by perturbative methods (very large coupling for localization and very small coupling for reducibility, depending on specific Diophantine conditions), powerful non-perturbative estimates are now available for the localization problem. For this reason, all the effective "hard analysis" we will do will be concentrated in the localization problem. However, those estimates by themselves are insufficient. We will need an additional soft analysis argument (again analytic extension), carried out for the reducibility problem under the assumption of non-Cantor spectrum, to improve (irrealistically) the localization results.

4.1. **Reducibility.** We say that $(\alpha, S_{\lambda,E})$ is reducible if it is analytically conjugate to a constant cocycle, that is, there exists an analytic map $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ such that $B(x+\alpha) \cdot S_{\lambda,E}(x) \cdot B(x)^{-1}$ is a constant A_*.

An important idea is that $(\alpha, S_{\lambda,E})$ is much more likely to be reducible if one assumes that $E \in \operatorname{int} \Sigma_{\lambda,\alpha}$, $0 < \lambda \leq 1$. Indeed most of reducibility is taken care by Theorem 2.1, which simplifies the problem to proving reducibility for an analytic cocycle of rotations. This is a much easier task, which reduces to consideration of the classical cohomological equation

$$\phi(x) = \psi(x+\alpha) - \psi(x), \tag{4.1}$$

which can be analysed via Fourier series: one has an explicit formula for the Fourier coefficients $\hat{\psi}(k) = \frac{1}{e^{2\pi i k \alpha} - 1} \hat{\phi}(k)$. The small divisors arise when $\|q\alpha\|_{\mathbb{R}/\mathbb{Z}}$ is small, where $\|\cdot\|_{\mathbb{R}/\mathbb{Z}}$ denotes the distance to the nearest integer.

This easily takes care of the case $\beta = 0$, but for $\beta > 0$ the information given by Theorem 2.1 is not quantitative enough to conclude. The analysis of the cohomological equation gives still the following interesting qualitative information.

Theorem 4.1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $0 < \lambda \leq 1$. Assume that $\beta < \infty$. Let $\Lambda_{\lambda,\alpha}$ be the set of $E \in \Sigma_{\lambda,\alpha}$ such that $(\alpha, S_{\lambda,E})$ is reducible. If $\Lambda_{\lambda,\alpha} \cap \operatorname{int} \Sigma_{\lambda,\alpha}$ has positive Lebesgue measure then $\Lambda_{\lambda,\alpha}$ has non-empty interior.*

The proof of this theorem uses again ideas from analytic extension.

Let $N = N_{\lambda,\alpha} : \mathbb{R} \to [0,1]$ be the integrated density of states. One of the key ideas of [22] is that if $(\alpha, S_{\lambda,E})$ is reducible for some $E \in \Sigma_{\lambda,\alpha}$ such that $N(E) \in \alpha\mathbb{Z} + \mathbb{Z}$ then E is the endpoint of an open gap. The argument is particular to the cosine potential, and involves Aubry duality. It, in fact, extends to the case of any analytic function such that the dual model (which in general will be long-range) has simple spectrum.

Since an open subset of $\Sigma_{\lambda,\alpha}$ must intersect $\{E \in \Sigma_{\lambda,\alpha},\ N(E) \in \alpha\mathbb{Z} + \mathbb{Z}\}$, we immediately obtain Cantor spectrum in the entire range of $\beta = 0$ just from the reducibility considerations alone. Note that $\beta = 0$ is strictly stronger than the Diophantine condition, and we did not use any localization result. As noted above, this $\beta = 0$ result extends to quasiperiodic potentials defined by analytic functions under the condition that the Lyapunov exponent is zero on the spectrum [1] and that the dual model has simple spectrum (it is actually enough to require that spectral multiplicities are nowhere dense).

For $0 < \beta < \infty$ it follows similarly that the hypothesis of the previous theorem must fail:

Corollary 4.2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $0 < \lambda \leq 1$. If $\beta < \infty$ then $\Lambda_{\lambda,\alpha} \cap \operatorname{int} \Sigma_{\lambda,\alpha}$ has zero Lebesgue measure.*

4.2. **Localization and reducibility.** Aubry duality gives the following relation between reducibility and localization. If $E \in \Sigma_{\lambda,\alpha}$ is such that $N(E) \notin \alpha\mathbb{Z} + \mathbb{Z}$ then the following are equivalent:

(1) $(\alpha, S_{\lambda,E})$ is reducible,
(2) There exists $\theta \in \mathbb{R}$, such that $2\theta \in \pm N(E) + 2\alpha\mathbb{Z} + 2\mathbb{Z}$ and $\lambda^{-1}E$ is a localized eigenvalue (an eigenvalue for which the corresponding eigenfunction exponentially decays) of $H_{\lambda^{-1},\alpha,\theta}$.

