ON THE REGULARIZATION OF CONSERVATIVE MAPS

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Abstract. We show that smooth maps are C^1-dense among C^1 volume preserving maps.

1. Introduction

Let M and N be C^∞ manifolds[1] and let $C^r(M,N)$ ($r \in \mathbb{N} \cup \{\infty\}$) be the space of C^r maps from M to N, endowed with the Whitney topology. It is a well known fact that C^∞ maps are dense in $C^r(M,N)$. Such a result is very useful in differentiable topology and in dynamical systems (as we will discuss in more detail). On the other hand, in closely related contexts, it is the non-existence of a regularization theorem that turns out to be remarkable: if homeomorphisms could always be approximated by diffeomorphisms then the whole theory of exotic structures would not exist.

Palis and Pugh [20] seem to have been the first to ask about the corresponding regularization results in the case of conservative and symplectic maps. Here one fixes C^∞ volume forms[2] (in the conservative case) or symplectic structures (symplectic case), and asks whether smoother maps in the corresponding class are dense with respect to the induced Whitney topology. The first result in this direction was due to Zehnder [27], who provided regularization theorems for symplectic maps, based on the use of generating functions. He also provided a regularization theorem for conservative maps, but only when $r > 1$ (he did manage to treat also non-integer r). The case $r = 1$ however has remained open since then (due in large part to intrinsic difficulties relating to the PDE's involved in Zehnder's approach, which we will discuss below), except in dimension 2, where it is equivalent to the symplectic case. This is the problem we address in this paper. Let $C^r_{\mathrm{vol}}(M,N) \subset C^r(M,N)$ be the subset of maps that preserve the fixed smooth volume forms.

Theorem 1. *C^∞ maps are dense in $C^1_{\mathrm{vol}}(M,N)$.*

Let us point out that the corresponding regularization theorem for conservative flows was obtained much earlier by Zuppa [28] in 1979. In fact, in a more recent approach of Arbieto-Matheus [1], it is shown that a result of Dacorogna-Moser [13] allows one to reduce to a local situation where the regularization of vector fields which are divergence free can be treated by convolutions. However, attempts to reduce the case of maps to the case of flows through a suspension construction have not been succesful.

Let us discuss a bit an approach to this problem which is succesful in higher regularity, and the difficulties that appear when considering C^1 conservative maps.

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[1] All manifolds will be assumed to be Hausdorff, paracompact and without boundary, but possibly not compact.

[2] For non-orientable manifolds, a volume form should be understood up to sign.

Let us assume for simplicity that M and N are compact, as all difficulties are already present in this case. Let $f \in C^r_{\mathrm{vol}}(M,N)$, and let ω_M and ω_N be the smooth volume forms. Approximate f by a smooth non-conservative map $\tilde{f}$. Then $\tilde{f}^*\omega_N$ is C^{r-1} close to ω_M. If we can solve the equation $h^*\tilde{f}^*\omega_N = \omega_M$ with h C^r close to *id* then the desired approximation could be obtained by taking $\tilde{f} \circ h$. Looking at the local problem one must solve to get h, it is natural to turn our attention to the C^r solutions of the equation $\det Dh = \phi$ where $\phi : \mathbb{R}^n \to \mathbb{R}$ is smooth and close to 1.

Unfortunately, though ϕ is smooth, we only know apriori that the C^{r-1} norm of ϕ is small. This turns out to be quite sufficient to get control on h if $r \geq 2$, according to the Dacorogna-Moser technique. But when $r = 1$, the analysis of the equation is different, as was shown by Burago-Kleiner [11] and McMullen [19]. This is well expressed in the following result, Theorem 1.2 of [11]: *Given* $c > 0$ *there exists a continuous function* $\phi : [0,1]^2 \to [1, 1+c]$*, such that there is no bi-Lipschitz map* $h : [0,1]^2 \to \mathbb{R}^2$ *with* $\det Dh = \phi$.

This implies that continuous volume forms on a C^∞ manifold need not be C^1 equivalent to smooth volume forms. This is in contrast with the fact that all smooth volume forms are C^∞ equivalent up to scaling [18], and the differential topology fact that all C^1 structures on a C^∞ manifold are C^1 equivalent.

