EXTREMAL LYAPUNOV EXPONENTS OF SMOOTH COCYCLES

ARTUR AVILA [1,2] AND MARCELO VIANA [2]

Abstract. A smooth cocycle is a skew-product map that acts by diffeomorphisms on the fibers. The smallest and largest Lyapunov exponents measure the minimum and maximum exponential rates of variation of the norm of the derivative along the fibers. We discuss conditions under which these numbers may vanish. The approach is based on a non-linear extension of a classical result of Ledrappier that we state and prove in here. The main applications in the present paper are for area preserving fiber bunched cocycles. Analyzing Lyapunov exponents as functions of the cocycle, we find that the points of discontinuity are rather rigid, even more so in the accessible case: in particular, the Oseledets decomposition must be continuous. In addition, we prove that for an open dense subset the Lyapunov exponents are different from zero and vary continuously with the cocycle.

Contents

Date: September 10, 2008.

[1] CNRS UMR 7599, Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, Boîte Postale 188, 75252 Paris Cedex 05, France.

[2] IMPA – Estrada D. Castorina 110, Jardim Botânico, 22460-320 Rio de Janeiro – Brazil.

1. Introduction

In the theory of partially hyperbolic dynamical systems, a core issue is the possible vanishing of the central Lyapunov exponents. Major progress on this question has been achieved recently in the simpler, but closely related setting of linear cocycles. In this paper we propose a unified approach to this problem, built around the notion of smooth cocycle. We develop an abstract framework and we illustrate its use in some skew-product situations. Other applications, including to partially hyperbolic systems which are not of skew-product type, are given in our joint papers with Santamaria [3] and Wilkinson [7].

In a few words, a smooth cocycle is a fiber bundle morphism that acts by diffeomorphisms on the fibers, with uniformly bounded norm. The fiber is assumed to have the structure of a Riemannian manifold. There are two important classes of examples that motivate this definition. The first one are the projectivizations

$$\mathbb{P}(L) : \mathbb{P}(\mathcal{V}) \to \mathbb{P}(\mathcal{V})$$

of linear cocycles $L : \mathcal{V} \to \mathcal{V}$ on a vector bundle $\mathcal{V}$. The second one are partially hyperbolic diffeomorphisms $f : M \to M$ admitting an invariant central foliation $\mathcal{F}^c$, which we may often view as cocycles on fiber bundles $\mathcal{E} \to M$ where the fiber of each point $x \in M$ is the corresponding central leaf.

On the one hand, there has been much recent progress on the theory of Lyapunov spectra for linear cocycles over "chaotic" transformations, dealing with such issues as the existence of non-zero Lyapunov exponents (see Bonatti, Gomez-Mont, Viana [11, 30]) or the simplicity of the spectrum (see Avila, Bonatti, Viana [5, 6, 13]), in line with the classical theory of random matrices developed by Furstenberg [20], Ledrappier [25], Guivarc'h, Raugi [22], Gol'dsheid, Margulis [21], and other authors. In a nutshell, the conclusion is that for generic linear cocycles the Lyapunov exponents are not all zero. Even more, at least in the so-called fiber bunched case, all Lyapunov exponents are generically distinct. As it turns out, the methods employed in proving these results are sufficiently non-linear that one may hope to apply them in much greater generality.

On the other hand, for partially hyperbolic systems, the understanding of the dynamics can be greatly enhanced when the Lyapunov exponents along the central direction are known not to vanish. See Alves, Bonatti, Viana [2, 12], for results on existence and uniqueness of physical measures, Shub, Wilkinson [29], concerning absolute continuity of central foliations, and Burns, Dolgopyat, Pesin [15], concerning accessibility and stable ergodicity. Baraviera, Bonatti [8] showed that, in the C^1 topology, one can perturb the system to make the sum of the central exponents non-zero. But just how common non-vanishing Lyapunov exponents are among more regular partially hyperbolic systems is not known. See Bochi, Viana [10] for a discussion of this and related topics.

Let us also mention the non-linear extension of Furstenberg theory carried out by Baxendale [9] in the special i.i.d. case. In discrete time, the stochastic flows of diffeomorphisms in [9] correspond to the particular case of smooth cocycles where the base dynamics is a Bernoulli shift and the cocycle depends only on one coordinate in shift space. The sharpest results in Baxendale [9] also assume the stationary measure to be absolutely continous on the fibers.

1.1. **Smooth cocycles.** Let $(\hat{M}, \hat{\mathcal{B}}, \hat{\mu})$ be a probability space and $\hat{f} : \hat{M} \to \hat{M}$ be a measurable transformation preserving $\hat{\mu}$. Let N be a Riemannian manifold, not necessarily complete, and let $\mathrm{Diff}^1(N)$ be endowed with a uniform C^1 norm. Let $\hat{P} : \hat{\mathcal{E}} \to \hat{M}$ be a measurable fiber bundle with fibers modeled on N. By this we mean $\hat{\mathcal{E}}$ comes with a countable system of bijections

$$\hat{P}^{-1}(U_n) \to U_n \times N \tag{1}$$

that map each fiber $\hat{\mathcal{E}}_{\hat{x}} = \hat{P}^{-1}(\hat{x})$ onto $\{\hat{x}\} \times N$, and all coordinate changes are measurable maps of the form

$$(U_m \cap U_n) \times N \to (U_m \cap U_n) \times N, \qquad (\hat{x}, \hat{\xi}) \mapsto (\hat{x}, g_{\hat{x}}(\hat{\xi})) \tag{2}$$

where $g_{\hat{x}} : N \to N$ is a diffeomorphism depending measurably on the base point $\hat{x}$ and such that both the derivative $Dg_{\hat{x}}(\hat{\xi})$ and its inverse are uniformly continuous and uniformly bounded. Then one may consider a Riemannian metric on the fibers, varying measurably with the base point, transported from N via these coordinates. This metric depends on the choice of the coordinates, but only up to a uniformly bounded factor, which does not affect the notions that follow.

A *smooth cocycle* over $\hat{f}$ is a measurable map $\hat{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ such that $\hat{P} \circ \hat{F} = \hat{f} \circ \hat{P}$, every

$$\hat{F}_{\hat{x}} : \hat{\mathcal{E}}_{\hat{x}} \to \hat{\mathcal{E}}_{\hat{f}(\hat{x})}$$

is a diffeomorphism depending measurably on $\hat{x}$, and the derivative $D\hat{F}_{\hat{x}}(\hat{\xi})$ and its inverse are uniformly bounded in norm. Then the functions

$$(\hat{x}, \hat{\xi}) \mapsto \log \|D\hat{F}_{\hat{x}}(\hat{\xi})\| \quad \text{and} \quad (\hat{x}, \hat{\xi}) \mapsto \log \|D\hat{F}_{\hat{x}}(\hat{\xi})^{-1}\|$$

are integrable, relative to any probability measure $\hat{m}$ on $\hat{\mathcal{E}}$. The *extremal Lyapunov exponents* of $\hat{F}$ at a point $(\hat{x}, \hat{\xi}) \in \hat{\mathcal{E}}$ are

$$\lambda_+(\hat{F}, \hat{x}, \hat{\xi}) = \lim_{n\to\infty} \frac{1}{n} \log \|D\hat{F}^n_{\hat{x}}(\hat{\xi})\| \,.$$
$$\lambda_-(\hat{F}, \hat{x}, \hat{\xi}) = \lim_{n\to\infty} \frac{1}{n} \log \|D\hat{F}^n_{\hat{x}}(\hat{\xi})^{-1}\|^{-1} \,.$$

The limits exist $\hat{m}$-almost everywhere if $\hat{m}$ is invariant under $\hat{F}$, by the subadditive ergodic theorem (Kingman [24]). Notice that

$$\lambda_-(\hat{F}, \hat{x}, \hat{\xi}) \le \lambda_+(\hat{F}, \hat{x}, \hat{\xi}),$$

because $\|D\hat{F}^n_{\hat{x}}(\hat{\xi})\| \|D\hat{F}^n_{\hat{x}}(\hat{\xi})^{-1}\| \ge 1$. Denote

$$\lambda_\pm = \lambda_\pm(\hat{F}, \hat{m}) = \int \lambda_\pm(\hat{F}, \hat{x}, \hat{\xi}) \, d\hat{m}(\hat{x}, \hat{\xi}).$$

If $(\hat{F}, \hat{m})$ is ergodic then $\lambda_\pm(\hat{F}, \hat{x}, \hat{\xi}) = \lambda_\pm$ for $\hat{m}$-almost every $(\hat{x}, \hat{\xi})$. Throughout, we shall only be interested in measures $\hat{m}$ that project down to μ under $\hat{P}$.

1.2. **Invariance criterion.** The main technical tool developed in this paper is a measurability criterion for the disintegration along the fibers of probability measures invariant under a cocycle. This is inspired by the main result in Ledrappier [25]: while Ledrappier's original formulation was for linear cocycles, ours applies to any deformation of a smooth cocycle, a notion that we also introduce here.

Take $(\hat{M}, \hat{\mathcal{B}}, \mu)$ to be a Lebesgue space, that is, a separable probability space which is complete mod 0. See Rokhlin [27, §2–§3]. Then any probability $\hat{m}$ on

$\hat{\mathcal{E}}$ such that $\hat{P}_*\hat{m} = \hat{\mu}$ admits a family $\{\hat{m}_{\hat{x}} : \hat{x} \in \hat{M}\}$ of probabilities such that $\hat{x} \mapsto \hat{m}_{\hat{x}}$ is $\hat{\mathcal{B}}$-measurable, every $\hat{m}_{\hat{x}}$ is supported inside the fiber $\hat{\mathcal{E}}_{\hat{x}}$ and

$$\hat{m}(E) = \int \hat{m}_{\hat{x}}(E)\, d\hat{\mu}(\hat{x})$$

for any measurable set $E \subset \hat{\mathcal{E}}$. Moreover, such a family is essentially unique. We call it the *disintegration* of $\hat{m}$ and refer to the $\hat{m}_{\hat{x}}$ as its *conditional probabilities* along the fibers.

Assume that $\hat{f}$ is invertible. A σ-algebra $\mathcal{B}_0 \subset \hat{\mathcal{B}}$ is *generating* if its iterates $\hat{f}^n(\mathcal{B}_0)$, $n \in \mathbb{Z}$ generate the whole $\hat{\mathcal{B}}$ mod 0. A *deformation of a smooth cocycle* $\hat{F}$ is a measurable transformation $\tilde{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ which is conjugated to $\hat{F}$,

$$\tilde{F} = H \circ \hat{F} \circ H^{-1},$$

by an invertible measurable map $H : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ of the form $H(\hat{x}, \hat{\xi}) = (\hat{x}, H_{\hat{x}}(\hat{\xi}))$ such that all the $H_{\hat{x}}^{-1}$, $\hat{x} \in \hat{M}$ are Hölder continuous, with uniform Hölder constants: there exist positive constants B and β such that

$$d(\hat{\xi}, \hat{\eta}) \le B d(H_{\hat{x}}(\hat{\xi}), H_{\hat{x}}(\hat{\eta}))^\beta \quad \text{for all } \hat{x} \in \hat{M} \text{ and } \hat{\xi},\, \hat{\eta} \in \mathcal{E}_{\hat{x}}. \tag{3}$$

To each $\hat{F}$-invariant probability measure $\hat{m}$ corresponds an $\tilde{F}$-invariant probability $\tilde{m} = H_*\hat{m}$, and $\hat{m}$ projects down to $\hat{\mu}$ if and only if $\tilde{m}$ does.

Theorem A. *Let $\tilde{F}$ be a deformation of a smooth cocycle $\hat{F}$. Let $\mathcal{B}_0 \subset \hat{\mathcal{B}}$ be a generating σ-algebra such that both $\hat{f}$ and $x \mapsto \tilde{F}_x$ are $\mathcal{B}_0$-measurable* mod 0*. Let $\hat{m}$ be an $\hat{F}$-invariant probability that projects down to $\hat{\mu}$. If $\lambda_-(\hat{F}, \hat{x}, \hat{\xi}) \ge 0$ for $\hat{m}$-almost every $(\hat{x}, \hat{\xi}) \in \hat{\mathcal{E}}$ then any disintegration $x \mapsto \tilde{m}_x$ of the corresponding $\tilde{F}$-invariant measure $\tilde{m}$ is $\mathcal{B}_0$-measurable* mod 0*.*

We get a dual result assuming that $\lambda_+(\hat{F}, \hat{x}, \hat{\xi}) \le 0$ for $\hat{m}$-almost every $(\hat{x}, \hat{\xi})$, and considering a σ-algebra $\mathcal{B}_0$ relative to which both maps $\hat{f}^{-1}$ and $\hat{x} \mapsto \tilde{F}_{\hat{x}}^{-1}$ are measurable mod 0. Indeed, it is clear that $\hat{F}$ has the same invariant probabilities as $\hat{F}^{-1}$, and $\tilde{F}$ is a deformation of $\hat{F}$ if and only if $\tilde{F}^{-1}$ is a deformation of $\hat{F}^{-1}$. Since

$$\lambda_+(\hat{F}, \hat{x}, \hat{\xi}) + \lambda_-(\hat{F}^{-1}, \hat{x}, \hat{\xi}) = 0,$$

the new assumption means that $\lambda_-(\hat{F}^{-1}, \hat{x}, \hat{\xi}) \ge 0$ for $\hat{m}$-almost every $(\hat{x}, \hat{\xi})$. Thus, we may apply Theorem A to the inverse cocycle, to obtain the same conclusion as before under this new assumption. See also Example 2.15 below.

Theorem B. *Let $\tilde{F}$ be a deformation of a smooth cocycle $\hat{F}$. Let $\mathcal{B}_0 \subset \hat{\mathcal{B}}$ be a generating σ-algebra such that both $\hat{f}$ and $x \mapsto \tilde{F}_x$ are $\mathcal{B}_0$-measurable* mod 0*. Let $(\hat{m}_k)_k$ be a sequence of $\hat{F}$-invariant probabilities projecting down to $\hat{\mu}$ and converging to some probability $\hat{m}$ in the weak* topology. Assume $\int \min\{0, \lambda_-(\hat{F}, \cdot)\}\, d\hat{m}_k \to 0$ when $k \to \infty$. Then any disintegration $x \mapsto \tilde{m}_x$ of the corresponding $\tilde{F}$-invariant measure $\tilde{m}$ is $\mathcal{B}_0$-measurable* mod 0*.*

Theorem A may be viewed as the special case when $m_k = m$ for all k: it is clear that $\int \min\{0, \lambda_-(\hat{F}, \cdot)\}\, d\hat{m} = 0$ if and only if $\lambda_-(\hat{F}, \cdot) \ge 0$ $\hat{m}$-almost everywhere.

1.3. **Hyperbolic homeomorphisms.** Next, we are going to derive more concrete versions of these results for continuous cocycles over hyperbolic homeomorphisms.

Let $\hat{M}$ be a complete metric space. Let $\hat{\mathcal{E}}$ be a continuous fiber bundle (the local coordinates (1) are defined on open sets and the coordinate changes (2) are homeomorphisms) where the diffeomorphisms $g_{\hat{x}}$ vary continuously with $\hat{x} \in \hat{M}$. Assume that a Riemannian metric has been chosen on each fiber, varying continuously with the base point. Moreover, let $\hat{F}$ be a smooth cocycle such that the diffeomorphisms $\hat{F}_{\hat{x}}$ vary continuously with $\hat{x} \in \hat{M}$.

We call a homeomorphism $\hat{f} : \hat{M} \to \hat{M}$ *hyperbolic* if there exist $\varepsilon > 0$, $\delta > 0$, $K > 0$, $\tau > 0$, and positive functions $\nu(\cdot)$ and $\nu_-(\cdot)$ such that

- (h1) $d(\hat{f}(\hat{y}_1), \hat{f}(\hat{y}_2)) \le \nu(\hat{x}) d(\hat{y}_1, \hat{y}_2)$ for all $\hat{y}_1, \hat{y}_2 \in W^s_\varepsilon(\hat{x})$, $\hat{x} \in \hat{M}$;
- (h2) $d(\hat{f}^{-1}(\hat{z}_1), \hat{f}^{-1}(\hat{z}_2)) \le \nu_-(\hat{x}) d(\hat{z}_1, \hat{z}_2)$ for all $z_1, z_2 \in W^u_\varepsilon(\hat{x})$, $\hat{x} \in \hat{M}$;
- (h3) $\nu_n(\hat{x}) := \nu(\hat{f}^{n-1}(\hat{x})) \cdots \nu(\hat{x}) \le K e^{-\tau n}$ for all $\hat{x} \in \hat{M}$ and $n \ge 1$;
- (h4) $\nu_{-n}(\hat{x}) := \nu_-(\hat{f}^{-n+1}(\hat{x})) \cdots \nu_-(\hat{x}) \le K e^{-\tau n}$ for all $\hat{x} \in \hat{M}$ and $n \ge 1$;
- (h5) if $d(\hat{x}_1, \hat{x}_2) \le \delta$ then $W^u_\varepsilon(\hat{x}_1)$ and $W^s_\varepsilon(\hat{x}_2)$ intersect at exactly one point, denoted $[\hat{x}_1, \hat{x}_2]$, and this point depends continuously on $(\hat{x}_1, \hat{x}_2)$;

where $W^s_\varepsilon(\hat{x})$ is the set of all $\hat{y} \in \hat{M}$ such that $d(\hat{f}^n(\hat{x}), \hat{f}^n(\hat{y})) \le \varepsilon$ for all $n \ge 0$, and $W^u_\varepsilon(\hat{x})$ is defined analogously, with $n \le 0$ instead. Then the stable and unstable sets of $\hat{x}$ are given by

$$W^s(\hat{x}) = \bigcup_{n \ge 0} \hat{f}^{-n}(W^s_\varepsilon(\hat{f}^n(\hat{x}))) \quad \text{and} \quad W^u(\hat{x}) = \bigcup_{n \ge 0} \hat{f}^{n}(W^u_\varepsilon(\hat{f}^{-n}(\hat{x}))).$$

Example 1.1. Let $\hat{f} : \hat{M} \to \hat{M}$ be the shift map on $\hat{M} = X^{\mathbb{Z}}$ where (X, d_X) is a complete metric space, and the metric $d(\cdot, \cdot)$ on $\hat{M}$ is defined by

$$d(\hat{x}, \hat{y}) = \sum_{n \in \mathbb{Z}} e^{-\tau |n|} \min\left\{1, d_X(x_n, y_n)\right\} \quad \text{for } \hat{x} = (x_n)_n \text{ and } \hat{y} = (y_n)_n.$$

Take $\varepsilon \in (0,1)$, $\delta \in (0,1)$, $K = 1$, and $\nu(\hat{x}) = \nu_-(\hat{x}) = e^{-\tau}$ for all $\hat{x} \in \hat{M}$.