Remark 4.1. When $N(E) \in \alpha\mathbb{Z} + \mathbb{Z}$, (1) still implies (2), but it is not clear that (2) implies (1) unless $\beta = 0$ (which covers the case treated in [22]). This is not however the main reason for us to avoid treating directly the case $N(E) \in \alpha\mathbb{Z} + \mathbb{Z}$.

Remark 4.2. The approach of [22] is to obtain a dense subset of $\{E \in \Sigma_{\lambda,\alpha},\ N(E) \in \alpha\mathbb{Z} + \mathbb{Z}\}$ for which $(\alpha, S_{\lambda,E})$ is reducible, for $0 < \lambda < 1$ and α satisfying the Diophantine condition $\ln q_{n+1} = O(\ln q_n)$, as a consequence of localization for $H_{\lambda^{-1},\alpha,0}$ and Aubry duality. Such a localization result (for $\theta = 0$) is however not expected to hold in the critical range of α, see more discussion in the next section.

[1] This condition holds for all analytic functions for sufficiently small λ (in a non-perturbative way) so that the result of [10] applies, thus by [9] $L(E)$ is zero on the spectrum for all irrational α.

Thus proving localization of $H_{\lambda^{-1},\alpha,\theta}$ for a large set of θ allows one to conclude reducibility of $(\alpha, S_{\lambda,E})$ for a large set of E. Coupled with Corollary 4.2, we get the following criterium for Cantor spectrum.

Theorem 4.3. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $0 < \lambda \leq 1$. Assume that $\beta < \infty$. If $H_{\lambda^{-1},\alpha,\theta}$ displays localization for almost every $\theta \in \mathbb{R}$ then $\Sigma_{\lambda,\alpha}$ (and hence $\Sigma_{\lambda^{-1},\alpha}$) is a Cantor set.*

5. A localization result

In order to prove the Main Theorem, it remains to obtain a localization result that covers the pairs $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda > 1$ which could not be treated by the Liouville method, namely the parameter region $\ln \lambda \geq 2\beta$.

In proving localization of $H_{\lambda,\alpha,\theta}$, two kinds of small divisors intervene,

(1) The usual ones for the cohomological equation, arising from $q \in \mathbb{Z} \setminus \{0\}$ for which $\|q\alpha\|_{\mathbb{R}/\mathbb{Z}}$ is small,
(2) Small denominators coming from $q \in \mathbb{Z}$ such that $\|2\theta + q\alpha\|_{\mathbb{R}/\mathbb{Z}}$ is small.

Notice that for any given α, a simple Borel-Cantelli argument allows one to obtain that for almost every θ the small denominators of the second kind satisfy polynomial lower bounds:

$$\|2\theta + q\alpha\|_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{(1+q)^2}. \tag{5.1}$$

When $\theta = 0$, or more generally $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$, which is the case linked to Cantor spectrum in [22], the small divisors of the second type are exactly the same as the first type.[2] When $\beta > 0$, where the small denominators of the first type can be exponentially small, $\theta = 0$ is thus much worse behaved than almost every θ, leading to a smaller range where one should be able to prove localization. More precisely, one expects that localization holds for almost every θ if and only if $\ln \lambda > \beta$, and for $\theta = 0$ if and only if $\ln \lambda > 2\beta$. Even with all the other tricks, this would leave out the parameters such that $\ln \lambda = 2\beta > 0$.

In any case, the following localization result is good enough for our purposes.

Theorem 5.1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Assume that $\ln \lambda > \frac{16}{9}\beta$. Then $H_{\lambda,\alpha,\theta}$ displays localization for almost every $\theta \in \mathbb{R}$.*

This is the most technical result of [2]. We use the general setup of [13], however our key technical procedure is quite different.

It is well known that to prove localization of $H_{\lambda,\alpha,\theta}$ it suffices to prove that all polynomially bounded solutions of $H_{\lambda,\alpha,\theta}\Psi = E\Psi$ decay exponentially.

[2] Actually there is an additional very small denominator, 0 of the second type, which leads to special considerations, but is not in itself a show stopper.

We will use the notation $G_{[x_1,x_2]}(x,y)$ for matrix elements of the Green's function $(H-E)^{-1}$ of the operator $H_{\lambda,\alpha,\theta}$ restricted to the interval $[x_1,x_2]$ with zero boundary conditions at x_1-1 and x_2+1.