Remark 1.1. One can define a C^r_{vol} structure on a manifold as a maximal atlas whose chart transitions are C^r maps preserving the usual volume of $\mathbb{R}^n$ (see [25], Example 3.1.12). Then Theorem 1 (and its equivalent for higher differentiability [27]) can be used to conclude that any C^r_{vol} structure is compatible with a C^∞_{vol} structure (unique up to C^∞_{vol}-diffeomorphism by [18]), by following the proof of the corresponding statement for C^r-structures (see Theorem 2.9 of [17]). For $r \geq 2$, a C^r_{vol} structure is the same as a C^r structure together with a C^{r-1} volume form by [13], but not all continuous volume forms on a C^∞ manifold arise from a C^1_{vol} structure, by Theorem 1.2 of [11] quoted above.

We notice also the following amusing consequence of Theorem 1.2 of [11], which we leave as an exercise: *A generic continuous volume form on a* C^1 *surface has no non-trivial symmetries, that is, the identity is the only diffeomorphism of the surface preserving the volume form.* This highlights that the correct framework to do C^1 conservative dynamics is the C^1_{vol} category (and not C^1 plus continuous volume form category).

The equation $\det Dh = \phi$ has been studied also in other regularity classes (such as Sobolev) by Ye [26] and Rivière-Ye [22], but this has not helped with the regularization theorem in the C^1 case.

The approach taken in this paper is very simple, ultimately constructing a smooth approximation by taking independent linear approximations (derivative) in a very dense set, and carefully modifying and gluing them into a global map (with a mixture of bare-hands technique and some results from the PDE approach *in high regularity*). A key point is to enforce that the choices involved in the construction are made through a *local* decision process. This is useful to avoid long-range effects, which if left out would lead us to a discretized version of the PDE approach in low regularity, with the associated difficulties. To ensure locality, we use the original unregularized map f as *background data* for making the decisions. The actual details of the procedure are best understood by going through the proof, since the difficulties of this problem lie in the details.

1.1. **Dynamical motivation.** In the discussion below, we restrict ourselves to diffeomorphisms of compact manifolds for definiteness.

There is a good reason why the regularization problem for conservative maps has been first introduced in a dynamical context. In dynamics, low regularity is often used in order to be able to use the strongest perturbation results, such as the Closing Lemma [21], the Connecting Lemma [16] and the simple but widely used Franks' Lemma [15]. Currently such results are only available precisely for the C^1 topology (even getting to $C^{1+\alpha}$ would be an amazing progress), except when considering one-dimensional dynamics. On the other hand higher regularity plays a fundamental role when distortion needs to be controlled, which is the case for instance when the ergodic theory of the maps is the focus ($C^{1+\alpha}$ is a basic hypothesis of Pesin theory, and for most results on stable ergodicity such as [12], though more regularity is necessary if KAM methods are involved [23]). While dynamics in the smooth and the low regularity worlds may often seem to be different altogether (compare [10] and [5]), it turns out that their characteristics can be often combined (both in the conservative and the dissipative setting), yielding for instance great flexibility in obtaining interesting examples: see the construction of non-uniformly hyperbolic Bernoulli maps [14] which uses C^1-perturbation techniques of [3].

In the dissipative and symplectic settings, regularization theorems have been an important tool in the analysis of C^1-generic dynamics: for instance, Zehnder's Theorem is used in the proof of [2] that ergodicity is C^1-generic for partially hyperbolic symplectic diffeomorphisms. [3] Thus it is natural to expect that Theorem 1 will lead to several applications on C^1-generic conservative dynamics. Indeed many recent results have been stated about certain properties of C^2-maps being dense in the C^1-topology, without being able to conclude anything about C^1-maps only due to the non-availability of Theorem 1. Thus it had been understood for some time that proving Theorem 1 would have many immediate applications. Just staying with examples in the line of [2], we point out that [9] now implies that ergodicity is C^1-generic for partially hyperbolic maps with one-dimensional center (see section 4 of [9]), and the same applies to the case of two-dimensional center, in view of the recent work [24].