Then there exist relative neighborhoods $B^s(\hat{x}) \subset W^s_\varepsilon(\hat{x})$ and $B^u(\hat{x}) \subset W^u_\varepsilon(\hat{x})$ of every $\hat{x} \in \hat{M}$ such that $\iota : (\hat{x}_1, \hat{x}_2) \mapsto [\hat{x}_1, \hat{x}_2]$ defines a homeomorphism from $B^s(\hat{x}) \times B^u(\hat{x})$ to some neighborhood $B(\hat{x})$ of every $\hat{x} \in \hat{M}$. We always consider $\hat{f}$-invariant probabilities $\hat{\mu}$ with *local product structure*: for every $\hat{x}$ in the support there exist measures μ^s and μ^u on $B^s(x)$ and $B^u(x)$, respectively, such that

$$\hat{\mu} \mid B(x) \sim \iota_*(\mu^u \times \mu^s), \tag{4}$$

meaning that the two measures have the same zero sets. This implies that the support is *su-saturated*, meaning it consists of entire stable leaves (*s-saturated* set) and of entire unstable leaves (*u-saturated* set). Moreover, $\hat{\mu}$ is locally ergodic, that is, its ergodic components are essentially open sets.

1.4. **Cocycles with holonomies.** An *s-holonomy* for $\hat{F}$ is a family h^s of β-Hölder homeomorphisms $h^s_{\hat{x},\hat{y}} : \hat{\mathcal{E}}_{\hat{x}} \to \hat{\mathcal{E}}_{\hat{y}}$, with uniform Hölder constant $\beta > 0$, defined for all $\hat{y} \in W^s(\hat{x})$ and satisfying

- (sh1) $h^s_{\hat{y},\hat{z}} \circ h^s_{\hat{x},\hat{y}} = h^s_{\hat{x},\hat{z}}$ and $h^s_{\hat{x},\hat{x}} = \mathrm{id}$
- (sh2) $\hat{F}_{\hat{y}} \circ h^s_{\hat{x},\hat{y}} = h^s_{\hat{f}(\hat{x}),\hat{f}(\hat{y})} \circ \hat{F}_{\hat{x}}$
- (sh3) $(\hat{x}, \hat{y}) \mapsto h^s_{\hat{x},\hat{y}}(\xi)$ is continuous, uniformly on ξ in any compact subset of N.

A disintegration $\{\hat{m}_{\hat{x}} : \hat{x} \in \hat{M}\}$ of an $\hat{F}$-invariant probability $\hat{m}$ is *s-invariant* if

$$(h^s_{\hat{x},\hat{y}})_* m_{\hat{x}} = m_{\hat{y}} \quad \text{for every } \hat{y} \in W^s(\hat{x}) \tag{5}$$

with $\hat{x}$ and $\hat{y}$ in the support of the projection of $\hat{m}$. Replacing $\hat{f}$ and $\hat{F}$ by their inverses, one obtains dual notions of u-holonomy h^u and u-invariant disintegration.

Theorem C. *Assume $\hat{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ admits s-holonomy and u-holonomy. Let $(\hat{m}_k)_k$ be a sequence of $\hat{F}$-invariant probability measures whose projection $\hat{\mu}$ has local product structure. Assume the sequence converges to some probability measure $\hat{m}$ in the weak* topology and $\int |\lambda_\pm(\hat{F}, \cdot)| \, d\hat{m}_k \to 0$ when $k \to \infty$. Then $\hat{m}$ admits a disintegration $\{\hat{m}_{\hat{x}} : \hat{x} \in \hat{M}\}$ which is s-invariant and u-invariant and whose conditional probabilities $\hat{m}_{\hat{x}}$ vary continuously with $\hat{x}$ on the support of $\hat{\mu}$.*

An extension for cocycles over certain partially hyperbolic maps will also be given in Theorem 4.10. A first application of Theorem C is given in the proposition that follows. It will be clear from the arguments that the hypotheses can be relaxed considerably.

Corollary D. *Let $\hat{f} : \hat{M} \to \hat{M}$ be the shift map on $\hat{M} = X^{\mathbb{Z}}$, where X is a complete metric space. Let $\hat{\mathcal{E}} = \hat{M} \times \mathbb{S}^1$ and $\hat{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ be a continuous smooth cocycle over $\hat{f}$ admitting invariant holonomies. Suppose $\hat{f}$ admits an invariant probability measure $\hat{\mu}$ and fixed points p and q in the support of $\hat{\mu}$ such that*

- *$\hat{F}_p : \mathbb{S}^1 \to \mathbb{S}^1$ has exactly two fixed points, an attractor a_p and a repeller r_p*
- *$\hat{F}_q : \mathbb{S}^1 \to \mathbb{S}^1$ has no periodic points of period less than 3.*

Then $\lambda_\pm(\hat{F}, \hat{m})$ are bounded away from zero, over all ergodic $\hat{F}$-invariant measures $\hat{m}$ that project down to $\hat{\mu}$.

1.5. **Volume preserving cocycles.** The main applications in the present paper are for area preserving cocycles over hyperbolic homeomorphisms satisfying certain partial hyperbolicity conditions.

From now on we take the fiber manifold N to be compact. Assume the continuous fiber bundle is Lipschitz, in the sense that the diffeomorphisms $g_{\hat{x}}$ in (2) depend in a Lipschitz fashion on the base point. Assume that the continuous cocycle is Lipschitz, in the sense that $\hat{F}_{\hat{x}}$ depends in a Lipschitz fashion on the point $\hat{x}$. We shall consider the following topology: two Lipschitz cocycles are close if they admit the same Lipschitz constant, they are uniformly close, and their actions on the fibers are close in the uniform C^1 norm.

Remark 1.2. For all our purposes it suffices to assume Hölder continuity, for some Hölder constant $\nu > 0$: up to replacing the metric on $\hat{M}$, one may always reduce the situation to the Lipschitz case $\nu = 1$.

We take the cocycle $\hat{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ to satisfy a normal hyperbolicity property similar to the center bunching condition of Burns, Wilkinson [16] and which was first introduced in [11] in the context of linear cocycles. We say that a Lipschitz smooth cocycle $\hat{F}$ is *dominated* if there exist $\ell \geq 1$ and $\theta < 1$ such that

$$\|(D\hat{F}^\ell_{\hat{x}}(\xi))^{-1}\| \, \nu_\ell(\hat{x}) \leq \theta \quad \text{and} \quad \|D\hat{F}^\ell_{\hat{x}}(\xi)\| \, \nu_{-\ell}(\hat{x}) \leq \theta \tag{6}$$

for every $(\hat{x}, \xi) \in \mathcal{E}$, and we say $\hat{f}$ is *fiber bunched* if, in addition to (6),

$$\|D\hat{F}^\ell_{\hat{x}}(\xi)\| \, \|(D\hat{F}^\ell_{\hat{x}}(\xi))^{-1}\| \, \nu_{\pm\ell}(\hat{x}) \leq \theta \tag{7}$$

for every $(\hat{x}, \xi) \in \mathcal{E}$. Interpretations of these conditions will be provided in Section 4. Let $\mathcal{B}(\hat{f})$ be the set of fiber bunched cocycles over $\hat{f}$. Observe that this is an open subset of Lipschitz cocycles, relative to the topology introduced above.

Let $\hat{m}_{\hat{y}}$ denote the normalized Riemannian volume on each fiber $\hat{\mathcal{E}}_{\hat{x}}$. We also take the cocycle $\hat{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ to be *volume preserving*, meaning that each

$$\hat{F}_{\hat{x}} : \hat{\mathcal{E}}_{\hat{x}} \to \hat{\mathcal{E}}_{\hat{f}(\hat{x})} \quad \text{maps } \hat{m}_{\hat{x}} \text{ to } \hat{m}_{\hat{f}(\hat{x})}.$$

Then the following probability measure $\hat{m}$ on $\hat{\mathcal{E}}$ is $\hat{F}$-invariant:

$$\hat{m}(B) = \int \hat{m}_{\hat{x}}(B \cap \hat{\mathcal{E}}_{\hat{x}})\, d\hat{\mu}(\hat{x}). \tag{8}$$

Let $\mathcal{B}_{\mathrm{vol}}(\hat{f})$ denote the subset of volume preserving fiber bunched cocycles.

1.6. **Continuity of Lyapunov exponents.** From now on we take $\hat{\mathcal{E}}$ to be compact and the fiber N to be a surface. Area preserving yields $\lambda_-(\hat{F}, \hat{x}, \xi) + \lambda_+(\hat{F}, \hat{x}, \xi) = 0$ at $\hat{m}$-almost every point. We call $\hat{F} \in \mathcal{B}_{\mathrm{vol}}(\hat{f})$ a *continuity point* for Lyapunov exponents if the functions

$$\mathcal{B}_{\mathrm{vol}}(\hat{f}) \ni \hat{G} \mapsto \lambda_\pm(\hat{G}, \hat{m})$$

are continuous at $\hat{F}$. Otherwise, $\hat{F}$ is a discontinuity point for Lyapunov exponents.

By analogy with Pugh, Shub [26], we say that a cocycle is *accessible* if any two points in the fiber bundle are joined by some path consisting of a finite number of legs each of which is either an s-holonomy path or a u-holonomy path (assuming the cocycle admits s-holonomy and u-holonomy).

Theorem E. *Let $\hat{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ be fiber bunched, area preserving, and ergodic. Let $\hat{F}$ be a discontinuity point for Lyapunov exponents. Then $\lambda_-(\hat{F}, \hat{m}) < 0 < \lambda_+(\hat{F}, \hat{m})$ and both Oseledets subspaces $E^-_{\hat{x},\xi}$ and $E^+_{\hat{x},\xi}$ are essentially invariant under the s-holonomy and the u-holonomy of the projective extension. If, in addition, the cocycle $\hat{F}$ is accessible then the Oseledets subspaces vary continuously with $(\hat{x}, \xi) \in \hat{\mathcal{E}}$.*

Our methods also reveal a remarkable connection between the behavior of Lyapunov exponents and the topology of the fiber, at least when the cocycle $\hat{F}$ is accessible. This is illustrated by the next corollary, which will follow from the more detailed statement in Theorem 5.6; see also Remark 5.7.

Corollary F. *Let $\hat{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ be fiber bunched, area preserving, and accessible. Assume the genus of the fiber N of $\hat{\mathcal{E}}$ is at least 2. Then $\lambda_-(\hat{F}, \hat{m}) < 0 < \lambda_+(\hat{F}, \hat{m})$ and $\hat{F}$ is a continuity point for the Lyapunov exponents $\lambda_\pm(\cdot, \hat{m})$.*

1.7. **Generic area preserving cocycles.** For the next theorem, let $\hat{f} : \hat{M} \to \hat{M}$ be a C^r Anosov diffeomorphism on a compact manifold, for some $r \geq 1$. Moreover, take the fiber bundle to be trivial, that is, $\hat{\mathcal{E}} = \hat{M} \times N$, and the cocycle $\hat{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ to be C^r. Recall we take N to be a compact surface. Let $\mathcal{B}^r_{\mathrm{vol}}(\hat{f})$ be the space of area preserving fiber bunched C^r cocycles, endowed with the uniform C^r topology. For simplicity, we also assume that the fiber N is orientable and $\hat{F}$ preserves the orientation of the fibers (the non-orientable case can be treated by considering a double cover).

Theorem G. *There is an open and dense set $\mathcal{U} \subset \mathcal{B}^r_{\mathrm{vol}}(\hat{f})$ such that every $\hat{G} \in \mathcal{U}$ is ergodic for $\hat{m}$ and the Lyapunov exponents $\hat{G} \mapsto \lambda_\pm(\hat{G}, \hat{m})$ vary continuously and never vanish on $\mathcal{U}$.*

In Section 2 we prove Theorems A and B. In Section 3 we apply them to continuous cocycles with holonomies, to deduce Theorem C and Corollary D. In Section 4 we show that holonomies do exist if the cocycle is fiber bunched. Theorem E and Corollary F are proved in Section 5. In Section 6 we prove Theorem G.

Acknowledgements. We are grateful to Jairo Bochi, Carlos Bocker, Jimmy Santamaría, Amie Wilkinson, and Jiagang Yang for several useful discussions. This work was started while we were visiting the Collège de France. It was partly conducted during the period A. A. served as a Clay Research Fellow. M. V. was partially supported by CNPq, FAPERJ, and PRONEX-Dynamical Systems.

2. Non-linear invariance criterion

In this section we prove Theorems A and B. It is no restriction to suppose that the fiber bundle $\hat{\mathcal{E}}$ is trivial, since the measurable trivialization domains U_n in (2) may always be chosen to be disjoint.

The first step is to reduce the proof to a natural extension situation similar to Example 2.13. As observed by Rokhlin [27, §1-§2], one may find a Lebesgue space $(M, \mathcal{B}, \mu)$ and a projection $\pi : \hat{M} \to M$ such that $\mathcal{B} = \pi_* \mathcal{B}_0$ and $\mu = \pi_* \hat{\mu}$. Here M is the quotient space obtained by identifying any two points of $\hat{M}$ which are not distinguished by any $B_0 \in \mathcal{B}_0$ and $\mathcal{B}$ and μ are characterized by the properties we just stated: $B \in \mathcal{B}$ if and only if $\pi^{-1}(B) \in \mathcal{B}_0$ and then $\mu(B) = \hat{\mu}(\pi^{-1}(B))$. Since $\hat{f}$ is $\mathcal{B}_0$-measurable mod 0, there exists a $\mathcal{B}$-measurable mod 0 transformation $f : M \to M$ such that $\pi \circ \hat{f} = f \circ \pi$. This transformation, which is usually non-invertible, preserves μ. Let $\mathcal{E} = M \times N$ and $P : \mathcal{E} \to M$ be the canonical projection. Since the deformation $\tilde{F}$ is $\mathcal{B}_0$-measurable mod 0, it may be written as $\tilde{F} = F \circ (\pi \times \mathrm{id})$ for some $\mathcal{B}$-measurable mod 0 fiber bundle morphism $F : \mathcal{E} \to \mathcal{E}$ over f. Since $\tilde{m} = H_* \hat{m}$ is a $\tilde{F}$-invariant probability projecting down to $\hat{\mu}$, the probability $m = (\pi \times \mathrm{id})_* \tilde{m}$ is F-invariant and projects down to μ.

Let κ be the dimension of N. Let

$$(F_x^{-1})_* m_{f(x)} = J(x, \cdot) m_x + \eta_x \tag{9}$$

be the Lebesgue decomposition of $(F_x^{-1})_* m_{f(x)}$ relative to m_x: the function

$$J(x, \xi) = \frac{d(F_x^{-1})_* m_{f(x)}}{dm_x}(\xi). \tag{10}$$

is integrable for m_x and the measure η_x is singular with respect to m_x. We call $J : \mathcal{E} \to [0, \infty)$ the *fibered Jacobian*, and define the *fibered entropy* to be

$$h = h(\tilde{F}, \tilde{m}) = \int -\log J \, dm. \tag{11}$$

The definition (9) implies $\int_{\{J>0\}} J \, dm = \int J \, dm \leq 1$. Then, by Jensen's inequality,

$$\int_{\{J>0\}} -\log J \, dm \geq 0. \tag{12}$$

The definition (11) means that h is the sum of this integral with the term $(+\infty) \cdot m(\{J = 0\})$ with the usual convention that the latter vanishes if $m(\{J = 0\}) = 0$.

Thus, h is always well-defined and non-negative. In our context h is finite, as we shall see later, and so $\{J = 0\}$ always has zero measure.

Proposition 2.1. *Let $\hat{m}$ be an $\hat{F}$-invariant probability measure projecting down to $\hat{\mu}$ and let $\tilde{m} = H_*\hat{m}$. Then*

$$0 \le \beta h(\tilde{F}, \tilde{m}) \le -\kappa \int \min\{0, \lambda_-(\hat{F}, \cdot)\}\, d\hat{m}.$$

This result may be seen as a coarse fibered version of the Ruelle inequality [28]. Indeed, Ruelle showed that the entropy of a diffeomorphism is bounded above by the integrated sum of the positive Lyapunov exponents. Considering the inverse map, we get that the entropy is also bounded by minus the integrated sum of the negative Lyapunov exponents. In view of this, Proposition 2.1 can probably be refined replacing $\kappa \min\{0, \lambda_-(\hat{F}, \cdot)\}$ by the sum of all negative exponents.

Proposition 2.2. *If $h(\tilde{F}, \tilde{m}) = 0$ then $\hat{x} \mapsto \tilde{m}_{\hat{x}}$ is $\mathcal{B}_0$-measurable* $\bmod 0$.

Theorem A is an immediate consequence of Propositions 2.1 and 2.2. Indeed, the assumption $\lambda_-(\hat{F}, \cdot) \ge 0$ means that $\min\{0, \lambda_-(\hat{F}, \cdot)\}$ vanishes identically. Then Proposition 2.1 yields $h(\tilde{F}, \tilde{m}) = 0$ and, by Proposition 2.2, it follows that the disintegration $\hat{x} \mapsto \tilde{m}_{\hat{x}}$ is $\mathcal{B}_0$-measurable $\bmod 0$, as claimed. This reduces the proof of Theorem A to proving Propositions 2.1 and 2.2.

For Theorem B we need the following version of Proposition 2.2 for sequences of measures. In what follows it is understood that $\tilde{m}_k = H_*\hat{m}_k$ and $m_k = (\pi \times \mathrm{id})_*\tilde{m}_k$.

Proposition 2.3. *Let $(\hat{m}_k)_k$ be a sequence of $\hat{F}$-invariant probability measures on $\hat{\mathcal{E}}$ that project down to $\hat{\mu}$ and converge to some probability $\hat{m}$ in the weak* topology. If $h(\tilde{F}_k, \tilde{m}_k)$ converges to 0 when $k \to \infty$ then the disintegration $\hat{x} \mapsto \tilde{m}_{\hat{x}}$ of $\tilde{m} = H_*\hat{m}$ is $\mathcal{B}_0$-measurable* $\bmod 0$.

In view of Proposition 2.1, the hypothesis of Theorem B implies that $h(\tilde{F}, \tilde{m}_k)$ converges to 0 when k goes to ∞. Then we may apply Proposition 2.3 to conclude that the disintegration $\hat{x} \mapsto \tilde{m}_{\hat{x}}$ is $\mathcal{B}_0$-measurable $\bmod 0$, as claimed. This reduces the proof of Theorem B to proving Propositions 2.1 and 2.3.

2.1. **Entropy zero means deterministic.** Let us prove Propositions 2.2 and 2.3.