It can be checked easily that values of any formal solution Ψ of the equation $H\Psi = E\Psi$ at a point $x \in I = [x_1,x_2] \subset \mathbb{Z}$ can be reconstructed from the boundary values via

$$\Psi(x) = -G_I(x,x_1)\Psi(x_1-1) - G_I(x,x_2)\Psi(x_2+1). \tag{5.2}$$

The strategy is to find, for every large integer x, a large interval $I = [x_1,x_2] \subset \mathbb{Z}$ containing x such that both $G(x,x_1)$ and $G(x,x_2)$ are exponentially small (in the length of I). Then, by using the "patching argument" of multiscale analysis, we can prove that $\Psi(x)$ is exponentially small in $|x|$. (The key property of Ψ, that it is a generalized eigenfunction, is used to control the boundary terms in the block-resolvent expansion.)

Fix $m>0$. A point $y \in \mathbb{Z}$ will be called (m,k)-*regular* if there exists an interval $[x_1,x_2]$, $x_2 = x_1+k-1$, containing y, such that

$$|G_{[x_1,x_2]}(y,x_i)| < e^{-m|y-x_i|}, \text{ and } \operatorname{dist}(y,x_i) \geq \frac{1}{40}k;\ i=1,2.$$

We now have to prove that every x sufficiently large is (m,k)-regular for appropriate m and k. The precise procedure to follow will depend strongly on the position of x with respect to the sequence of denominators q_n (we assume that $x>0$ for convenience). Let $b_n = \max\{q_n^{8/9}, \frac{1}{20}q_{n-1}\}$. Let n be such that $b_n < x \leq b_{n+1}$. We distinguish between the two cases:

1. **Resonant:** meaning $|x-\ell q_n| \leq b_n$ for some $\ell \geq 1$ and
2. **Non-resonant:** meaning $|x-\ell q_n| > b_n$ for all $\ell \geq 0$.

Theorem 5.1 is a consequence then of the following estimates:

Lemma 5.2. *Assume that θ satisfies (5.1). Suppose x is non-resonant. Let $s \in \mathbb{N}\cup\{0\}$ be the largest number such that $sq_{n-1} \leq \operatorname{dist}(x,\{\ell q_n\}_{\ell\geq 0})$. Then for any $\epsilon>0$ for sufficiently large n,*

1. *If $s \geq 1$ and $\ln\lambda > \beta$, x is $(\ln\lambda - \frac{\ln q_n}{q_{n-1}} - \epsilon, 2sq_{n-1}-1)$-regular.*
2. *If $s=0$ then x is either $(\ln\lambda-\epsilon, 2[\frac{q_{n-1}}{2}]-1)$ or $(\ln\lambda-\epsilon, 2[\frac{q_n}{2}]-1)$ or $(\ln\lambda-\epsilon, 2q_{n-1}-1)$-regular.*

Lemma 5.3. *Let in addition $\ln\lambda > \frac{16}{9}\beta$. Then for sufficiently large n, every resonant x is $(\frac{\ln\lambda}{50}, 2q_n-1)$-regular.*

Each of those estimates is proved following a similar scheme, though the proof of Lemma 5.3 needs additional bootstrapping from the proof of Lemma 5.2. All small denominators considerations are entirely captured through the following concept:

We will say that the set $\{\theta_1,\ldots,\theta_{k+1}\}$ is ϵ-*uniform* if

$$\max_{z\in[-1,1]} \max_{j=1,\dots,k+1} \prod_{\substack{\ell=1\\ \ell\neq j}}^{k+1} \frac{|z-\cos 2\pi\theta_\ell)|}{|\cos 2\pi\theta_j - \cos 2\pi\theta_\ell)|} < e^{k\epsilon} \tag{5.3}$$

The uniformity of some specific sequences can then be used to show that some $y \in \mathbb{Z}$ is regular following the scheme of [13]. In this approach, the goal is to find two non-intersecting intervals, I_1 around 0 and I_2 around y, of combined length $|I_1|+|I_2| = k+1$, such that we can establish the uniformity of $\{\theta_i\}$ where $\theta_i = \theta + (x + \frac{k-1}{2})\alpha$, $i = 1, \dots, k+1$, for x ranging through $I_1 \cup I_2$.

The actual proof of uniformity depends on the careful estimates of trigonometric products along arithmetic progressions $\theta + j\alpha$. Since $\int \ln|E - \cos 2\pi\theta| d\theta = -\ln 2$ for any $|E| \leq 1$ such estimates are equivalent to the analysis of large deviations in the appropriate ergodic theorem. A simple trigonometric expansion of (5.3) shows that uniformity involves equidistribution of the θ_i along with cumulative repulsion of $\pm\theta_i(\text{mod } 1)$'s, and thus involves both kinds of small divisors previously mentioned.

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