Though we do not aim to be exhaustive in the discussion of applications here, we give a few other examples which were pointed out to us by Bochi and Viana:

1. Any C^1_{vol}-robustly transitive diffeomorphism admits a dominated splitting (conjectured, e.g., in [7], page 365), a result obtained for $C^{1+\alpha}$ diffeomorphisms in [1] using a Pasting Lemma. (We note also that this work also allows one to extend the Pasting Lemma of [1] itself, and hence its other consequences, to the C^1 case.)
2. A generic C^1-generic consevative non-Anosov diffeomorphism has only hyperbolic sets of zero Lebesgue measure . Zehnder's Theorem has been used in [3] and [5] to achieve this conclusion in the symplectic case, and such a result is necessary for the conclusion of the central dychotomy of [3]. It is based on a statement about C^2 conservative maps obtained in [6], so the conclusion for conservative maps now follows directly from Theorem 1. We hope that results in this direction will play a role in further strengthenings of [5].
3. The existence of locally generic non-uniformly hyperbolic ergodic conservative diffeomorphisms with non-simple Lyapunov spectrum (a proof, conditional to the existence of regularization, is sketched in page 260 of [8]).

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[3] It was this fact indeed that convinced the author to work on Theorem 1.

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2. EXTENDING CONSERVATIVE MAPS

Fix two connected open sets with smooth boundary $B_1, B_2 \subset \mathbb{R}^n$ with $\overline{B}_1 \subset B_2$ and $\overline{B}_1 \setminus B_2$ smoothly diffeomorphic to $\partial B_1 \times [0,1]$. For the proof of Theorem 1, we will need the following slight variations of Theorems 2 and 1 of Dacorogna-Moser [13].

Theorem 2. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function with $\int \phi = 0$ supported inside B_1. Then there exists $v \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ supported inside B_2 with* $\mathrm{div} v = \phi$. *Moreover, if ϕ is C^∞ small then v is C^∞ small.*

Proof. Theorem 2 of [13] states in a more general context, that there exists $w : \overline{B}_2 \to \mathbb{R}^n$ with $\mathrm{div} w = \phi$ and $w|\partial B_2 = 0$, and if ϕ is C^∞ small then w is also C^∞ small. It is thus enough to find some C^∞ $u : \overline{B}_2 \to \mathbb{R}^n$ (small if ϕ is small) with $\mathrm{div} u = 0$ and $u|\overline{B}_2 \setminus B_1 = w$, and let $v|\overline{B}_2 = w - u$, $v|\mathbb{R}^n \setminus \overline{B}_2 = 0$. This procedure is the standard one already used in [13].

There is a duality between smooth vector fields u and smooth $n-1$-forms u^*, given by $u^*(x)(y_1, ..., y_{n-1}) = \det(u(x), y_1, ..., y_{n-1})$. The duality transforms the equation $\mathrm{div} u = 0$ into $du^* = 0$. The form w^* is thus closed in $\overline{B}_2 \setminus B_1$, and the boundary condition $w|\partial B_2 = 0$ implies that it is exact in $\overline{B}_2 \setminus B_1$. Solve the equation $d\alpha = w^*$ in $\overline{B}_2 \setminus B_1$ and extend α smoothly to $\overline{B}_2$ (notice that α can be required to be small if w is small). Let u be a vector field on $\overline{B}_2$ given by $d\alpha = u^*$. Then $u|\overline{B}_2 \setminus B_1 = w$, and since $du^* = 0$ in $\overline{B}_2 \setminus B_1$, we have $\mathrm{div} u|\overline{B}_2 \setminus B_1 = 0$. □

Theorem 3. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function with $\int \phi = 0$ supported inside B_1. Then there exists $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\psi - id$ supported inside B_1 such that* $\det \psi = 1 + \phi$. *Moreover, if ϕ is C^∞-small then $\psi - id$ is C^∞ small.*

Proof. As in [13], the solution is given explicitly as $\psi = \psi_1$ where $\psi_t(x)$ is the solution of the differential equation $\frac{d}{dt}\psi_t(x) = v(\phi_t(x))/(t + (1-t)(1 + \phi(\psi_t(x))))$ with $\psi_0(x) = x$ and v comes from the previous theorem. □

Corollary 4. *Let K be a compact set, U be a neighborhood of K and let $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be C^∞ close to the identity and such that $f|U$ is volume preserving. Assume that for every bounded connected component W of $\mathbb{R}^n \setminus K$, W and $f(W)$ have the same volume. Then there exists a C^∞ conservative map close to the identity such that $\tilde{f} = f$ on K.*