Lemma 2.4. *The disintegrations $\{\tilde{m}_{\hat{x}} : \hat{x} \in \hat{M}\}$ and $\{m_x : x \in M\}$ of $\tilde{m}$ and $m = (\pi \times \mathrm{id})_*\tilde{m}$, respectively, are related by*

$$\tilde{m}_{\hat{x}} = \lim_{n\to\infty} (F^n_{x(n)})_* m_{x(n)} \text{ where } x(n) = \pi(\hat{f}^{-n}(\hat{x})), \quad \text{at } \hat{\mu}\text{-almost every } \hat{x} \in \hat{M}.$$

Proof. Let m_0 be the probability defined on $\mathcal{B}_0$ by $\pi_* m_0 = m$. The disintegration of m_0 is just $\hat{x} \mapsto m_{\pi(\hat{x})}$. The relation $\pi_*\tilde{m} = m$ implies that $\tilde{m} \mid \mathcal{B}_0 = m_0$ or, in other words, $E(\hat{x} \mapsto \tilde{m}_{\hat{x}} \mid \mathcal{B}_0) = [\hat{x} \mapsto m_{\pi(\hat{x})}]$. Next, the relation $\tilde{F}_*\tilde{m} = \tilde{m}$ implies that

$$E(\hat{x} \mapsto \tilde{m}_{\hat{x}} \mid \hat{f}^{\,n}(\mathcal{B}_0)) = E(\hat{x} \mapsto (\hat{F}^n_{\hat{x}(n)})_* \tilde{m}_{\hat{x}(n)} \mid \mathcal{B}_0),$$

with $\hat{x}(n) = \hat{f}^{-n}(\hat{x})$, and so

$$E(\hat{x} \mapsto \tilde{m}_{\hat{x}} \mid \hat{f}^{\,n}(\mathcal{B}_0)) = [\hat{x} \mapsto (F^n_{x(n)})_* m_{x(n)}].$$

Any of these expressions defines a martingale of probability measures, relative to the sequence of σ-algebras $\hat{f}^{\,n}(\mathcal{B}_0)$. Since $\mathcal{B}_0$ is generating and the sequence $\hat{f}^n(\mathcal{B}_0)$ is increasing, the limit of the left hand side is

$$[\hat{x} \mapsto \tilde{m}_{\hat{x}}] = E(\hat{x} \mapsto \tilde{m}_{\hat{x}} \mid \hat{\mathcal{B}}).$$

It follows that $(F^n_{x(n)})_* m_{x(n)}$ converges and the limit coincides with $\tilde{m}_{\hat{x}}$ at $\hat{\mu}$-almost every point. □

Lemma 2.5. *If $h(\tilde{F}, \tilde{m}) = 0$ then $(F_x)_* m_x = m_{f(x)}$ for μ-almost every $x \in M$.*

Proof. The definition (9) implies that $\int J(x,\xi)\, dm_x(\xi) \le 1$ for μ-every x. So, by Jensen's inequality, $\int -\log J(x,\xi)\, dm_x(\xi) \ge 0$ for μ-every x. Moreover, the equalities hold if and only if $J(x,\xi) = 1$ for m_x-almost every ξ. This implies that $h \ge 0$, and $h = 0$ if and only if $J(x,\xi) = 1$ for m_x-almost every ξ and μ-almost x. In particular, $h = 0$ implies $m_{f(x)} = (F_x)_* m_x$ for μ-almost x, as claimed. □

Lemma 2.5 implies $(F^n_{x(n)})_* m_{x(n)} = m_{x(0)}$ for every $n \ge 0$ and $\hat{\mu}$-almost every $\hat{x}$. Then Lemma 2.4 yields $\tilde{m}_{\hat{x}} = m_{x(0)}$ for $\hat{\mu}$-almost every $\hat{x}$. Since $x(0) = \pi(\hat{x})$, this implies that $\hat{x} \mapsto \tilde{m}_{\hat{x}}$ is $\mathcal{B}_0$-measurable. The proof of Proposition 2.2 is complete.

Next, we prove Proposition 2.3. Let $(F_x^{-1})_* m_{k,f(x)} = J_k(x,\cdot) m_{k,x} + \eta_{k,x}$ be the Lebesgue decomposition for each m_k: in particular, $J_k : \mathcal{E} \to \mathbb{R}$ is the fibered Jacobian. We denote by $\|\xi\|$ the total variation of a signed measure ξ.

Lemma 2.6. *$\int |J_k(x,\xi) - 1|\, dm_k(x,\xi) \to 0$ and $\int \|\eta_{k,x}\|\, d\mu(x) \to 0$ when $k \to \infty$.*

Proof. Since the $m_{k,y}$ are probabilities,

$$\eta_{k,x}(\mathcal{E}_x) = m_{k,f(x)}(\mathcal{E}_{f(x)}) - \int J_k(x,\cdot)\, dm_{k,x} = \int (1 - J_k(x,\cdot))\, dm_{k,x}.$$

Integrating with respect to μ, we obtain $\int \|\eta_{k,x}\|\, d\mu = \int (1 - J_k)\, dm_k$ and so the second claim is a consequence of the first one. Next, define $\phi(x) = x - \log(1+x)$ for $x > -1$. Then $\phi(x) \ge 0$ for all x and, given any $\delta > 0$, there exists $c(\delta) > 0$ such that $\phi(x) \ge c(\delta)|x|$ whenever $|x| \ge \delta$. Let $\delta > 0$ be fixed. Denote $a_k = \int J_k\, dm_k$ for each $k \ge 0$. Using Jensen's inequality,

$$h(\tilde{F}, \tilde{m}_k) \ge -\log a_k \ge 0,$$

and so a_k converges to 1 when $n \to \infty$. Assume k is large enough that $h(\tilde{F}, \tilde{m}_k)$ and $a_k - 1$ are both less than $\delta c(\delta)$. Then, by the definition of ϕ,

$$\int -\log J_k\, dm_k = \int (1 - J_k)\, dm_k + \int \phi(J_k - 1)\, dm_k.$$

The first integral is less than $\delta c(\delta)$ and the second one is $1 - a_k > -\delta c(\delta)$. The third integral is bounded below by

$$\int_{\{|J_k - 1| > \delta\}} \phi(J_k - 1)\, dm_k \ge c(\delta) \int_{\{|J_k - 1| > \delta\}} |J_k - 1|\, dm_k.$$

This implies

$$\int |J_k - 1|\, dm_k \le \delta + \int_{\{|1 - J_k| > \delta\}} |J_k - 1|\, dm_k \le 3\delta$$

for all large k. This completes the proof of the lemma. □

For each $k \geq 1$, let $\check{m}_k$ be the probability measure on $\mathcal{E}$ that projects down to $\hat{\mu}$ and whose conditional measures along the fibers are given by

$$\check{m}_{k,\hat{x}} = m_{k,x} \quad \text{for all } x = \pi(\hat{x}).$$

Up to taking a subsequence, we may assume $\check{m}_k$ to converge to some measure $\check{m}$, whose disintegration $\hat{x} \mapsto \check{m}_x$ along the fibers is $\mathcal{B}_0$-measurable mod 0. Clearly, $(\pi \times \mathrm{id})_* \check{m}_k = m_k$ for every k. Taking the limit as $k \to \infty$, we conclude that $\pi_* \check{m} = m$.

Lemma 2.7. *The total variation $\|\tilde{F}_*^{-1}\check{m}_k - \check{m}_k\|$ converges to 0 as $k \to \infty$.*

Proof. Given any measurable set $B \subset \hat{\mathcal{E}}$, we denote $B_{\hat{x}} = B \cap \hat{\mathcal{E}}_{\hat{x}}$ for each $\hat{x} \in \hat{M}$. Then

$$\begin{aligned}\big(\tilde{F}_*^{-1}\check{m}_k - \check{m}_k\big)(B) &= \int \check{m}_{k,\hat{y}}(\tilde{F}(B)_{\hat{y}})\, d\hat{\mu}(\hat{y}) - \int \check{m}_{k,\hat{x}}(B_{\hat{x}})\, d\hat{\mu}(\hat{x}) \\ &= \int \check{m}_{k,\hat{f}(\hat{x})}(\tilde{F}_{\hat{x}}(B_{\hat{x}}))\, d\hat{\mu}(\hat{x}) - \int \check{m}_{k,\hat{x}}(B_{\hat{x}})\, d\hat{\mu}(\hat{x})\end{aligned}$$

because $\hat{\mu}$ is invariant under $\hat{f}$. Since $\tilde{F}_{\hat{x}}$ and $\check{m}_{k,\hat{x}}$ are both $\mathcal{B}_0$-measurable, the last term may be rewritten as

$$\begin{aligned}&\int m_{k,f(x)}(F_x(B_{\hat{x}}))\, d\hat{\mu}(\hat{x}) - \int m_{k,x}(B_{\hat{x}})\, d\hat{\mu}(\hat{x}) \\ &\qquad = \int \Big(\int_{B_{\hat{x}}} (J_k(x,\cdot) - 1)\, dm_{k,x} + \eta_{k,x}(B_{\hat{x}})\Big) d\hat{\mu}(\hat{x})\end{aligned}$$

These relations imply that

$$\big|\big(\tilde{F}_*^{-1}\check{m}_k - \check{m}_k\big)(B)\big| \leq \int \Big(\int |J_k(x,\cdot) - 1|\, dm_{k,x} + \|\eta_{k,x}\|\Big) d\hat{\mu}(\hat{x})$$

for every $B \subset \hat{\mathcal{E}}$, and so $\|\tilde{F}_*^{-1}\check{m}_k - \check{m}_k\| \leq \int |J_k - 1|\, dm_k + \int \|\eta_{k,x}\|\, d\mu(x)$. Now the claim follows from Lemma 2.6. □

Taking the limit in Lemma 2.7 we conclude that the measure $\check{m}$ is invariant under $\tilde{F}$. It follows that $\check{m} = \tilde{m}$: any two $\tilde{F}$-invariant measures that project down to m under π must coincide, because the σ-algebra $\mathcal{B}_0$ is generating. This proves that the disintegration of $\tilde{m}$ is $\mathcal{B}_0$-measurable mod 0, as claimed. The proof of Proposition 2.3 is complete.

2.2. **Entropy is smaller than exponents.** We are left to prove Proposition 2.1. We begin by reducing the proof to the ergodic case. Let $\{\hat{m}_\alpha\}$ be the ergodic decomposition of $\hat{m}$ and $d\alpha$ denote the corresponding quotient measure:

$$\int \varphi\, d\hat{m} = \int \Big(\int \varphi\, d\hat{m}_\alpha \Big)\, d\alpha$$

for any integrable function φ. Then $\tilde{m}_\alpha = H_* \hat{m}_\alpha$ and $m_\alpha = (\pi \times \mathrm{id})_* \tilde{m}_\alpha$ define the ergodic decompositions of $\tilde{m} = H_* \hat{m}$ and $m = (\pi \times \mathrm{id})_* \tilde{m}$, respectively, with the same quotient measure. If $\lambda_-(\hat{F}, \hat{x}, \xi) \geq 0$ at $\hat{m}$-almost every point then the same is true at $\hat{m}_\alpha$-almost every point, for $d\alpha$-almost every ergodic component. Assuming the proposition holds for ergodic measures, it follows that

$$0 \leq \beta \int -\log J\, dm_\alpha \leq -\kappa \int \min\{0, \lambda_-(\hat{F}, \cdot)\}\, d\hat{m}_\alpha$$

for $d\alpha$-almost every α. Integrating with respect to $d\alpha$, we obtain that

$$0 \le \beta h(\tilde{F}, \tilde{m}) \le -\kappa \int \min\{0, \lambda_-(\hat{F}, \cdot)\}\, d\hat{m},$$

as claimed. Hence, it is no restriction to assume that $\hat{m}$ is ergodic for $\hat{F}$, and we do so in what follows. Then $\tilde{m}$ and m are ergodic for $\tilde{F}$ and F, respectively. Moreover, $\min\{0, \lambda_-(\hat{F}, \cdot)\}$ is constant $\hat{m}$-almost everywhere. Let $-\lambda$ denote this constant.

Now we begin the proof of the proposition in the ergodic case. Given $\varepsilon > 0$, define $J_\varepsilon = J + \varepsilon$ and $h_\varepsilon = -\int \log J_\varepsilon \, dm$. Notice that $h_\varepsilon \to h$ as $\varepsilon \to 0$, by the monotone convergence theorem. Our goal is to prove that $h \le \kappa\beta^{-1}\lambda$. The proof is by contradiction. Assume this inequality is false. Then we may choose some small $\varepsilon > 0$ such that

$$h_\varepsilon - 10\varepsilon \ge \kappa\beta^{-1}(\lambda + 10\varepsilon). \tag{13}$$

Lemma 2.8. *There exists a sequence of countable partitions $P_n(x, \cdot)$ of each fiber $\mathcal{E}_x$, depending measurably on $x \in M$, a sequence of measurable subsets W_n of $\mathcal{E}$ with $m(W_n) \to 1$ such that, for every large n,*

(a) $\operatorname{diam} P_n(x, \xi) \le e^{-\beta^{-1}(\lambda+5\varepsilon)n}$ *for every* $(x, \xi) \in \mathcal{E}$
(b) *each* $W_n \cap \mathcal{E}_x$ *is covered by not more than* $e^{\kappa\beta^{-1}(\lambda+8\varepsilon)n}$ *atoms of* $P_n(x, \cdot)$
(c) $m_x(\partial P_n(x, \xi)) = 0$ *for every* $(x, \xi) \in \mathcal{E}$.

Proof. Since N is a manifold, we may choose a sequence of countable partitions Q_n with relatively compact atoms with diameter bounded by $e^{-\beta^{-1}(\lambda+6\varepsilon)n}$, and an increasing sequence of subsets V_n exhausting N and such that V_n is covered by not more than $e^{\kappa\beta^{-1}(\lambda+8\varepsilon)n}$ atoms of Q_n. Of course, we may take these to be the first atoms of Q_n with respect to some ordering of the partition. This defines ordered countable partitions $Q_n(x, \cdot)$ of each fiber $\mathcal{E}_x$, and sets $W_n \subset \mathcal{E}$ exhausting every fiber, such that $\operatorname{diam} Q_n(x, \xi) \le \text{const}\, e^{-\beta^{-1}(\lambda+6\varepsilon)n}$ for every (x, ξ), and every $W_n \cap \mathcal{E}_x$ is covered by the first $e^{\kappa\beta^{-1}(\lambda+8\varepsilon)n}$ atoms of $Q_n(x, \cdot)$. For each atom Q of $Q_n(x, \cdot)$, let $B_1, \ldots, B_k$ be a finite covering of the boundary of Q by open sets with diameter less than $e^{-\beta^{-1}(\lambda+6\varepsilon)n}$ and such that $m_x(\partial B_j) = 0$ for all j. Let $\tilde{Q}_n(x, \cdot)$ be the family of all $\tilde{Q} = Q \cup B_1 \cup \cdots \cup B_k$ obtained in this way. Notice that $m_x(\partial \tilde{Q}) = 0$. Removing from each $\tilde{Q}$ the union of the elements of $\tilde{Q}_n(x, \cdot)$ that precede it, relative to the ordering inherited from $Q_n(x, \cdot)$, one obtains a new ordered partition $P_n(x, \cdot)$ such that the diameter of its atoms is bounded by $\text{const}\, e^{-\beta^{-1}(\lambda+6\varepsilon)n}$, the first $e^{\kappa\beta^{-1}(\lambda+8\varepsilon)n}$ atoms cover $W_n \cap \mathcal{E}_x$, and the boundary of every atom has zero m_x-measure. Replacing 6ε by 5ε in the exponent and assuming n is large, one gets rid of the constant. This finishes the construction. □

For $0 \le k < n$, define $P_{n,k}(\cdot, \cdot)$ as the pullback of $P_n(\cdot, \cdot)$ by F^{n-k}, that is,

$$P_{n,k}(x, \xi) = (F_x^{n-k})^{-1}\big(P_n(F^{n-k}(x, \xi))\big).$$

Let also $P_{n,n}(\cdot, \cdot) = P_n(\cdot, \cdot)$. For each $0 \le k < n$, define

$$J_n(x, \xi) = \frac{m_{f^n(x)}(P_n(F^n(x, \xi)))}{m_x(P_{n,0}(x, \xi))} \quad \text{and} \quad J_{n,k}(x, \xi) = \frac{m_{f(x)}(P_{n,k+1}(F(x, \xi)))}{m_x(P_{n,k}(x, \xi))}.$$

Then $J_n(x,\xi) = \prod_{k=0}^{n-1} J_{n,k}(F^k(x,\xi))$. Moreover, let

$$J_{n,k,\varepsilon} = J_{n,k} + \varepsilon \quad \text{and} \quad J_{n,\varepsilon}(x,\xi) = \prod_{k=0}^{n-1} J_{n,k,\varepsilon}(F^k(x,\xi)).$$

Notice that $J_{n,\varepsilon} \geq J_n$ because $J_{n,k,\varepsilon} \geq J_{n,k}$ for every k. The key ingredient in the proof of Proposition 2.1 is the following lemma, whose proof we postpone for a while:

Lemma 2.9. *We have* $\lim_{n\to\infty} \sup_{0\leq k<n} \|\log J_{n,k,\varepsilon} - \log J_\varepsilon\|_{L^1(m)} = 0$.

As a consequence of this lemma and the ergodic theorem,

$$\lim \frac{1}{n} \log J_{n,\varepsilon} = \lim \frac{1}{n} \sum_{k=0}^{n-1} \log J_\varepsilon \circ F^k = \int \log J_\varepsilon dm = -h_\varepsilon$$

in $L^1(m)$ and, hence, in measure. In particular, for every large n there exists $E_n \subset \mathcal{E}$ with $m(E_n) \geq 1 - \varepsilon$ such that

$$\frac{1}{n} \log J_n(x,\xi) \leq \frac{1}{n} \log J_{n,\varepsilon}(x,\xi) \leq -h_\varepsilon + 5\varepsilon \quad \text{for all } (x,\xi) \in E_n.$$

Using Lemma 2.8 and the definition of J_n, we conclude that the fiber of $F^n(E_n)\cap W_n$ over $f^n(x)$ is covered by at most $e^{\kappa\beta^{-1}(\lambda+8\varepsilon)n}$ atoms of $P_n(f^n(x),\cdot)$ all with $m_{f^n(x)}$-measure at most $e^{(-h_\varepsilon+5\varepsilon)n}$. By (13), this implies $m(F^n(E_n)\cap W_n)$ goes to zero as $n \to \infty$, contradicting the fact that both $m(W_n)$ and $m(E_n)$ are close to 1. This contradiction reduces the proof of Proposition 2.1 to proving Lemma 2.9.

For every $l \geq 1$, define $\omega_l(\hat{x},\xi) = \log \|(D\hat{F}^l_{\hat{x}}(\xi))^{-1}\|^{-1}$ and

$$\Omega_l(\hat{x},\xi) = \liminf_{n\to\infty} \inf_{0\leq k<n} \frac{1}{n} \sum_{j=k}^{n-1} \frac{1}{l} \omega_l(\hat{F}^{jl}(\hat{x},\xi)).$$

Lemma 2.10. *We have* $\sup_{l\geq 1} \Omega_l(\hat{x},\xi) \geq -\lambda$ *for every* $(\hat{x},\xi)$ *in some full* $\hat{m}$-*measure set* $\hat{Z} \subset \hat{\mathcal{E}}$.

Proof. We begin by claiming that $\sup_{l\geq 1} \Omega_l$ is constant along orbits. Indeed, since the norms of $DF^{\pm 1}$ are uniformly bounded, there exists some constant $A > 0$ such that $|\omega_l(\hat{F}(\hat{y},\eta)) - \omega_l(\hat{y},\eta)| \leq A$ for every $(\hat{y},\eta)$. This implies

$$|\Omega_l(\hat{F}(\hat{x},\xi)) - \Omega_l(\hat{x},\xi)| \leq \frac{A}{l} \quad \text{for every } (\hat{x},\xi). \tag{14}$$

Similarly, since, $\omega_{2l}(\hat{y},\eta) \geq \omega_l(\hat{y},\eta) + \omega_l(\hat{F}(\hat{y},\eta))$ for every $(\hat{y},\eta)$, we have $\Omega_{2l}(\hat{x},\xi) \geq \Omega_l(\hat{x},\xi)$ for every $(\hat{x},\xi)$ and every $l \geq 1$. This implies

$$\sup_l \Omega_l(\hat{x},\xi) = \limsup_{l\to\infty} \Omega_l(\hat{x},\xi) \quad \text{for every } (\hat{x},\xi). \tag{15}$$

The relations (14) and (15) imply our claim.

Next, by ergodicity and the definition of smallest Lyapunov exponent,

$$\lambda_- = \lim_l \frac{1}{l} \omega_l(\hat{x},\xi) = \sup_l \frac{1}{l} \omega_l(\hat{x},\xi) \quad \text{for } \hat{m}\text{-almost every } (\hat{x},\xi).$$

Given $\varepsilon > 0$, fix $s \geq 1$ large enough so that $\hat{m}(E_{s,\varepsilon}) > 1 - \varepsilon$, where

$$E_{s,\varepsilon} = \{(\hat{y},\eta) : \frac{1}{s} \omega_s(\hat{y},\eta) \geq \lambda_- - \varepsilon\}.$$

By ergodicity, for $\hat{m}$-almost every $(\hat{x}, \xi)$ the number of iterates $0 \le i < ns$ for which $F^i(\hat{x}, \xi) \notin E_{s,\varepsilon}$ is less than $2\varepsilon ns$, assuming n is large enough. Then there exists $0 \le r < s$ such that the number of iterates $0 \le j < n$ for which $F^{js+r}(\hat{x}, \xi) \notin E_{s,\varepsilon}$ is less than $2\varepsilon n$. Let $B > 0$ be an upper bound for the absolute value of $\log \|D\hat{F}^{-1}\|$. Then $|\omega_l(\hat{y}, \eta)| \le Bl$ for every $(\hat{y}, \eta)$ and every $l \ge 1$. It follows that, given any $0 \le k < n$,

$$\frac{1}{n}\sum_{j=k}^{n-1}\frac{1}{s}\omega_s(\hat{F}^{js+r}(\hat{x},\xi)) \ge \frac{1}{n}\Big[(\lambda_- - \varepsilon)\#\{k \le j < n : \hat{F}^{js+r}(\hat{x},\xi) \in E_{s,\varepsilon}\} - 2n\varepsilon B\Big]$$
$$\ge -\lambda - \varepsilon(1+2B).$$

Since this holds for every $0 \le k < n$ and every n sufficiently large, we conclude that

$$\sup_l \Omega_l(\hat{F}^r(\hat{x},\xi)) \ge \Omega_s(\hat{F}^r(\hat{x},\xi)) \ge -\lambda - \varepsilon(1+2B).$$

So, in view of the claim in the first paragraph, $\sup_l \Omega_l(\hat{x}, \xi) \ge -\lambda - \varepsilon(1 + 2B)$. Since $\varepsilon > 0$, this completes the proof of the lemma. □

The next results provides the main estimate for the proof of Lemma 2.9. Let

$$d_x(\xi,\eta) = \sup\{d(H_{\hat{x}}^{-1}(\xi)), H_{\hat{x}}^{-1}(\eta)) : \hat{x} \in \pi^{-1}(x)\}$$

for each $x \in M$ and $\xi, \eta \in \mathcal{E}_x$. This defines a metric d_x on each fiber $\mathcal{E}_x$ which, by (3), relates to the Riemannian distance d through $d_x(\xi, \eta) \le Bd(\xi, \eta)^\beta$. Then let $\Delta_{n,k}(x, \xi)$ denote the d_x-diameter of each atom $P_{n,k}(x, \xi)$.

Lemma 2.11. *We have* $\lim_{n\to\infty} \sup_{0\le k<n} \Delta_{n,k} = 0$ *at m-almost every point.*

Proof. It suffices to show that $\lim_{n\to\infty} \sup_{0\le k<n} \Delta_{n,k}(x, \xi) = 0$ holds for every (x, ξ) in the full m-measure set $Z = (\pi \times \mathrm{id})(\hat{Z})$. To this end, consider any $\hat{x} \in \pi^{-1}(x) \cap \hat{Z}$. We claim that, given any $\delta > 0$, there exists $m_0 \ge 1$ such that (balls are with respect to the Riemannian metric along the fiber)

$$\hat{F}^m(B_\delta(\hat{x},\xi)) \supset B_{e^{-(\lambda+4\varepsilon)m}}(\hat{F}^m(\hat{x},\xi)) \tag{16}$$

for every $m > m_0$. Assume this fact for a while. It implies that

$$\hat{F}^{n-k}(B_\delta(\hat{x},\xi)) \supset B_{e^{-(\lambda+4\varepsilon)n}}(\hat{F}^{n-k}(\hat{x},\xi)) \tag{17}$$

for all $0 \le k < n$, as long as n is large enough: when $n - k \ge m_0$ this is a direct consequence of (16); otherwise, use the fact that $\hat{F}^{\pm j}$, $1 \le j \le m_0$ are uniformly continuous along fibers, and take n to be large enough. By the Hölder property (3) and Lemma 2.8, we also have $H_{\hat{x}}^{-1}(P_n(F^{n-k}(x, \xi))) \subset B_{e^{-(\lambda+4\varepsilon)n}}(\hat{F}^{n-k}(\hat{x}, \xi))$, as long as n is large enough (to make the radius of the last neighborhood sufficiently small). Combined with (17), this gives

$$P_n(F^{n-k}(x,\xi)) \subset F_x^{n-k}(H_{\hat{x}}(B_\delta(\hat{x},\xi))), \text{ that is, } P_{n,k}(x,\xi) \subset H_{\hat{x}}(B_\delta(\hat{x},\xi))$$

for all $0 \le k < n$, as long as n is large enough. Since $H_{\hat{x}}$ is continuous, this implies the conclusion of the lemma.

To prove the claim (16), begin by fixing $l \ge 1$ such that $\Omega_l(\hat{x}, \xi) \ge -(\lambda + \varepsilon)$. Then define $\delta_{j,n}$, $0 \le k \le n$ by

$$\log \delta_{n,n} = -(\lambda+3\varepsilon)ln \quad \text{and} \quad \log \delta_{k,n} = \log \delta_{k+1,n} + \varepsilon - \omega_l(\hat{F}^{kl}(\hat{x},\xi)).$$

Then $\log\delta_{j,n} \le -\varepsilon n$ for all $0 \le k \le n$, because

$$\log\delta_{j,n} = \log\delta_{n,n} + \sum_{j=k}^{n-1} \varepsilon - \omega_l(\hat{F}^{jl}(\hat{x},\xi)) \le -(\lambda+3\varepsilon)ln + (\varepsilon - \Omega_l(\hat{x},\xi))ln.$$

Since the derivatives $D\hat{F}^{\pm l}_{\hat{y}}$ are uniformly bounded and uniformly continuous, we conclude from the definition of ω_l that

$$\hat{F}^l(B_{\delta_{k,n}}(\hat{F}^{kl}(\hat{x},\xi))) \supset B_{\delta_{k+1,n}}(\hat{F}^{(k+1)l}(\hat{x},\xi))$$

for every $0 \le k < n$, as long as n is large enough (to make $e^{-\varepsilon n}$ sufficiently small). In particular,

$$\hat{F}^{ln}(B_{\delta_{n,0}}(\hat{x},\xi)) \supset B_{\delta_{n,n}}(\hat{F}^n(\hat{x},\xi)).$$

This gives a version of (16) for the iterates that are multiples of l: given any $\delta > 0$ there exists $n_0 \ge 1$ such that

$$\hat{F}^{ln}(B_\delta(\hat{x},\xi)) \supset B_{e^{-(\lambda+3\varepsilon)ln}}(\hat{F}^{ln}(x,\xi))$$

for every $n \ge n_0$. To complete the proof it suffices to note that, since the derivatives $D\hat{F}^{\pm j}_{\hat{y}}$, $0 \le j < l$ are bounded,

$$\hat{F}^j(B_{e^{-(\lambda+3\varepsilon)ln}}(\hat{F}^{ln}(x,\xi))) \supset B_{e^{-(\lambda+4\varepsilon)ln}}(\hat{F}^{ln+j}(x,\xi))$$

for all $0 \le j < l$, as long as n is large enough. This finishes the proof of (16) and of the lemma. □

We also need the following abstract result:

Lemma 2.12. *Let K be a complete metric space and μ_0 and μ_1 be probability measures on K with $\mu_1 \ge \alpha\mu_0$ for some $\alpha > 0$. Let $\phi = d\mu_1/d\mu_0$ and, given any countable partition P of K, define*

$$\phi_P(x) = \frac{\mu_1(P(x))}{\mu_0(P(x))}.$$

Then $\int \log\phi(x)d\mu_0(x) \le \int \log\phi_P(x)d\mu_0(x) \le 0$. Moreover, given $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\log\phi_P - \log\phi\|_{L^1(\mu_0)} \le \varepsilon$ for any countable partition P of K such that the total measure of the atoms with diameter larger than δ is smaller than δ.

Proof. By convexity, $\int \log\phi_P\, d\mu_0 \le \log\int \phi_P\, d\mu_0 = 0$. Similarly,

$$\int_{P(x)} \log\phi\, d\mu_0 \le \log\phi_P(x)\mu_0(P(x))$$

for every atom $P(x)$, and so $\int \log\phi\, d\mu_0 \le \int \log\phi_P\, d\mu_0$. This proves the first claim.

Next, notice that the functions ϕ_P satisfy a uniform integrability condition: for all $X \subset K$ with $\mu_0(X) < 1/e$,

$$\int_X |\log\phi_P| d\mu_0 \le -\mu_0(X)(\log\mu_0(X) + \log\alpha). \tag{18}$$

Indeed, the assumption implies $-\log\phi_P \le -\log\alpha$ and so the claim is trivial if $\log\phi_P$ happens to be negative on X. When $\log\phi_P \ge 0$ on the set X, the claim follows from convexity:

$$\int_X \log\phi_P d\frac{\mu_0}{\mu_0(X)} \le \log\int_X \phi_P d\frac{\mu_0}{\mu_0(X)} \le \log\frac{\mu_1(\tilde{X})}{\mu_0(X)} \le \log\frac{1}{\mu_0(X)}$$

($\tilde{X}$ denotes the union of all atoms of P that intersect X). The general case is handled by splitting X into two subsets where $\log\phi_P$ has constant sign.

We also use the following fact: if R refines Q then

$$\|\log\phi_R - \log\phi_Q\|_{L^1(\mu_0)} \le \|\log\phi - \log\phi_Q\|_{L^1(\mu_0)}. \tag{19}$$

To see that this is so, write

$$\int |\log\phi_R - \log\phi_Q|\, d\mu_0 = \sum_{r\subset q} \int_r |\log\phi_R - \log\phi_Q|\, d\mu_0,$$

where the sum is over the pairs of atoms $r \in R$ and $q \in Q$ with $r \subset q$. Since ϕ_R and ϕ_Q are constant on r, this may be rewritten as

$$\begin{aligned}\sum_{r\subset q} \mu_0(r)\, |\log\phi_R - \log\phi_Q| &= \sum_{r\subset q} \Big| \int_r \log\phi\, d\mu_0 - \mu_0(r)\log\phi_Q \Big| \\ &\le \sum_{r\subset q} \int_r |\log\phi - \log\phi_Q|\, d\mu_0\,.\end{aligned}$$

The combination of these two relations proves (19).

Let $(Q_n)_n$ be any refining sequence of partitions with diameter decreasing to zero. Then $\phi_{Q_n} \to \phi$ at μ_0-almost every point (martingale convergence theorem). By uniform integrability (18), it follows that $\log\phi_{Q_n} \to \log\phi$ in $L^1(\mu_0)$. Assume, in what follows, that the sequence was chosen so that $\mu_0(\partial Q_n(x)) = 0$ for every x and every n (this can be obtained using the argument in Lemma 2.8(c)). Given $\varepsilon > 0$, fix n sufficiently large so that

$$\|\log\phi_{Q_n} - \log\phi\|_{L^1(\mu_0)} < \varepsilon/4.$$

Let $R = P \vee Q_n$ (the coarsest partition that refines both P and Q_n) and let Δ be the set of all x such that $P(x) \not\subset Q_n(x)$. By (19),

$$\|\log\phi_R - \log\phi\|_{L^1(\mu_0)} < \varepsilon/2.$$

Choosing $\delta > 0$ small, we also ensure that the measure of Δ is small, so that $-\mu_0(\Delta)(\log\mu_0(\Delta) + \log\alpha) < \varepsilon/4$. Clearly, $P(x) = R(x)$ for every x in the complement of Δ. So, using (18),

$$\|\log\phi_P - \log\phi_R\|_{L^1(\mu_0)} = \int_\Delta |\log\phi_P - \log\phi_R|\, d\mu_0 < \varepsilon/2.$$

From these two relations it follows that $\|\log\phi_P - \log\phi\|_{L^1(\mu_0)} < \varepsilon$, as claimed in the second part of the lemma. □

We are ready to prove Lemma 2.9. Lemmas 2.8 and 2.11 ensure that the hypotheses of Lemma 2.12 are satisfied for $K = N$ and $\mu_0 = m_x$ and $\mu_1 = (F_x^{-1})_* m_{f(x)} + \varepsilon m_x$ ($\alpha = \varepsilon$) and $P = P_{n,k}(x,\cdot)$. Notice that

$$\phi = J(x,\cdot) + \varepsilon = J_\varepsilon \quad \text{and} \quad \phi_P = J_{n,k,\varepsilon}(x,\cdot).$$

From Lemma 2.12 we conclude that

$$\sup_k \|\log J_{n,k,\varepsilon} - \log J_\varepsilon\|_{L^1(m_x)} \to 0$$

for μ-almost every x. Since $\int |\log J_{n,k,\varepsilon}| dm_x \le -2\log\varepsilon$ for μ-almost every x, this implies that $\|\log J_{n,k,\varepsilon} - \log J_\varepsilon\|_{L^1(m)} \to 0$, as claimed in Lemma 2.9.

The proof of Proposition 2.1 is complete, finishing the proofs of Theorems A/B.

2.3. **Examples.** A few simple examples illustrate the contents of Theorems A/B.

Example 2.13. Given any (non-invertible) measure-preserving map $f : M \to M$ in a probability space $(M, \mathcal{B}, \mu)$, define $\hat{M}$ to be the space of all sequences $(x_n)_{n\le 0}$ in M such that $f(x_n) = x_{n+1}$ for all $n < 0$, and consider the *natural extension* of f,

$$\hat{f} : \hat{M} \to \hat{M}, \quad \hat{f}(\dots, x_n, \dots, x_0) = (\dots, x_n, \dots, x_0, f(x_0)).$$

Then $\hat{f}$ is invertible and $\pi \circ \hat{f} = f \circ \pi$, where $\pi : \hat{M} \to M$ is the projection to the zeroth term. Denote $\mathcal{B}_0 = \pi^{-1}(\mathcal{B})$ and let $\hat{\mathcal{B}}$ be the σ-algebra on $\hat{M}$ generated by the iterates $\hat{f}^n(\mathcal{B}_0)$, $n \ge 0$. Then $\hat{f}$ is measurable with respect to $\mathcal{B}_0$ and to $\hat{\mathcal{B}}$. Let μ_0 be the probability measure defined on $\mathcal{B}_0$ by $\pi_*\mu_0 = \mu$. There is a unique $\hat{f}$-invariant probability $\hat{\mu}$ on $(\hat{M}, \hat{\mathcal{B}})$ such that $\pi_*\hat{\mu} = \mu$: it is characterized by

$$E\big(\hat{\mu} \mid \hat{f}^{\,n}(\mathcal{B}_0)\big) = \hat{f}^{\,n}_*\mu_0 \quad \text{for every } n \ge 0. \tag{20}$$

To any smooth cocycle $F : \mathcal{E} \to \mathcal{E}$ over f, defined on a fiber bundle $P : \mathcal{E} \to M$, we may associate the smooth cocycle $\hat{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}$ over $\hat{f}$ defined by $\hat{\mathcal{E}}_{\hat{x}} = \mathcal{E}_{\pi(\hat{x})}$ and $\hat{F}_{\hat{x}} = F_{\pi(\hat{x})}$. Their extremal Lyapunov exponents are related by

$$\lambda_\pm(\hat{F}, \hat{x}, \hat{\xi}) = \lambda_\pm(F, \pi(\hat{x}), \hat{\xi}).$$

Clearly, $\hat{x} \mapsto \hat{F}_{\hat{x}}$ is $\mathcal{B}_0$-measurable. We denote by $\pi \times \mathrm{id}$ the natural projection from $\hat{\mathcal{E}}$ to $\mathcal{E}$ (this terminology is motivated by the case when $\hat{\mathcal{E}} = \hat{M} \times N$ and $\mathcal{E} = M \times N$). Given any F-invariant probability m, there is exactly one $\hat{F}$-invariant probability $\hat{m}$ with $(\pi \times \mathrm{id})_*\hat{m} = m$: it is characterized by

$$E\big(\hat{x} \mapsto \hat{m}_{\hat{x}} \mid \hat{f}^n(\mathcal{B}_0)\big) = [\hat{x} \mapsto (\hat{F}^n_{\hat{x}})_* m_{\pi(\hat{x})}] \quad \text{for every } n \ge 0 \tag{21}$$

(see Lemma 2.4 below), where $\{\hat{m}_{\hat{x}} : \hat{x} \in \hat{M}\}$ and $\{m_x : x \in M\}$ are the disintegrations of $\hat{m}$ and m, respectively. If $P_*m = \mu$ then $\hat{P}_*\hat{m} = \hat{\mu}$.