Proof. We may modify f away from K so that $f - id$ is compactly supported and $\det f - 1$ is supported inside some open set with smooth boundary $\tilde{B}_1$ disjoint from some neighborhood of K. Let m be the number of connected components of $\tilde{B}_1$. We can assume that each connected component of $\mathbb{R}^n \setminus K$ contains at most one connected component of $\tilde{B}_1$. For each connected component of B_1^i of $\tilde{B}_1$, select a small ϵ-neighborhood B_i^2 of $\overline{B}_1^i$. Let ϕ_i be given by $\phi_i|B_1^i = \det f - 1$ and $\phi|\mathbb{R}^n \setminus B_1^i = 0$. Then $\int \phi_i = 0$. Indeed, if B_1^i is contained in a bounded connected components of $\mathbb{R}^n \setminus K$, this follows immediately from f preserving the volumes of such sets, and if B_1^i is contained in the unbounded component W of $\mathbb{R}^n \setminus K$, one uses that f preserves the volume of $W \cap B$ for all sufficiently large balls B (to see this one uses that $f - id$ is compactly supported). Applying the previous theorem, one gets maps ψ_i with $\psi_i - id$ supported inside B_1^i. We then take $\tilde{f} = f \circ \psi_1^{-1} \circ \cdots \circ \psi_m^{-1}$. $\square$

3. Moving mass

In this section we will consider the L^∞ norm in $\mathbb{R}^n$. The closed ball of radius $r > 0$ around $p \in \mathbb{R}^n$ will be denoted by $B(p, r)$ (this ball is actually a cube). The canonical basis of $\mathbb{R}^n$ will be denoted by $e_1, ..., e_n$.

Lemma 3.1. *Fix $0 < \delta < 1/10$. Let $S \subset \{1, ..., n\}$ be a subset with $0 \leq k \leq n-1$ elements. Let $P \subset \mathbb{R}^n$ be the set of all p of the form $\sum_{i \notin S} u_i e_i$ with $u_i = \pm 1$. Let $B = \cup_{p \in P} B(p, 1)$, and let B' be the open δ-neighborhood of B. Let W be a Borelian set whose δ-neighborhood is contained in B and which contains $B(0, \delta)$. If $F \in C^1_{\mathrm{vol}}(B', \mathbb{R}^n)$ is C^1 close to the identity, then there exists $s \in C^\infty_{\mathrm{vol}}(\mathbb{R}^n, \mathbb{R}^n)$ such that*

1. *$s|\mathrm{int}B(0, 10)$ is C^∞ close to the identity,*
2. *$\mathrm{vol}F(s(W)) \cap B(p, 1) = \mathrm{vol}W \cap B(p, 1)$ for $p \in P$,*
3. *s is the identity outside the δ-neighborhood the subspace generated by the $\{e_i\}_{i \in S}$.*

Proof. Notice that there are 2^{n-k} elements in P. Call two elements $p, p' \in P$ adjacent if $p - p' = \pm 2e_k$ for some $1 \leq k \leq n$.

Let p, p' be adjacent. Let $q = q(p, p') = \delta(p' + p)/4$, and let $C = C(p, p')$ be the set of all $z \in \mathbb{R}^n$ of the form $z + t(p' - p)$ where $t \in \mathbb{R}$ and $z \in B(q, \delta/4)$. Let $\phi = \phi^{p,p'} : \mathbb{R}^n \to [0, 1]$ be a C^∞ function such that $\phi(q) = 1$, $\phi|\mathbb{R}^n \setminus C = 0$ and $\phi(z + t(p - p')) = \phi(z)$ for $t \in \mathbb{R}$. For $z \in \mathbb{R}$, let $s_t = s_t^{p,p'} \in C^\infty_{\mathrm{vol}}(\mathbb{R}^n, \mathbb{R}^n)$ be given by $s_t(z) = z + t\phi(z)(p - p')$.