Example 2.14. Ledrappier [25] deals with the particular case when the cocycle is actually linear or, more precisely, projective: $\mathcal{E} = \hat{M} \times \mathbb{P}(\mathbb{R}^d)$ and each $\hat{F}_{\hat{x}}$ is the diffeomorphism induced on the projective space $N = \mathbb{P}(\mathbb{R}^d)$ by some linear map $A(\hat{x}) \in \mathrm{GL}(d, \mathbb{R})$. Denote

$$\lambda_+(\hat{x}) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(\hat{x})\| \quad \text{and} \quad \lambda_-(\hat{x}) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(\hat{x})^{-1}\|^{-1}.$$

The subadditive ergodic theorem [24] ensures that these two limits exist almost everywhere, and it is clear that $\lambda_+(\hat{x}) \ge \lambda_-(\hat{x})$ at μ-almost every $\hat{x}$. Theorem 1 in [25] assumes that

$$\int \lambda_+ \, d\hat{\mu} = \int \lambda_- \, d\hat{\mu} \tag{22}$$

or, equivalently, $\lambda_+(\hat{x}) = \lambda_-(\hat{x})$ for $\hat{\mu}$-almost every $\hat{x}$. This implies the hypothesis of Theorem A. To see this, notice that, locally, the points of $\mathbb{P}(\mathbb{R}^d)$ may be represented by unit vectors $\hat{\xi}$. Then

$$\hat{F}^n_{\hat{x}}(\hat{\xi}) = \frac{A^n(\hat{x})\hat{\xi}}{\|A^n(\hat{x})\hat{\xi}\|}$$

for every $\hat{x}$, $\hat{\xi}$, and n. It follows that,

$$D\hat{F}^n_{\hat{x}}(\hat{\xi})\dot{\xi} = \frac{\mathrm{proj}_{A^n(\hat{x})\hat{\xi}}(A^n(\hat{x})\dot{\xi})}{\|A^n(\hat{x})\hat{\xi}\|},$$

where $\operatorname{proj}_u v = v - u(u \cdot v)/(u \cdot u)$ is the projection of v to the orthogonal complement of u. This implies that

$$\|D\hat{F}^n_{\hat{x}}(\hat{\xi})\| \le \|A^n(\hat{x})\|/\|A^n(\hat{x})\hat{\xi}\| \le \|A^n(\hat{x})\|\|A^n(\hat{x})^{-1}\| \tag{23}$$

for every x, $\hat{\xi}$, and n. Replacing n by $-n$ and $(\hat{x}, \hat{\xi})$ by an appropriate iterate, it also follows that

$$\|D\hat{F}^n_{\hat{x}}(\hat{\xi})^{-1}\| \le \|A^n(\hat{x})^{-1}\|\|A^n(\hat{x})\| \tag{24}$$

for every x, $\hat{\xi}$, and n. The last two inequalities imply that

$$\lambda_-(\hat{x}) - \lambda_+(\hat{x}) \le \lambda_-(\hat{F}, \hat{x}, \hat{\xi}) \le \lambda_+(\hat{F}, \hat{x}, \hat{\xi}) \le \lambda_+(\hat{x}) - \lambda_-(\hat{x}).$$

Hence, (22) implies $\lambda_+(\hat{F}, \hat{x}, \hat{\xi}) = \lambda_-(\hat{F}, \hat{x}, \hat{\xi}) = 0$ for m-almost every $(\hat{x}, \hat{\xi})$.

Similar observations apply in the more general case of projective cocycles on *Grassmannian bundles,* such as considered in [5, 6].

Example 2.15. Let $F : M \times \mathbb{P}(\mathbb{R}^2) \to M \times \mathbb{P}(\mathbb{R}^2)$ be the projective cocycle defined by some $A : M \to \mathrm{SL}(2, \mathbb{R})$ over a non-invertible system (f, μ). Let $\hat{F}$ and $(\hat{f}, \hat{\mu})$ be the natural extensions and the σ-algebra $\mathcal{B}_0$ be as in Example 2.13. Assume the Lyapunov exponents are distinct

$$\lambda_-(\hat{x}) < \lambda_+(\hat{x}) \quad \text{at almost every point}$$

and let $E^-_{\hat{x}}$ and $E^+_{\hat{x}}$ be the Oseledets subspaces, viewed as elements of the projective space. Notice that

$$\lambda_-(\hat{F}, \hat{x}, \hat{\xi}) = \lambda_+(\hat{F}, \hat{x}, \hat{\xi}) = \begin{cases} \lambda_+(\hat{x}) - \lambda_-(\hat{x}) & \text{for } \hat{\xi} = E^-_{\hat{x}} \\ \lambda_-(\hat{x}) - \lambda_+(\hat{x}) & \text{for } \hat{\xi} = E^+_{\hat{x}}. \end{cases}$$

Consider the $\hat{F}$ invariant measures m^- and m^+ whose conditional probabilities along the fibers are the Dirac masses at $\hat{\xi} = E^-_{\hat{x}}$ and $\hat{\xi} = E^+_{\hat{x}}$, respectively, and whose projections down to M coincide with $\hat{\mu}$. Then $\lambda_-(\hat{F}, \hat{x}, \hat{\xi}) > 0$ at m^--almost every point and so we may use Theorem C to conclude that the disintegration $\hat{x} \mapsto \delta_{E^-_{\hat{x}}}$ is $\mathcal{B}_0$-measurable. This conclusion is also an immediate consequence of the observation that the contracting subspace $E^-_{\hat{x}}$ depends only on the future iterates. On the other hand, $\lambda_+(\hat{F}, \hat{x}, \hat{\xi}) < 0$ at m^--almost every point and yet $\hat{x} \mapsto \delta_{E^+_{\hat{x}}}$ is usually not $\mathcal{B}_0$-measurable: the expanding subspace is determined by the past, not the future iterates of the cocycle.

3. Cocycles with invariant holonomies

For simplicity, **from now on we write M, $\mathcal{B}$, μ, f, $\mathcal{E}$, P, F, m in the place of $\hat{M}$, $\hat{\mathcal{B}}$, $\hat{\mu}$, $\hat{f}$, $\hat{\mathcal{E}}$, $\hat{P}$, $\hat{F}$, $\hat{m}$.** In this section we prove Theorem C and Corollary D. Let $\hat{M}$ be a complete metric space, $\hat{\mathcal{E}}$ be a continuous fiber bundle endowed with a continuous Riemannian metric, and $\hat{F}$ be a continuous smooth cocycle.

3.1. **Holonomy invariance.** An *s-lamination* for the transformation $f : M \to M$ is a partition $W^s = \{W^s(x) : x \in M\}$ of M such that there exist ε, K, $\tau > 0$, a family $W^s_\varepsilon = \{W^s_\varepsilon(x) : x \in M\}$ of closed subsets of M, and a function $\nu(\cdot)$ such that

(sl1) $f(W^s_\varepsilon(x)) \subset W^s_\varepsilon(f(x))$ and $W^s(x) = \cup_{n \ge 0} f^{-n}(W^s_\varepsilon(f^n(x)))$;
(sl2) $d(f(y_1), f(y_2)) \le \nu(x) d(y_1, y_2)$ for all $y_1, y_2 \in W^s_\varepsilon(x)$;

(sl3) $\nu_n(x) := \nu(f^{n-1}(x)) \cdots \nu(x) \le K e^{-\tau n}$ for all $x \in M$ and $n \ge 1$.

Suppose f admits some s-lamination W^s. A subset of M is *s-saturated* (relative to W^s) if it consists of entire leaves of W^s. An *s-holonomy* for F (relative to W^s) is a family h^s of homeomorphisms $h^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$ defined for all $y \in W^s(x)$ and satisfying conditions (sh1), (sh2), (sh3) in Section 1.4.

Let $\mathcal{M}(\mu)$ be the set of probabilities on $\mathcal{E}$ that project down to μ. Suppose F admits an s-holonomy h^s (relative to W^s). An F-invariant probability measure $m \in \mathcal{M}(\mu)$ is called an *s-state* (relative to h^s) if it admits some disintegration $\{m_x : x \in \operatorname{supp}\mu\}$ which is *essentially s-invariant*, meaning that

$$(h^s_{x,y})_* m_x = m_y \quad \text{for every } y \in W^s(x) \tag{25}$$

with x and y in some full μ-measure subset E. Then the same is true for any other disintegration of m. The full measure set may always be taken to be s-saturated: just consider the union E' of all W^s leaves through E and modify the disintegration on the (zero measure) set $E' \setminus E$ so as to enforce (25) for all points of E'.

Replacing f and F by their inverses, one obtains dual notions of u-lamination W^u, u-saturated set, u-holonomy h^u, u-invariant disintegration, and u-state. We say that the cocycle F *admits invariant holonomies* if it admits both s-holonomy and u-holonomy. Then we call an invariant probability an *su-state* if it is both an s-state and a u-state.

Remark 3.1. In some cases existence of s-states or u-states can be ensured a priori. For instance, if the fiber N is compact then one may start with a product measure $\mu \times \nu$ on $\mathcal{E}$ and consider Cesaro limits of its forward/backward iterates: any such limit is a u-state/s-state. See [13, Proposition 4.2] for a similar construction.

Proposition 3.2. *Assume F admits s-holonomy. Let $m \in \mathcal{M}(\mu)$ be such that there exists a sequence $(m_k)_k$ of F-invariant probability measures converging to m in the weak* topology, such that $\int \min\{0, \lambda_-(F,\cdot)\}\, dm_k \to 0$ as $k \to \infty$. Then m is an s-state.*

The proof of Proposition 3.2 occupies Sections 3.2 and 3.3. Let us also register the following particular case, corresponding to $m_k \equiv m$:

Corollary 3.3. *Assume F admits s-holonomy. Let $m \in \mathcal{M}(\mu)$ be an F-invariant probability measure such that $\lambda_-(F, x, \xi) \ge 0$ at m-almost every point. Then m is an s-state.*

Replacing F by its inverse one obtains dual statements for cocycles admitting u-holonomy. In particular: if $m \in \mathcal{M}(\mu)$ is an F-invariant probability measure such that $\lambda_+(F, x, \xi) \le 0$ at m-almost every point then m is a u-state. Compare Example 2.15.

3.2. **Local Markov property.** We need the following local Markov property of the s-lamination. Fix $\ell \ge 1$ such that $K e^{-\tau \ell} < 1/4$ and let $g = f^\ell$.

Lemma 3.4. *There exists $\delta > 0$ and for every $x \in M$ there exists a partition $Q = Q_x$ of M such that*

1. *if $g^j(Q(z))$ intersects $Q(w)$ then $g^j(Q(z)) \subset Q(w)$*
2. *every $Q(y)$ is contained in some $W^s_\varepsilon(z)$*
3. *$W_\varepsilon(y) \cap B(x, \delta) \subset Q(y) \subset W^s_\varepsilon(y)$ for every $y \in B(x, \delta)$.*

Proof. Pick $\delta < \varepsilon/4$ and write $V = B(x, \delta)$. For the sake of clearness, we split the proof into three steps:

Step 1: We claim that, for any $z, w \in V$ and $k \geq 0$, if $g^k(W^s_\varepsilon(w))$ intersects $W^s_\varepsilon(z) \cap V$ then either $k = 0$ and $W^s_\varepsilon(w) \cap V = W^s_\varepsilon(z) \cap V$ or $g^k(W^s_\varepsilon(w)) \subset W^s_\varepsilon(z)$. We call this the pre-Markov property. For the proof, consider any $z, w, p \in V$ such that $p \in g^k(W^s_\varepsilon(w)) \cap W^s_\varepsilon(z) \cap V$. Suppose first that $k = 0$. Let q be any point in $W^s_\varepsilon(w) \cap V$. Since $W^s(z)$ and $W^s(w)$ intersect each other (at p), they must coincide. It follows that $q \in W^s(z)$. Given our choices of ℓ and δ, and using that $p \in W^s_\varepsilon(z) \cap V$ and $p, q \in W^s_\varepsilon(w) \cap V$,

$$\begin{aligned} d(g^j(z), g^j(q)) &\leq d(g^j(z), g^j(p)) + d(g^j(p), g^j(q)) \\ &\leq Ke^{-j\ell} d(z, p) + Ke^{-j\ell} d(p, q) \leq 4^{-j} 4\delta \leq \varepsilon \end{aligned}$$

for all $j \geq 0$. This proves that $q \in W^s_\varepsilon(z)$, and so we have shown that $W^s_\varepsilon(w) \cap V \subset W^s_\varepsilon(z) \cap V$. The converse inclusion follows as well, by symmetry. Now suppose $k \geq 1$. Let q be any point in $g^k(W^s_\varepsilon(w))$. Since $W^s(g^k(w))$ and $W^s(z)$ intersect each other, they must coincide. It follows that $q \in W^s(z)$. Since $p \in W^s_\varepsilon(z)$ and $p, q \in g^k(W^s_\varepsilon(w))$

$$\begin{aligned} d(g^j(z), g^j(q)) &\leq d(g^j(z), g^j(p)) + d(g^j(p), g^j(q)) \\ &\leq 4^{-j} 2\delta + 4^{-(j+k)} 2\varepsilon \leq \varepsilon \end{aligned}$$

for all $j \geq 0$. This shows that $q \in W^s_\varepsilon(z)$, which completes the proof of the claim.

Step 2: Call P a stable pre-piece of rank $k \geq 0$ if $P = g^k(W^s_\varepsilon(y) \cap V)$ for some $y \in V$. By the pre-Markov property, two stable pre-pieces of rank 0 either coincide or are disjoint. Since g is invertible, the same holds for any two stable pre-pieces of the same rank $k \geq 0$. Call $P_0, ..., P_n$ a chain if the P_i are stable pre-pieces and $P_i \cap P_{i-1} \neq \emptyset$ for $1 \leq i \leq n$. Denote by k_j the rank of each p_j and call n the length of the chain. We claim that if $P_0, ..., P_n$ is a chain then

$$\bigcup_{j=0}^{n} P_j \subset g^{k_s}(W^s_\varepsilon(g^{-k_s}(z))) \tag{26}$$

for any $z \in P_s$ with rank $k_s = \min_{0 \leq j \leq n} k_j$. To see this, we argue by induction on n. The case $n = 0$ is obvious. Assume the claim is true for every $m < n$, and let $P_0, ..., P_n$ be any chain as above. If $0 < s < n$ then both $P_0, \ldots, P_s$ and $P_s, \ldots, P_n$ are chains with smaller lengths, and so the conclusion follows immediately from the induction hypothesis. Hence, we may suppose either $s = 0$ or $s = n$. In what follows we deal with the former case, the latter being entirely analogous. We may also assume that $k_s = 0$, up to replacing z and all the P_j by their pre-images under g^{k_s}. The definition of chain implies that $P_0 = W^s_\varepsilon(z) \cap V$ intersects the union of the other P_j, $1 \leq j \leq n$. By the induction hypothesis,

$$\bigcup_{j=1}^{n} P_j \subset g^{k_r}(W^s_\varepsilon(g^{-k_r}(\zeta)))$$

for some $\zeta \in P_r$, $1 \leq r \leq n$ with rank $k_r = \min_{1 \leq j \leq n} k_j$. If $k_r = 0$ then the pre-Markov property implies that $W^s_\varepsilon(\zeta) \cap V = W^s_\varepsilon(z) \cap V$, and so the union of all P_j, $0 \leq j \leq n$ is indeed contained in $W^s_\varepsilon(z)$. Similarly, if $k_r > 0$ then the pre-Markov property implies

$$g^{k_r}(W^s_\varepsilon(g^{-k_r}(\zeta))) \subset W^s_\varepsilon(z)$$

and so the union of all P_j, $0 \le j \le n$ is again contained in $W^s_\varepsilon(z)$. This completes the induction step.

Step 3: Define the stable piece of a point $y \in M$ to be the set $Q(y)$ all $\xi \in M$ such that there exists a chain $P_0, ..., P_n$ with $y \in P_0$ and $\xi \in P_n$. If no such point ξ exists, just let $Q(y) = \{y\}$ instead. It is clear that the image under g of a stable piece is contained in a stable piece, and any two stable pieces that intersect must coincide. This implies property (1) in the lemma. Next, the property in (26) gives that $Q(y) \subset g^k(W^s_\varepsilon(g^{-k}(z))) \subset W^s_\varepsilon(z)$ for any z in a pre-piece with minimum rank k. This gives (2). If $y \in V$ then $Q(y) \supset W^s_\varepsilon(y) \cap V$, because one may always take $P_0 = W^s_\varepsilon(y) \cap V$. Moreover, in this case $k = 0$ and $Q(y) \subset W_\varepsilon(y)$. This proves (3) and the lemma. □

Remark 3.5. The support of μ admits countable covers by balls of radius r, for any $r > 0$. Let X be a maximal subset of the support such that the balls of radius $r/2$ are pairwise disjoint. Maximality implies that the balls of radius r centered at the points of X cover the support. On the other hand, X is countable, because every $B(x, r/2)$, $x \in X$ has positive measure, and the sum of these measures is finite.

Lemma 3.6. *There exists a measurable function $\pi : M \to M$ constant on every stable piece $Q(y)$ and such that $\pi(Q(y))$ is in the closure of $Q(y)$ and, consequently, $d(y, \pi(y)) \le 2\varepsilon$, for every $y \in M$.*

Proof. For each $k \ge 1$ let $\{B^j_k : j \ge 1\}$ be a countable cover of the support of μ by closed balls (recall Remark 3.5). Define $Q_0(y)$ to be the closure of $Q(y)$ and, for $k \ge 1$,

$$Q_k(y) = Q_{k-1}(x) \cap B^j_k$$

where $j = j(k)$ is smallest such that the intersection is non-empty. These $Q_k(y)$ are a nested sequence of non-empty closed subsets of $W^s_\varepsilon(y)$ with diameters going to zero. Since M is complete, their intersection contains exactly one point: define $\pi(y)$ to this point. Measurability is clear, since the definition is constructive. By construction, $\pi(y)$ belongs to the closure of $Q(y)$. Using part (2) of Lemma 3.4, we find that y and $\pi(y)$ are contained in the same $W^s_\varepsilon(z)$ and so the distance between them is bounded by 2ε. □

3.3. **Lyapunov exponents and holonomy invariance.** In this section we conclude the proof of Proposition 3.2. Let $\ell \ge 1$ and $\delta > 0$ be as in Lemma 3.4. The main remaining step is to show that every disintegration of m is essentially s-invariant restricted to the δ-neighborhood of any point $x \in M$. This will be done by applying Theorem B to a deformation of the cocycle $G = F^\ell$ over $g = f^\ell$, more precisely, to a cocycle which is conjugate to G via s-holonomies. Covering M with these neighborhoods we obtain a disintegration of m which is essentially s-invariant on the whole space.

Clearly, m is invariant under G and $\lambda_-(G, x, \xi) = \ell\lambda_-(F, x, \xi) \ge 0$ for m-almost every point. Given any $x \in M$, let V be its δ-neighborhood, and let $Q = Q_x$ be the partition constructed in Lemma 3.4. Let $\mathcal{B}_0$ be the σ-algebra of measurable subsets of M that are unions of entire atoms of Q. In other words, a measurable subset E belongs to $\mathcal{B}_0$ if and only if every stable piece is either contained in or disjoint from E. Notice that g is $\mathcal{B}_0$-measurable, because the image of any stable piece is contained in a stable piece. Let $\pi : M \to M$ be as in Lemma 3.6.

Let $\tilde{G} = \tilde{G}(x) : \mathcal{E} \to \mathcal{E}$ be the transformation defined by

$$\tilde{G}_y = h^s_{g(\pi(y)),\pi(g(y))} \circ G_{\pi(y)}.$$

Then $\tilde{G}$ is $\mathcal{B}_0$-measurable, because π is constant on stable pieces and the image of every stable piece is contained in a stable piece. Property (sh2) applied to G (the properties in the definition of s-holonomy remain valid when one replaces the cocycle by any forward iterate) yields

$$\tilde{G}_y = h^s_{g(y),\pi(g(y))} \circ G_y \circ h^s_{\pi(y),y} \tag{27}$$

for every $y \in M$. This relation can be rewritten as $\tilde{G} = \Phi \circ G \circ \Phi^{-1}$, where $\Phi : \mathcal{E} \to \mathcal{E}$ is given by $\Phi(y, \xi) = (y, h^s_{y,\pi(y)}(\xi))$. Then $\tilde{G}$ is a deformation of G, since Φ and its inverse are β-Hölder continuous on every fiber. Let $\tilde{m}$ be the probability measure on $\mathcal{E}$ defined by $\tilde{m} = \Phi_*(m)$. Clearly, it is invariant under $\tilde{G}$, it projects down to μ, and its conditional probabilities along the fibers are given by

$$\tilde{m}_y = (h^s_{y,\pi(y)})_* m_y. \tag{28}$$

So, we are in a position to apply Theorem B to conclude that the disintegration $\{\tilde{m}_y\}$ is $\mathcal{B}_0$-measurable: there exists a full μ-measure subset restricted to which $\tilde{m}_y$ is constant on stable pieces $Q(y)$. Through (27), this gives rise to a disintegration $\{m_y\}$ of m which is s-invariant on each stable piece:

$$m_z = (h^s_{\pi(z),z})_* \tilde{m}_z = (h^s_{\pi(w),z})_* \tilde{m}_w = (h^s_{w,z})_* (h^s_{\pi(w),w})_* \tilde{m}_w = (h^s_{w,z})_* m_w$$

for any z and w in any $Q(y)$. In particular, the disintegration of m is essentially s-invariant restricted to the ball $V = V(x)$ of radius δ around every $x \in M$. Now, consider any countable set $\{x_n\} \subset \operatorname{supp} \mu$ such that the balls of radius $\delta/2$ around these x_n cover the support of μ (see Remark 3.5). For each n, let B_n be a zero μ-measure subset of the ball $V(x_n)$ of radius δ around x_n such that

$$m_w = (h^s_{z,w})_* m_z \quad \text{for all } z, w \in V(x_n) \setminus B_n \text{ in the same stable piece.} \tag{29}$$

Let E be the set of all point $x \in \operatorname{supp} \mu$ whose orbits never meet $\cup_n B_n$ and such that

$$m_{f^l(x)} = (F^l_x)_* m_x \quad \text{for all } l \in \mathbb{Z}. \tag{30}$$

Then E has full μ-measure in M. By properties (sl1)-(sl3) in the definition of s-lamination, given any $x, y \in E$ with $y \in W^s(x)$ there exists $k \geq 1$ such that $d(f^l(x), f^l(y)) \leq \delta/2 \quad (\leq \varepsilon)$ for all $l \geq k$. Fix n such that $f^k(x) \in B(x_n, \delta/2)$. Then $f^k(y) \in V(x_n) \cap W^s_\varepsilon(f^k(x))$. So, by Lemma 3.4, the two points $z = f^k(x)$ and $w = f^k(y)$ belong to the same stable piece (associated to x_n). Since they are outside B_n, we may combine (29) and (30) with property (sh2) of s-holonomies to conclude that $m_y = (h^s_{x,y})_* m_x$. This proves that the disintegration of m is essentially s-invariant, as claimed. This finishes the proof of Proposition 3.2.

3.4. **Product structure and continuity.** In particular, if the cocycle F admits invariant holonomies, then any invariant measure $m \in \mathcal{M}(\mu)$ for which Lyapunov exponents vanish almost everywhere is an su-state. This means that m has some disintegration which is s-invariant on a full measure s-saturated set and some disintegration which is u-invariant on a full measure u-saturated set. We are now going to discuss additional conditions under which these two disintegrations may be taken to coincide. Based on this we prove Theorem C and Corollary D.

Assume the s-lamination and the u-lamination of $f : M \to M$ satisfy a local product condition like (h5) in Section 1.3: *there is $\delta > 0$ such that if $d(x_1, x_2) \le \delta$ then $W^u_\varepsilon(x_1)$ and $W^s_\varepsilon(x_2)$ intersect at exactly one point, and this point $[x_1, x_2]$ depends continuously on (x_1, x_2).* Given any $x \in M$, let $B^s(x)$ and $B^u(x)$ be the balls of radius $\delta/2$ around x inside $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$, respectively. Then

$$\phi : B^s(x) \times B^u(x) \to M, \quad (x_1, x_2) \mapsto [x_1, x_2]$$

defines a homeomorphism from $B^s(x) \times B^u(x)$ to a neighborhood $B(x)$ of x. Indeed, assumption (h5) ensures that ϕ is injective, its image covers a neighborhood of x, and its inverse is continuous. Assume also that μ has local product structure. As observed before, this implies that the support of μ is su-saturated.

Proposition 3.7. *Assume F admits s-holonomy and u-holonomy. Assume f and μ have local product structure, as described above. If m is an su-state then it admits a disintegration which is s-invariant and u-invariant and whose conditional probabilities m_x vary continuously with x in the support of μ.*

Proof. It suffices to prove that given any $z \in \operatorname{supp} \mu$ there exists a disintegration of m which satisfies the conclusion restricted to $B(z)$. By assumption, there exists some s-invariant disintegration $\{m^s_x\}$ and some u-invariant disintegration $\{m^u_x\}$. By essential uniqueness, $m^u_x = m^s_x$ for every x in some full μ-measure set $E \subset M$. Using product structure, there exists $x^u \in B^u(x)$ such that E intersects $\{x^u\} \times B^s(x)$ on a full μ^s-measure set. Define

$$m_x = (h^u_{y,x})_* m^s_y \quad \text{for every } x \in B(z).$$

Note that $y \mapsto m^s_y$ is continuous on $\{x^u\} \times B^s(x)$, because m^s_x is s-invariant and s-holonomies vary continuously with the point. Using that u-holonomies also vary continuously with the point, we get that $x \mapsto m_x$ is continuous on $B(z)$. By construction, m_x is u-invariant. Moreover, it coincides with m^u_x almost everywhere, due to product structure and the choice of x^u. Hence, m_x also coincides with m^s_x almost everywhere. By continuity, it follows that m_x is also s-invariant. □

Proof of Theorem C. Clearly, the assumption $\int |\lambda_\pm(F, \cdot)| \, dm_k \to 0$ is stronger than $\int \min\{0, \lambda_-(F, \cdot)\} \, dm_k \to 0$. So, by Proposition 3.2, the measure m is an s-state. Since the assumption is symmetric under time reversion, we may apply the proposition to F^{-1} as well, to conclude that m is also a u-state. Now the conclusion of the theorem follows from Proposition 3.7. □

Remark 3.8. The same arguments yield a more local version Theorem C. Let $U \subset M$ be an invariant, s-saturated, u-saturated set, with positive Lebesgue measure. An *su-state over U* is an invariant measure that projects down to the normalized restriction of the Lebesgue measure to U and which admits some essentially s-invariant disintegration and some essentially u-invariant disintegration. Then it admits a disintegration which is su-invariant and continuous.

Proof of Corollary D. Since the fiber is 1-dimensional, $\lambda_-(F, \cdot) = \lambda_+(F, \cdot)$ wherever they are defined. Suppose there is a sequence $(m_k)_k$ of ergodic probability measures projecting down to μ and such that $\lambda_\pm(F, m_k) \to 0$ as $k \to \infty$. By ergodicity, this is the same as the condition in the assumption of Theorem C. Since the fiber is compact, the sequence must have some accumulation point. Every accumulation point m is also an invariant measure that projects down to μ. By Theorem C, the measure m admits a disintegration $\{m_x : x \in M\}$ which is s-invariant, u-invariant, and continuous on the support of μ. By invariance and continuity, this disintegration satisfies

$$(F_x)_* m_x = m_{f(x)}$$

for every point in $\operatorname{supp}\mu$. In particular,

$$(F_p)_* m_p = m_p \quad \text{and} \quad (F_q)_* m_q = m_q. \tag{31}$$

The first equality implies that the support of m_p is contained in the subset $\{a_p, r_p\}$ of the circle. Let $z \in W^u(p) \cap W^s(q)$. By invariance of the conditional probabilities,

$$m_q = (h^s_{z,q} \circ h^u_{p,z})_* m_p.$$

Consequently, $\operatorname{supp} m_q$ contains at most two points. The second equality in (31) implies that the support is invariant under F_q. It follows that F_q has periodic points of period 1 or 2, which contradicts the assumption of the corollary. This contradiction proves that the exponent is indeed bounded away from zero. □

4. Domination and fiber bunching

Here we introduce a number of ideas that will be useful for analyzing the dependence of Lyapunov exponents on the cocycle. We take the fiber manifold N to be compact; towards the end of the section, we assume the fiber bundle $\mathcal{E}$ itself is compact. In addition to the conditions in the previous section, we assume the fiber bundle and the cocycle to be Lipschitz, in the sense of Section 1.5. We also consider the Lipschitz topology in the space of Lipschitz cocycles introduced in Section 1.5.

4.1. **Existence of holonomies.** Assume f admits an s-lamination W^s. We call a cocycle F *s-dominated* (relative to W^s) if there exist $\ell \geq 1$ and $\theta < 1$ such that

$$\|DF^\ell_x(\xi)^{-1}\| \, \nu_\ell(x) \leq \theta \quad \text{for all } (x,\xi) \in \mathcal{E}. \tag{32}$$

In other words, the strongest contraction of F^ℓ along the fibers is strictly weaker than the weakest contraction of f^ℓ along the leaves of W^s. Replacing f and F by its inverses, we obtain the dual notion of *u-dominated cocycle*. Denote by $\mathcal{D}^s(f)$ the subset of s-dominated cocycles and by $\mathcal{D}^u(f)$ the subset of u-dominated cocycles. It is clear from the definition that these are open sets for the topology we have just introduced. When f admits both an s-lamination and a u-lamination, we let $\mathcal{D}(f) = \mathcal{D}^s(f) \cap \mathcal{D}^u(f)$ be the subset of *dominated* cocycles.

The s-domination condition is designed so that the usual graph transform argument yields a "strong-stable" lamination for the map F (there is a dual statement for u-dominated cocycles):

Proposition 4.1. *If the cocycle F is s-dominated then there exists a unique partition $\mathcal{W}^s = \{\mathcal{W}^s(x,\xi) : (x,\xi) \in \mathcal{E}\}$ of the fiber bundle $\mathcal{E}$ such that*

1. *every $\mathcal{W}^s(x,\xi)$ is a Lipschitz graph over $W^s(x)$, with Lipschitz constant uniform on x;*
2. *$F(\mathcal{W}^s(x,\xi)) \subset \mathcal{W}^s(F(x,\xi))$ for all $(x,\xi) \in \mathcal{E}$;*
3. *the map $h^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$ defined by $(y, h^s_{x,y}(\xi)) \in \mathcal{W}^s(x,\xi)$, for $y \in W^s(x)$, coincides with the uniform limit of $(F^n_y)^{-1} \circ F^n_x$ as $n \to \infty$;*
4. *the family of maps $h^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$ is an s-holonomy for F.*

Proof. The claims follow from the same partial hyperbolicity methods (see Hirsch, Pugh, Shub [23]) used before to obtain similar results for linear cocycles [11, 13, 30], and so we just sketch the main ingredients. Existence (1) and invariance (2) of the family $\mathcal{W}^s$ follow from a standard application of the graph transform argument [23].

Notice that, for every x and y in the same stable manifold, and every $n \geq 0$,

$$h^s_{x,y} = (F^n_y)^{-1} \circ h^s_{f^n(x),f^n(y)} \circ F^n_x \tag{33}$$

and the uniform distance from $h^s_{f^n(x),f^n(y)}$ to the identity map is bounded by $Cd(f^n(x), f^n(y))$, where C is the uniform Lipschitz constant in (1). Putting these observations together, we find that

$$\begin{aligned} d_{C^0}(h^s_{x,y}, (F^n_y)^{-1} \circ F^n_x) &\leq \operatorname{Lip}\left((F^n_y)^{-1}\right) d_{C^0}(h^s_{f^n(x),f^n(y)}, \mathrm{id}) \\ &\leq C \sup_\xi \|DF^n_y(\xi)^{-1}\| \, d(f^n(x), f^n(y)). \end{aligned}$$

Fix ℓ as in the domination condition (32) and write $k = [n/\ell]$. Clearly,

$$d(f^n(x), f^n(y)) \leq K d(f^{k\ell}(x), f^{k\ell}(y)) \leq K \prod_{i=0}^{k-1} \nu_\ell(f^{i\ell}(y)) d(x, y)$$

and $\|DF^n_y(\xi)^{-1}\|$ is similarly bounded above by a product of norms of the derivative of $(F^\ell)^{-1}$ along the orbit of y. Using the domination condition (32) we conclude that

$$d_{C^0}(h^s_{x,y}, (F^n_y)^{-1} \circ F^n_x) \leq \operatorname{const} \theta^k \leq \operatorname{const} \theta^{n/\ell},$$

where the constants are independent of n, x, y. This proves (3).

Conditions (sl1)-(sl3) in the definition of s-holonomy are direct consequences of the definition of $h^s_{x,y}$. Thus, to prove (4) we only have to check that the maps $h^s_{x,y}$ are Hölder continuous, with uniform exponential Hölder constant. The arguments are quite standard, see for instance [1, 23]. In view of (33), and the fact that the F_z and their inverses are Lipschitz, it is no restriction to assume that x and y are in the same local stable set. For each $n \geq 0$, denote $x_n = f^n(x)$ and $y_n = f^n(y)$. Given $\xi, \eta \in \mathcal{E}_x$, denote $\xi_n = F^n_x(\xi)$ and $\eta_n = F^n_x(\eta)$. By domination, there exists $K > 0$ and

$$n \leq -K \log d(\xi, \eta) \tag{34}$$

such that $d(x_n, y_n) \leq d(\xi_n, \eta_n)$. Then

$$d(h^s_{x_n,y_n}(\xi), h^s_{x_n,y_n}(\eta)) \leq d(\xi_n, \eta_n) + 2Cd(x_n, y_n) \leq 3Cd(\xi_n, \eta_n),$$

where C is the uniform Lipschitz constant in (1). Let L be a uniform upper bound for the norms of $\|DF^{\pm 1}_z\|$. The previous inequality yields

$$L^{-n} d(h^s_{x,y}(\xi), h^s_{x,y}(\eta)) \leq 3CL^n d(\xi, \eta).$$

In view of (34), we have $L^{2n} \leq d(\xi, \eta)^{-\theta}$ for some uniform constant $\theta \in (0, 1)$. Then the previous inequality gives

$$d(h^s_{x,y}(\xi), h^s_{x,y}(\eta)) \leq 3Cd(\xi, \eta)^{1-\theta},$$

which proves our claim. □

4.2. **Continuity of holonomies.** We are going to see that s-holonomies vary continuously with the cocycle on $\mathcal{D}^s(f)$. Of course, there is a dual statement for u-holonomies on $\mathcal{D}^u(f)$. Let $\mathcal{W}^s(G) = \{\mathcal{W}^s(G,x,\xi) : (x,\xi) \in \mathcal{E}\}$ denote the "strong-stable" lamination of a cocycle $G \in \mathcal{D}^s$, as in Proposition 4.1, and $h^s_G = h^s_{G,x,y}$ be the corresponding s-holonomy:

$$(y, h^s_{G,x,y}(\xi)) \in \mathcal{W}^s(G,x,\xi). \tag{35}$$

Recall $\mathcal{W}^s(G,x\xi)$ is a graph over $W^s(x)$. We also denote by $\mathcal{W}^s_\varepsilon(G,x,\xi)$ the part of the graph located over $W^s_\varepsilon(x)$, that is, the set of all points $(y, h^s_{G,x,y}(\xi))$ with $y \in W^s_\varepsilon(x)$.

Proposition 4.2. *Let $(F_k)_k$ be a sequence of cocycles converging to F in $\mathcal{D}^s(f)$. Then*

(1) *every $\mathcal{W}^s(F_k,x,\xi)$ is a Lipschitz graph, with Lipschitz constant uniform on x, ξ, and k*
(2) *$\mathcal{W}^s_\varepsilon(F_k,x,\xi)$ converges to $\mathcal{W}^s_\varepsilon(F,x,\xi)$, as graphs over the same domain, uniformly on $(x,\xi) \in \mathcal{E}$*
(3) *$h^s_{F_k,x,y}(\xi)$ converges to $h^s_{F,x,y}(\xi)$ for every $x \in M$, $y \in W^s(x)$, and $\xi \in \mathcal{E}_x$, and the converse is uniform over all $y \in W_\varepsilon(x)$.*

Proof. This is another standard consequence of the classical graph transform argument [23]. Indeed, the assumptions imply that the graph transform of F_k converges to the graph transform of F in an appropriate sense, so that the corresponding fixed points converge as well. This yields (1) and (2). Part (3) is a direct consequence of (2) and the definition (35), in the case $y \in W^s_\varepsilon(y)$. The general statement follows, using the invariance property (sh2):

$$H^s_{F_k,x,y} = (F^n_{k,y})^{-1} \circ h_{F_k,f^n(x),f^n(y)} \circ F^n_{k,x}.$$

See also [30, Section 4] where stronger results are proved in detail using similar methods, in the context of linear cocycles. □

Corollary 4.3. *The subset of cocycles admitting some su-state is closed in $\mathcal{D}(f)$.*

Proof. If $F_k \to F$ in $\mathcal{D}(f)$ and m_k are su-states for F_k projecting down to μ then any weak limit m of the sequence m_k is an su-state for F projecting down to μ. □

4.3. **Projective extension.** Let $F : \mathcal{E} \to \mathcal{E}$ be a dominated cocycle. It will be convenient to think of F as a transformation in its own right, and to consider a certain smooth cocycle $\mathbb{P}(F)$ over F that we call projective extension. Here we define $\mathbb{P}(F)$ and discuss a stronger domination condition, called fiber bunching, that ensures robust existence of holonomies for $\mathbb{P}(F)$.

The partition $\mathcal{W}^s = \mathcal{W}^s(F)$ given by Proposition 4.1 is an s-lamination for F: in particular, since strong-stable leaves are Lipschitz graphs (Proposition 4.1) and local charts are Lipschitz, we have

$$d(F^n(y,\eta), F^n(z,\zeta)) \le C_0 \nu_n(x)\, d((y,\xi),(z,\zeta)) \tag{36}$$

for every $(y,\eta),(z,\zeta) \in \mathcal{W}^s(x,\xi)$, where $C_0 > 1$ is a uniform constant. Analogously, the "strong-unstable" lamination $\mathcal{W}^u = \mathcal{W}^u(F)$ is a u-lamination for F. In addition, we consider the *c-lamination*

$$\mathcal{W}^c = \{\mathcal{W}^c(x,\xi) = \mathcal{E}_x : (x,\xi) \in \mathcal{E}\}.$$

Let $\mathbb{P}(\mathcal{E})$ be the projective tangent bundle of $\mathcal{E}$, that is, the fiber bundle over $\mathcal{E}$ such that the fiber of each (x, ξ) is the projectivization of the tangent space $T_\xi \mathcal{E}_x$. The *projective extension* of F is the smooth cocycle $\mathbb{P}(F) : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ over $F : \mathcal{E} \to \mathcal{E}$ defined by

$$\mathbb{P}(F)(x, \xi, [v]) = (f(x), F_x(\xi), [DF_x(\xi)v]), \quad \text{for each } [v] \in \mathbb{P}(T_\xi \mathcal{E}_x).$$

Notice $\mathbb{P}(\mathcal{E})$ is also a fiber bundle over M, with fiber $\mathbb{P}(T\mathcal{E}_x)$, and one may think of $\mathbb{P}(F)$ as a cocycle over $f : M \to M$ instead. However, this will usually not be our point of view: instead, most of the time, we think of $\mathbb{P}(F)$ as a cocycle over F itself.

Assume the cocycle F is s-dominated. We say that F is *s-fiber bunched* if there exist $\ell \geq 1$ and $\theta < 1$ such that

$$\|DF_x^\ell(\xi)\| \|(DF_x^\ell(\xi))^{-1}\| \, \nu_\ell(x) \leq \theta \quad \text{for every } (x, \xi) \in \mathcal{E}. \tag{37}$$

The product of the first two factors bounds the norm of the derivative $\mathbb{P}(F)^\ell$ and its inverse. Recall (23) and (24). Thus, this condition means that the strongest contraction of $\mathbb{P}(F)^\ell$ along the fibers $\mathbb{P}(T_\xi \mathcal{E}_x)$ is strictly weaker than the weakest contraction of f^ℓ along the leaves of W^s.

It is easy to see that s-fiber bunching implies that $\mathbb{P}(F)$ is s-dominated relative to the s-lamination $\mathcal{W}^s$ of F and, consequently, has s-holonomy. Indeed, (37) implies

$$\|DF_x^{k\ell}(\xi)\| \|(DF_x^{k\ell}(\xi))^{-1}\| \, C_0 \nu_{k\ell}(x) \leq C_0 \theta^k$$

and so, in view of (36), it suffices to fix $k \geq 1$ such that $C_0 \theta^k < 1$.

Remark 4.4. Under condition (37), a computation similar to Proposition 4.1(3) shows that $(F_y^n)^{-1} \circ F_x^n$ converges to $h^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$ in the C^1 *topology.* In particular, in this case the s-holonomy maps are diffeomorphisms between the fibers of $\mathcal{E}$. The projectivizations

$$\mathbb{P}(Dh^s_{x,y}(\xi)) : \mathbb{P}(T_\xi \mathcal{E}_x) \to \mathbb{P}(T_\eta \mathcal{E}_y), \quad \eta = h^s_{x,y}(\xi)$$

of the derivatives are precisely the s-holonomy maps of $\mathbb{P}(F)$.

A u-dominated cocycle F is *u-fiber bunched* if its inverse F^{-1} is s-fiber bunched. Then $\mathbb{P}(F)$ is u-dominated (relative to the u-lamination $\mathcal{W}^u$ of F), and so it admits u-holonomies. Let $\mathcal{B}^s(f) \subset \mathcal{D}^s(f)$ be the subspace of s-fiber bunched cocycles, and $\mathcal{B}^u(f) \subset \mathcal{D}^u(f)$ be the subspace of u-fiber bunched cocycles. We call a dominated cocycle *fiber bunched* if it belongs to $\mathcal{B}(f) = \mathcal{B}^s(f) \cap \mathcal{B}^u(f)$.

Remark 4.5. Let $F : \mathcal{E} \to \mathcal{E}$ be a fiber bunched cocycle and m be an F-invariant probability such that $\lambda_-(F, x, \xi) = \lambda_+(F, x, \xi) = 0$ at m-almost every $x \in M$. Then every $\mathbb{P}(F)$-invariant probability η projecting down to m is an su-state. This follows directly from Corollary 3.3 applied to the cocycle $\mathbb{P}(F)$ over the transformation F, and to its inverse.

4.4. **Accessibility.** In order to handle the construction in the previous section we shall need a few facts about cocycles over partially hyperbolic systems, that we present in here. Propositions 4.6 and 4.7 below are special versions of much more general results of Pugh, Shub [26] and Avila, Santamaria, Viana [3], respectively. We include the proofs since the arguments are much simpler in our setting, namely skew-products with differentiable stable and unstable holonomies.

Let F be a smooth cocycle admitting stable and unstable holonomies. The *accessibility class* of a point $(x,\xi) \in \mathcal{E}$ is the set of all $(y,\eta) \in \mathcal{E}$ such that there exist $(z_0,\zeta_0) = (x,\xi)$, (z_1,ζ_1), $\dots$, (z_{n-1},ζ_{n-1}), $(z_n,\zeta_n) = (y,\eta)$ in $\mathcal{E}$ satisfying

$$(z_{j+1},\zeta_{j+1}) \in \mathcal{W}^s(z_j,\zeta_j) \cup \mathcal{W}^u(z_j,\zeta_j) \quad \text{for every } j = 0,\dots,n-1.$$

It is easy to see that any accessibility class with non-empty interior is open. We say that F is *accessible* if the whole $\mathcal{E}$ is an accessibility class.

Proposition 4.6. *If F is a fiber bunched volume preserving cocycle and Z is an accessibility class with positive m-measure then there exists $n \ge 1$ such that $F^n(Z) = Z$ and $F^n \mid Z$ is ergodic for m. In particular, if F is accessible then it is ergodic.*

Proof. The first claims are immediate: Z must intersect $F^n(Z)$ for some $n \ge 1$, since $m(Z) > 0$, and then the two sets must coincide. We are left to prove that, given any continuous function $\varphi : \mathcal{E} \to \mathbb{R}$, the time averages

$$\varphi^{\pm} = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ F^{\pm jn}$$

are constant m-almost everywhere on Z. Given $c \in \mathbb{R}$, let A_c be the set of points $z \in Z$ for which $\varphi^{\pm}(z)$ are well-defined and satisfy $\varphi^+(z) = \varphi^-(z) \le c$. All we have to do is prove that every A_c has either zero of full m-measure in Z. Let c be such that A_c has positive m-measure and let m^c be the normalized restriction of m to A_c. Since φ^+ is constant on s-leaves, the set A_c is essentially s-saturated; for similar reasons it is also essentially u-saturated. This implies that m^c is an su-state and projects down to μ. Then, by Proposition 3.7, the measure m^c admits a continuous s-invariant and u-invariant disintegration $\{m^c_x : x \in \operatorname{supp}\mu\}$. Using also that the holonomies of F are area preserving diffeomorphisms, we obtain that the density of m^c_x with respect to Lebesgue measure on the fiber is constant along s-leaves and along u-leaves, over the support of μ. It follows that the density is constant on the whole accessibility class Z, over the support of μ. This can only happen if A_c has full m-measure in Z. □

Let $\mathcal{M} \to \mathcal{E}$ denote the fiber bundle where the fiber of each $z = (x,\xi) \in \mathcal{E}$ is the space of probability measures in the projective fiber $\mathbb{P}(\mathcal{E})_z = \mathbb{P}(T_\xi \mathcal{E}_x)$. Let $H^s_{z,w} : \mathbb{P}(\mathcal{E})_z \to \mathbb{P}(\mathcal{E})_w$ be the s-holonomy maps of the projective extension $\mathbb{P}(F)$: if $w = (y,\eta)$ with $\eta = h^s_{x,y}(\xi)$ then $H^s_{z,w} : \mathbb{P}(T_\xi\mathcal{E}_x) \to \mathbb{P}(T_\eta\mathcal{E}_y)$ is the projectivization of the derivative of $h^s_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$ at the point ξ.

Through the end of this section, we assume the ambient space $\mathcal{E}$ to be compact.

Proposition 4.7. *Let F be a fiber bunched accessible volume preserving cocycle. Then any invariant su-state of $\mathbb{P}(F)$ projecting down to m admits a disintegration which is s-invariant and u-invariant and whose conditional probabilities vary continuously with the base point on the support of μ.*

Proof. Let ζ be any invariant su-state of $\mathbb{P}(F)$ projecting down to m. We begin by thinking of $\mathbb{P}(F)$ as a cocycle over the hyperbolic transformation $f : M \to M$. It is clear that ζ is an su-state of this cocycle as well. Then, by Proposition 3.7, there exists a disintegration $\{\zeta_x : x \in M\}$, along the fibers of $\mathbb{P}(\mathcal{E}) \to M$ which is su-invariant and continuous. To proceed with the proof, let $\{\zeta_{(x,\xi)} : \xi \in \mathcal{E}_x\}$ be any disintegration of ζ_x along the fibers of $\mathbb{P}(T\mathcal{E}_x) \to \mathcal{E}_x$, for every $x \in M$. Consider the section $\psi : \mathcal{E} \to \mathcal{M}$ defined by $\psi(x,\xi) = \zeta_{(x,\xi)}$ We call $z = (x,\xi) \in \mathcal{E}$ a point of measurable continuity for ψ if there exists some probability measure ν on $\mathbb{P}(T_\xi \mathcal{E}_x)$ such that z is a Lebesgue density point of $\psi^{-1}(U)$ for any neighborhood U of ν (use any local trivialization of the fiber bundle $\mathcal{E}$; the definition does not depend on the particular choice). Notice that ν is unique when it exists, and the set $\mathrm{MC}(\psi)$ of points of measurable continuity has full m-measure. In that case define $\tilde{\psi}(z) = \nu$.

Lemma 4.8. $\mathrm{MC}(\psi)$ *is su-saturated and the section* $\tilde{\psi} : \mathrm{MC}(\psi) \to \mathcal{M}$ *is su-invariant on* $\mathrm{MC}(\psi)$.

Proof. The fact that $\{\zeta_x : x \in M\}$ is s-invariant means that $(\hat{H}^s_{x,y})_* \zeta_x = \zeta_y$ for every x and y on the same stable leaf of f, where $\hat{H}^s$ denotes the s-holonomy of the cocycle $\mathbb{P}(F)$ over f (which fibers over the s-holonomy H^s of the cocycle $\mathbb{P}(F)$ over F). Consequently, $\left(H^s_{z,w}\right)_* \zeta_z = \zeta_w$ for Lebesgue almost every $\xi \in \mathcal{E}_x$, where $w = (y,\eta)$ with $\eta = h^s_{x,y}(\xi)$. Since the holonomy maps $h^s_{x,y}$ are diffeomorphisms, and the $H^s_{z,w}$ are the fibers of the continuous map $\hat{H}^s_{x,y}$, it follows that measurable continuity points of the section ψ are preserved by the H^s. This proves s-saturation and s-invariance; the arguments for u-saturation and u-invariance are analogous. □

Since we assume accessibility, this gives that $\mathrm{MC}(\psi)$ is the whole $\mathcal{E}$ and $\tilde{\psi}$ is *su*-invariant. Since $\tilde{\psi}$ coincides with ψ almost everywhere, it defines a disintegration of ζ. To conclude the proof we only have to check that $\tilde{\psi}$ is continuous. Given $z \in \mathcal{E}$, let us denote by $\mathcal{B}(z,N) \subset \mathcal{E}$ the set of points which are accessible from z through an *su*-path with not more than N legs, all of them contained in local stable or instable manifolds.

Lemma 4.9. *There exists* $N \geq 1$ *such that* $\mathcal{B}(z,N) = \mathcal{E}$ *for every* $z \in \mathcal{E}$.

Proof. First, notice that, given any $\varepsilon > 0$ there exists $N(\epsilon) \geq 1$ such that $\mathcal{B}(z,N)$ is ε-dense in $\mathcal{E}$. Indeed, otherwise there would exist $\varepsilon > 0$ and sequences z_N and w_N such that $\mathcal{B}(z_N,N)$ avoids the ball $B(w_N,\varepsilon)$ for every N. By compactness, it would follow that there exist z and w such that $w \notin \mathcal{B}(z,N)$ for every N. This would contradict the assumption of accessibility. Now fix $z_0 \in \mathcal{E}$. Clearly from the definition, $\mathcal{B}(z_0,N)$ is compact for every $N \geq 1$. Since $\bigcup_{N\geq 1} \mathcal{B}(z_0,N) = \mathcal{E}$, and $\mathcal{E}$ is compact, there exists N_0 such that $\mathcal{B}(z_0,N)$ has non-empty interior. Hence it contains some ϵ-ball for some $\epsilon > 0$. Thus for every $z \in \mathcal{E}$, $\mathcal{B}(z,N(\epsilon)) \cap \mathcal{B}(z_0,N_0) \neq \emptyset$. It follows that for every $z \in \mathcal{E}$, $\mathcal{B}(z,N) = \mathcal{E}$ with $N = 2(N_0 + N(\epsilon))$. □

Using this lemma, we can now upgrade measurable continuity to *uniform* measurable continuity, as follows. Fix any metric on the fibers of $\mathcal{M}$ compatible with the weak* topology. We claim that for every $\epsilon > 0$ and every sufficiently small ball B on any fiber $\mathcal{E}_x$ (with respect to a fixed, but arbitrary Riemannian metric depending continuously on the fiber) there exists a subset W of B, with $\mathrm{Leb}(W) > (1-\epsilon)\,\mathrm{Leb}(B)$, such that $\tilde{\psi}(W)$ is contained in the ϵ-ball around $\tilde{\psi}(z)$ in $\mathcal{M}$. Indeed, for any two points $z \in \mathcal{E}_x$ and $w \in \mathcal{E}_y$, there exists a composition $H : \mathcal{E}_x \to \mathcal{E}_y$ of at most N local holonomy maps such that $H(z) = w$. It follows that H has uniformly bounded derivative, and the corresponding projective extension $\hat{H} : \mathbb{P}(\mathcal{E}_x) \to \mathbb{P}(\mathcal{E}_y)$ is uniformly continuous. Thus the quantifiers for measurable continuity at any two points are related with bounded distortion, yielding the claimed uniformity. Finally, it is easy to see that any uniformly measurable continuous function is in fact uniformly continuous in the fiber. Thus, the image under $\tilde{\psi}$ of any small ball in any fiber has small diameter in $\mathcal{M}$. Since ζ_x depends continuously on x, it follows that $\tilde{\psi}$ is continuous. □

Combining Remark 4.5 with Proposition 4.7 one immediately obtains

Theorem 4.10. *Let F be a fiber bunched accessible volume preserving cocycle. If $\lambda_-(F, x, \xi) = \lambda_+(F, x, \xi) = 0$ at m-almost every $(x, \xi) \in \mathcal{E}$ then every $\mathbb{P}(F)$-invariant probability that projects down to m admits a disintegration which is s-invariant, u-invariant, and whose conditional probabilities vary continuously with the base point on the support of μ.*

5. Continuity and positivity of exponents

Here we start our analysis of area preserving cocycles, to prove Theorem E and Corollary F. Let us begin by observing that *every cocycle volume preserving cocycle admits some su-state*, namely, the measure m defined by (8). Indeed, it is clear that m is an F-invariant probability. Moreover, its disintegration m_x is invariant under s-holonomy and u-holonomy because, by part (4) of Proposition 4.1, all holonomy maps are volume preserving if the cocycle is. This means that, unlike the situation in Corollary D for instance, the methods we developed in the previous sections can not be applied directly to cocycles $F \in \mathcal{B}_{\text{vol}}(f)$.

Nevertheless, we are going to show that those criteria remain useful to obtain information on the Lyapunov exponents of F. The strategy is to apply them to the projective extension $\mathbb{P}(F)$ instead. As observed in Section 4.3, the fiber bunching condition ensures that $\mathbb{P}(F)$ is dominated and, hence, admits holonomies in a robust fashion. A fiber bunched cocycle F is called *bundle free* if its projective extension admits no *su*-states. Corollary 4.3 implies that this is an open condition (recall that at this point we take the fiber N to be compact). More generally, given any invariant *su*-saturated set U with positive Lebesgue measure, we say that F *is bundle free over* U if the projective extension has no *su*-state over U.

5.1. Discontinuity points.

We are going to prove Theorem E. Let $F \in \mathcal{B}_{\text{vol}}(f)$ be ergodic for m and a discontinuity point for the Lyapunov exponents $\lambda_\pm(F, m)$. Recall that $\lambda_+(F, m) + \lambda_-(F, m) = 0$. It is well-known that the upper exponent $\lambda_+(\cdot, m)$ is upper semi-continuous and the lower exponent $\lambda_-(\cdot, m)$ is lower semi-continuous. Thus, if F is a discontinuity point then we must have

$$\lambda_-(F, m) < 0 < \lambda_+(F, m).$$

By ergodicity, this means that $\lambda_-(F, x, \xi) < 0 < \lambda_+(F, x, \xi)$ for m-almost every (x, ξ). Let $T_{x,\xi}\mathcal{E} = E^s_{x,\xi} \oplus E^u_{x,\xi}$ be the Oseledets decomposition of F. For $* \in \{s, u\}$, denote by η_* the probability measure on $\mathbb{P}(\mathcal{E})$ which projects down to m under the fibration $\mathbb{P}(\mathcal{E}) \to \mathcal{E}$ and whose conditional probability measure on the fiber of each (x, ξ) is the Dirac mass at the Oseledets space $E^*_{x,\xi}$. Equivalently,

$$\eta_*(B) = m\big(\{(x, \xi) : (x, \xi, E^*_{x,\xi}) \in B\}\big)$$

for every measurable set $B \subset \mathbb{P}(\mathcal{E})$. Notice that η_u is an invariant u-state and η_s is an invariant s-state of $\mathbb{P}(F)$. Let $\mathcal{M}(m)$ denote the space of probability measures η on $\mathbb{P}(\mathcal{E})$ that are mapped to m under the fibration $\mathbb{P}(\mathcal{E}) \to \mathcal{E}$ and, hence, project down to μ under $\mathbb{P}(\mathcal{E}) \to M$.

Lemma 5.1. *A measure $\eta \in \mathcal{M}(m)$ is $\mathbb{P}(F)$-invariant if and only if it is a convex combination of η_u and η_s, that is, if $\eta = \alpha\eta_u + \beta\eta_s$ for some F-invariant functions $\alpha, \beta : M \to [0,1]$ such that $\alpha + \beta = 1$.*

Proof. The 'if' part is trivial. For the converse just notice that every compact subset of $\mathbb{P}(\mathcal{E})$ disjoint from $\{E^u, E^s\}$ accumulates on E^u in the future and on E^s in the past. ☐

Lemma 5.2. *The exponent $\lambda_+(F,m)$ coincides with the maximum of*

$$\int \log \|DF_x(\xi)v\| \, d\eta(x,\xi,v)$$

over all $\mathbb{P}(F)$-invariant probability measures $\eta \in \mathcal{M}(m)$. When $\lambda_+(F,m) > 0$, the probability measure $\eta = \eta^u$ realizes the maximum.

Proof. Clearly, for any probability η that projects down to m,

$$\frac{1}{n}\int \log \|DF_x^n(\xi)v\| \, d\eta(x,\xi,v) \le \frac{1}{n}\int \log \|DF_x^n(\xi)\| \, dm(x,\xi).$$

The right hand side converges to $\lambda_+(F,m)$ when $n \to \infty$. The left hand side coincides with

$$\frac{1}{n}\int \sum_{j=0}^{n-1} \log \|DF_{x_j}(\xi_j)v_j\| \, d\eta(x,\xi,v) = \int \log \|DF_x(\xi)v\| \, d\eta(x,\xi,v),$$

where $(x_j, \xi_j, v_j) = \mathbb{P}(F)^j(x,\xi,v)$ and we take η to be $\mathbb{P}(F)$-invariant. Combining these observations, one obtains the upper bound in the statement.

Now we only have to check that η_u realizes the maximum. To this end, notice

$$\frac{1}{n}\int \log \|DF_x^n(\xi)v\| \, d\eta^u(x,\xi,v) = \frac{1}{n}\int \log \|DF_x^n(\xi)v_u\| \, dm(x,\xi),$$

where $v_u = v_u(x,\xi)$ is a unit representative of $E^u_{x,\xi}$. By the previous arguments, the left hand side coincides with $\int \log \|DF_x(\xi)v\| \, d\eta^u(x,\xi,v)$, for every $n \ge 1$. By dominated convergence, the right hand side goes to

$$\int \lim \frac{1}{n} \log \|DF_x^n(\xi)v_u\| \, dm(x,\xi) = \int \lambda_+(F,x,\xi) \, dm(x,\xi) = \lambda_+(F,m)$$

when $n \to \infty$. This proves our claim. ☐

Proposition 5.3. *Let F be fiber bunched and ergodic. If F is a point of discontinuity for the Lyapunov exponent then every $\mathbb{P}(F)$-invariant probability $\eta \in \mathcal{M}(m)$ is an su-state for $\mathbb{P}(F)$. In particular, F is not bundle free.*

Proof. The assumption implies there exists a sequence $(F_k)_k$ of cocycles converging to F in $\mathcal{B}_{\mathrm{vol}}(f)$ such that $\lim_k \lambda_+(F_k,m) < \lambda_+(F,m)$ (the other inequality always holds, by semi-continuity of the largest exponent). Then, by Lemma 5.2, there exists some invariant u-state η_k for each $\mathbb{P}(F_k)$, such that

$$\lim_k \int \log \|DF_{k,z}(\xi)v\| \, d\eta_k < \lambda_+(F,m).$$

We may assume that $(\eta_k)_k$ converges to some probability measure η. Clearly, η is an invariant u-state for $\mathbb{P}(F)$. By Lemma 5.1,

$$\eta = \alpha\eta_u + \beta\eta_s$$

where α and β are constants (by ergodicity). Moreover,

$$\int \log \|DF_z(\xi)v\| \, d\eta = \lim_k \int \log \|DF_{k,z}(\xi)v\| \, d\eta_k < \lambda_+(F,m).$$

This implies that $\eta \neq \eta_u$ and, thus, β is not zero. It follows that η_s is a u-state for $\mathbb{P}(F)$, since η and η_u are. Analogously, η_u is an s-state for $\mathbb{P}(F)$. Therefore, η is an su-state for $\mathbb{P}(F)$. □

Corollary 5.4. *If F is fiber bunched, ergodic, and bundle free then it is a point of continuity for the Lyapunov exponents and satisfies $\lambda_-(F,m) < 0 < \lambda_+(F,m)$.*

Remark 5.5. In the non-ergodic case we find that if F is a discontinuity point for the Lyapunov exponents then there exists an s-saturated positive measure set $Z^s \subset \mathcal{E}$ where $E^-_{x,\xi}$ is s-invariant and a u-saturated positive measure set $Z^u \subset \mathcal{E}$ where $E^+_{x,\xi}$ is u-invariant.

We are ready to finish the proof of Theorem E. We have seen in Proposition 5.3 that, under the theorem's assumptions, every $\mathbb{P}(F)$-invariant probability in $\mathcal{M}(m)$ is an su-state. From Proposition 4.7 we conclude that it admits some disintegration which is su-invariant and continuous. This completes the proof.

5.2. **Topological obstructions.** In this section we observe that the topology of the fiber imposes certain restrictions on the behavior of the Lyapunov exponents. Corollary F is a consequence of the following result:

Theorem 5.6. *Let $F : \mathcal{E} \to \mathcal{E}$ be a fiber bunched area preserving cocycle admitting some open accessibility class C. If F is not bundle free over $U = \cup_{n \in \mathbb{Z}} F^n(C)$ then either*

(1) *F is accessible, $N = \mathbb{S}^2$ or $N = \mathbb{T}^2$, and there exists a continuous Riemannian metric on the fibers, inducing the same area form, and which is invariant under both F and the invariant holonomies,*
(2) *or F admits either an invariant continuous line field over U or an invariant pair of transverse continuous line fields over U.*

Proof. Let η be an su-state for the projective extension, and $\{\eta_z : z \in \operatorname{supp} m\}$ be a continuous, $\mathbb{P}(F)$-invariant and su-invariant disintegration of η. Observe that $\operatorname{supp} m = \operatorname{supp} \mu \times N$. For each $x \in M$, let U_x be the intersection of U with each fiber $\mathcal{E}_x = N$. Then U_x is an open subset of N. The definition implies that, given any points (x_0, ξ_0) and (x_1, ξ_1) in U there exist homeomorphisms $U_{x_0} \to U_{x_1}$ obtained by concatenating cocycle iterates and stable and unstable holonomies and mapping ξ_0 to ξ_1. These homeomorphisms preserve the family of conditional probabilities.

Suppose first that for some (and, hence, for any) $z \in \operatorname{supp} m$, the probability η_z admits some atom with mass at least $1/2$. Either such an atom is unique or there exist exactly two, that exhaust the total mass of the conditional probability. In the first case, the family of conditional probabilities defines a continuous map assigning to each point in U_x a point in projective space, that is, a continuous line field on U_x. Moreover, the line field is preserved by the cocycle and its invariant holonomies. The second case is analogous, except that one gets an invariant pair of line fields instead. This gives part (2) of the theorem.

Now suppose that every η_z admits no atom with mass 1/2 or larger. Then, by Douady, Earle [17, Section 2], the conditional measure has a well-defined conformal barycenter $\xi(z) \in \mathbb{D}$ and, consequently, it defines a conformal structure on the tangent space to the fiber at z. This endows every U_x with a Riemman surface structure. Together with the area form, this conformal structure defines a Riemannian metric on the tangent space to the leaves, which is invariant under the cocycle and its holonomies. In particular, the group of isometries acts transitively on every U_x. Thus (see Farkas, Kra [18, Theorem V.4]), U_x must be one of five exceptional surfaces: the sphere $\mathbb{S}$, the plane $\mathbb{C}$, the punched plane $\mathbb{C}^*$, the hyperbolic disk $\mathbb{D}$, or the torus $\mathbb{T}$. Moreover, the plane, the punched plane, and the disk may be excluded, since U_x has finite area. It follows that U_x is either the sphere or the torus and, in either case, coincides with the whole fiber N. In particular, F is accessible. □

Remark 5.7. Part (2) of Theorem 5.6 can be strengthened considerably, if the cocycle is sufficiently regular: the fiber is $N = \mathbb{T}^2$ and the cocycle is conjugate to a sheer $(\xi, \eta) \mapsto (\xi + t(x)\eta, \eta)$ on the fibers. This fact is neither proved nor used in this paper.

Proof of Corollary F. The assumptions of the corollary ensure we are in the setting of Theorem 5.6, with $C = U = \mathcal{E}$. The hypothesis on the genus excludes alternative 1 in the conclusion of the theorem. Alternative 2 is also similarly excluded: since the Euler characteristic of the fiber is non-zero, there can be no continuous vector field, nor pair of vector fields, over the whole $\mathcal{E}$. This proves that F must be bundle free. Now the conclusion follows from Corollary 5.4. □

6. Generic area preserving cocycles

Here we prove Theorem G: every $F \in \mathcal{B}_{\mathrm{vol}}(f)$ is approximated by open sets where the Lyapunov exponents vary continuously and do not vanish. We begin with an outline of the arguments.

We have seen in Corollary 5.4 that if a cocycle F is bundle free and ergodic then its Lyapunov exponents are non-zero and they are continuous at F. We also know, from Corollary 4.3, that every bundle free cocycle is stably bundle free. In [4] we prove that every accessible cocycle with 2-dimensional fiber is stably accessible. By Proposition 4.6, every accessible fiber bunched volume preserving cocycle is ergodic. In [4] we also prove that every fiber bunched cocycle with 2-dimensional fiber is approximated by a (stably) accessible one. Thus, it suffices to prove that every accessible $F \in \mathcal{B}_{\mathrm{vol}}(f)$ is approximated by a (stably) bundle free cocycle.

By Theorem 5.6, if the fiber N is a hyperbolic surface then F itself is bundle free, and so there is nothing to prove. Indeed, to finish the proof we only have to explain how to perturb the cocycle in each of the situations left open by Theorem 5.6, in order to make it bundle free. We use a simple mechanism to ensure the bundle free property: creation of non-degenerate elliptic periodic points of F on some periodic fiber. A few explanations are in order, before giving the details.

A periodic point ζ of an area preserving map $h : N \to N$ is *elliptic* if the eigenvalues of $Dh^n(p)$ are not real, where n denotes the period. We call the elliptic periodic point ζ *non-degenerate* if there exists $\kappa \neq 0$, $\epsilon > 0$, and a Diophantine number $\alpha \in \mathbb{R}$, such that h^n is locally conjugate to

$$r \exp(2\pi i\theta) \mapsto r \exp(2\pi i\theta + \alpha + \kappa r^2) + O(r^5)$$

by some C^∞ diffeomorphism mapping ζ to $0 \in \mathbb{R}^2$. Then, by the Kolmogorov-Arnold-Moser theorem, there are arbitrarily small neighborhoods V of p which are C^∞ embedded disks invariant under h^n such that $h^n \mid \partial V$ is conjugate to an irrational rotation and $\|Dh^n(x)\|$ grows linearly with n for every $x \in \partial V$. Consequently, *h can not be an isometry* with respect to any continuous Riemannian metric, and *h can not preserve any continuous line field* on N either.

Choose some periodic point $p \in M$ of the transformation f once and for all. Note that periodic points do exist, indeed they are dense in the support of μ: this follows from the Poincaré recurrence theorem, using the shadowing lemma (see Bowen [14]) to close recurrent trajectories. For simplicity we take the period to be 1.

Let us consider first the case when $N = \mathbb{S}^2$ and the cocycle and its holonomies are isometries with respect to some continuous Riemannian metric on the fibers. Since $F_p : \mathcal{E}_p \to \mathcal{E}_p$ is an orientable homeomorphism of the sphere, it has some fixed point $\zeta \in \mathcal{E}_p$. Since F_p is an isometry, this fixed point must be elliptic and degenerate. Perturb F near the fiber of p so as to make ζ non-degenerate. By the previous observations and Theorem 5.6, the new cocycle is bundle free.

Now let us consider the case when $N = \mathbb{T}^2$ and the cocycle and its holonomies are isometries. We claim that, perturbing the cocycle if necessary, the map F_p has some periodic point $\zeta \in \mathcal{E}_p$. If F_p is not homotopic to the identity then existence of a periodic point follows for topological reasons. If F_p is homotopic to the identity, then consider the rotation number

$$\rho(F_p) = \int \big(F_p - \mathrm{id}\big)\, d\,\mathrm{Leb}\,.$$

Perturbing F near the fiber over p in such a way that F_p is replaced by $F_p + v$ for some convenient $v \in \mathbb{T}^2$, we can ensure that the rotation number is rational. Then, by Franks [19], the map F_p has some periodic point. This proves the claim. From now on the argument is analogous to the sphere case: perturbing the cocycle once more, we can make the periodic point non-degenerate, and then the new cocycle must be bundle free.

Next, assume $N = \mathbb{T}^2$ and the cocycle admits a continuous invariant line field. Let V be the (open) set of all $x \in \mathbb{T}^2$ such that there exists K_x and ϵ_x such that for every w which is ϵ_x close to x, and every $k \ge 0$ such that $F_p^k(w)$ is ϵ_x-close to w, we have $\|DF_p^k(w)\| < K_x$. If there exists a periodic point of F_p in V, it must be elliptic and we can argue as before. So, assume that there is no periodic point in V. Similarly to what we did in the proof of Theorem 5.6, we can define on V a locally bounded measurable Riemannian metric inducing the same area form, which is F_p-invariant. Thus, V gets the structure of a one-dimensional complex manifold, possibly disconnected, on which F_p acts holomorphically. By Poincaré recurrence, all connected components of V are periodic. From the classification of conformal automorphisms (see [18, Chapter V]) we see that any automorphism of a Riemann surface which satisfies Poincaré recurrence admits a periodic point, unless the Riemann surface is an annulus and the automorphism is an irrational rotation, or the Riemann surface is the torus and the automorphism is not periodic. The torus case is covered by previous arguments (F_p is necessarily homotopic to the identity and then a periodic point can be created by a small perturbation). So, we are left with the annulus case only.

For every $x \notin V$, choose sequences $w_n \to x$ and $k_n \to \infty$ such that $F_p^{k_n}(w_n) \to x$ and $\|DF_p^{k_n}(w_n)\| \to \infty$, the direction s_n most contracted under $DF_p^n(w_n)$ converges to some limit $s(x)$, and the image u_n of the direction most expanded under $DF_p^n(w_n)$ converges to some limit $u(x)$. Though the choice is not canonical, we fix it once and for all. Then $l(x) \in \{s(x), u(x)\}$, because every line bundle is attracted to $u(\cdot)$ under iteration, unless it coincides with $s(\cdot)$. Let h be the holonomy map associated to an arbitrary homoclinic loop of p. Since h preserves area, there exists a connected component V_0 of V and some $k \geq 1$ such that $h^k(V_0) \cap V_0 \neq \emptyset$. Using area preservation again, we conclude that $h^k(\partial V_0) \cap \partial V_0 \neq \emptyset$, which implies that there exists $z \in \mathbb{T}^2 \setminus V$ such that $h^k(z) \in \mathbb{T}^2 \setminus V$. Up to perturbing the dynamics without touching F_p, we may assume that $Dh^k(z) \cdot \{s(z), u(z)\} \cap \{s(h^k(z)), u(h^k(z))\} \neq \emptyset$. For the perturbed system, no line field or pair of transverse invariant line fields is invariant under both the dynamics and the invariant holonomies.

Finally, assume $N = \mathbb{T}^2$ and the cocycle admits a pair of transverse continuous invariant line fields $\{l_1(\xi), l_2(\xi)\}$, but no continuous invariant line field. Then F_p lifts to a map $\tilde{F}_p : \tilde{\mathcal{E}}_p \to \tilde{\mathcal{E}}_p$, where $\tilde{\mathcal{E}}_p$ is the set of all (x, ξ) with $x \in \mathbb{T}^2$ and $\xi \in \{l_1(x), l_2(x)\}$. The assumption that there is no invariant line field ensures that $\tilde{\mathbb{T}}^2$ is connected, and so it is a torus. Let $\pi : \tilde{\mathbb{T}}^2 \to \mathbb{T}^2$ be the projection on the first coordinate. Let $\tilde{V}$ be the set of all $x \in \tilde{\mathbb{T}}^2$ such that there exists K_x and ϵ_x such that for every w which is ϵ_x close to x, and every $k \geq 0$ such that $\tilde{F}_p^k(w)$ is ϵ_x close to w we have $\|D\tilde{F}_p^k(w)\| < K_x$. Then the proof proceeds just as in the previous case, with $V = \pi(\tilde{V})$. In the present situation one gets $\{l_1(x), l_2(x)\} = \{s(x), u(x)\}$. This completes the proof of Theorem G.

References

[1] F. Abdenur and M. Viana. Flavors of partial hyperbolicity. Preprint www.preprint.impa.br 2008.

[2] J. F. Alves, C. Bonatti, and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent. Math.*, 140:351–398, 2000.

[3] A. Avila, J. Santamaria, and M. Viana. Linear cocycles over partially hyperbolic maps. Preprint www.preprint.impa.br 2008.

[4] A. Avila and M. Viana. On stable accessibility with 2-dimensional center. In preparation.

[5] A. Avila and M. Viana. Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture. *Acta Math.*, 198:1–56, 2007.

[6] A. Avila and M. Viana. Symplicity of Lyapunov spectra: A sufficient criterion. *Portugaliae Mathematica*, 64:311–376, 2007.

[7] A. Avila, M. Viana, and A. Wilkinson. Absolute continuity, Lyapunov exponents, and rigidity. In preparation.

[8] A. Baraviera and C. Bonatti. Removing zero central Lyapunov exponents. *Ergod. Th. & Dynam. Sys.*, 23:1655–1670, 2003.

[9] P. Baxendale. Lyapunov exponents and relative entropy for a stochastic flow of diffeomorphisms. *Probab. Theory Related Fields*, 81:521–554, 1989.

[10] J. Bochi and M. Viana. Lyapunov exponents: how frequently are dynamical systems hyperbolic? In *Modern dynamical systems and applications*, pages 271–297. Cambridge Univ. Press, 2004.

[11] C. Bonatti, X. Gómez-Mont, and M. Viana. Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20:579–624, 2003.

[12] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.

[13] C. Bonatti and M. Viana. Lyapunov exponents with multiplicity 1 for deterministic products of matrices. *Ergod. Th. & Dynam. Sys*, 24:1295–1330, 2004.

[14] R. Bowen. A horseshoe with positive measure. *Invent. Math.*, 29:203–204, 1975.
[15] K. Burns, D. Dolgopyat, and Ya. Pesin. Partial hyperbolicity, Lyapunov exponents and stable ergodicity. *J. Statist. Phys.*, 108:927–942, 2002. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.
[16] K. Burns and A. Wilkinson. On the ergodicity of partially hyperbolic systems. *Annals of Math.*
[17] A. Douady and J. C. Earle. Conformally natural extension of homeomorphisms of the circle. *Acta Math.*, 157:23–48, 1986.
[18] H. Farkas and I. Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, second edition, 1992.
[19] J. Franks. Realizing rotation vectors for torus homeomorphisms. *Trans. Amer. Math. Soc.*, 311:107–115, 1989.
[20] H. Furstenberg. Non-commuting random products. *Trans. Amer. Math. Soc.*, 108:377–428, 1963.
[21] I. Ya. Gol'dsheid and G. A. Margulis. Lyapunov indices of a product of random matrices. *Uspekhi Mat. Nauk.*, 44:13–60, 1989.
[22] Y. Guivarc'h and A. Raugi. Products of random matrices : convergence theorems. *Contemp. Math.*, 50:31–54, 1986.
[23] M. Hirsch, C. Pugh, and M. Shub. *Invariant manifolds*, volume 583 of *Lect. Notes in Math.* Springer Verlag, 1977.
[24] J. Kingman. The ergodic theorem of subadditive stochastic processes. *J. Royal Statist. Soc.*, 30:499–510, 1968.
[25] F. Ledrappier. Positivity of the exponent for stationary sequences of matrices. In *Lyapunov exponents (Bremen, 1984)*, volume 1186 of *Lect. Notes Math.*, pages 56–73. Springer, 1986.
[26] C. Pugh and M. Shub. Stably ergodic dynamical systems and partial hyperbolicity. *J. Complexity*, 13:125–179, 1997.
[27] V. A. Rokhlin. On the fundamental ideas of measure theory. *A. M. S. Transl.*, 10:1–52, 1952. Transl. from Mat. Sbornik 25 (1949), 107–150.
[28] D. Ruelle. An inequality for the entropy of differentiable maps. *Bull. Braz. Math. Soc.*, 9:83–87, 1978.
[29] M. Shub and A. Wilkinson. Pathological foliations and removable zero exponents. *Invent. Math.*, 139:495–508, 2000.
[30] M. Viana. Almost all cocycles over any hyperbolic system have non-vanishing Lyapunov exponents. *Annals of Math.*, 167:643–680, 2008.

E-mail address: artur@math.sunysb.edu
E-mail address: viana@impa.br