Notice that for $|t| < \delta/100$, we have $\mathrm{vol}s_t(W) \cap B(p, 1) - \mathrm{vol}W \cap B(p, 1) = t \int_{B(0,1/2)} \phi(z) dz$. Let B'' be the $\delta/2$ open neighborhood of B. We claim that if $\tilde{F} \in C^1_{\mathrm{vol}}(B'', \mathbb{R}^n)$ is C^1 close to the identity then there exists $t \in \mathbb{R}$ small such that $\mathrm{vol}\tilde{F}(s_t(W)) \cap B(p, 1) = \mathrm{vol}W \cap B(p, 1)$ while for $\tilde{p} \in P \setminus \{p, p'\}$ we have $\mathrm{vol}\tilde{F}(s_t(W)) \cap B(\tilde{p}, 1) = \mathrm{vol}\tilde{F}(W) \cap B(\tilde{p}, 1)$.

Notice that for $|t| < \delta/100$, $\mathrm{vol}\tilde{F}(s_t(W)) \cap B(p, 1) - \mathrm{vol}s_t(W) \cap B(p, 1)$ is small if $\tilde{F}$ is close to the identity. For $|t| < \delta/100$, we can directly compute $\mathrm{vol}s_t(W) \cap B(p, 1) - \mathrm{vol}W \cap B(p, 1) = t \int_{B(0,1/4)} \phi(z) dz$. Thus the first part of the claim follows from the obvious continuity of $t \mapsto \mathrm{vol}\tilde{F}(s_t(W)) \cap B(p, 1)$ for $|t| < \delta/100$. The second part of the claim is easy, since s_t is the identity in $B(\tilde{p}, 1 + \delta/4)$.

As a graph, P is just an hypercube, so there exists an ordering $p^1, ..., p^{2^{n-k}}$ of the elements of P such that for $1 \leq i \leq 2^{n-k} - 1$, p_i and p_{i+1} are adjacent. Given F, we define sequences $F_{(k)} \in C^1_{\mathrm{vol}}(B'', \mathbb{R}^n)$, $s_{(k)} \in C^\infty_{\mathrm{vol}}(\mathbb{R}^n, \mathbb{R}^n)$, $0 \leq k \leq 2^{n-k} - 1$ by induction as follows. We let $s_{(0)} = id$, $F_{(k)} = F \circ s_{(k)}$ for $0 \leq k \leq 2^{n-k} - 1$, and for $1 \leq k \leq 2^{n-k} - 1$ we let $s_{(k)} = s_t^{p,p'} \circ s_{(k-1)}$ for $1 \leq k \leq 2^{n-k} - 1$, where $p = p_i$, $p' = p_{i+1}$, and t is given by the claim applied with $\tilde{F} = F_{(k-1)}$. As long as F is sufficiently close to the identity, we get inductively that $F_{(k)}$ is close to the identity, so this construction can indeed be carried out.

Let us show that $s = s^{(2^{n-k}-1)}$ has all the required properties. Properties (1) and (3) are rather clear. By construction, we get inductively that $\mathrm{vol} F_{(k)}(W) \cap B(p,1) = \mathrm{vol} W \cap B(p,1)$ for $p \in \{p_1, ..., p_k\}$, so it is clear that $\mathrm{vol} F(s(W)) \cap B(p,1) = \mathrm{vol} W \cap B(p,1)$ except possibly for $p = p_{2^{n-k}}$. But $\sum_{p \in P} \mathrm{vol} F(s(W)) \cap B(p,1) = \mathrm{vol} F(s(W)) \cap B = \mathrm{vol} W \cap B = \sum_{p \in P} \mathrm{vol} W \cap B(p,1)$, so we must have $\mathrm{vol} F(s(W)) \cap B(p,1) = \mathrm{vol} W \cap B(p,1)$ also for $p = p_{2^{n-k}}$, and property (2) follows. $\square$

4. Proof of Theorem 1

4.1. **Charts.** If $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^n$ is a bounded C^r map with bounded derivatives up to order r, we let $\|f\|_{C^r}$ be the natural C^r norm.

Theorem 5. *Let W be an open subset of $\mathbb{R}^n$ and let $f \in C^1_{\mathrm{vol}}(W, \mathbb{R}^n)$ be a map with bounded uniformly continuous derivative. Let $K_0 \subset W$ be a compact set such that f is C^∞ in a neighborhood of K_0. Let $U \subset W$ be open. Then for every $\epsilon > 0$ there exists $\tilde{f} \in C^1_{\mathrm{vol}}(W, \mathbb{R}^n)$ such that $\tilde{f}|U$ is C^∞, $\tilde{f}$ coincides with f in $W \smallsetminus U$ and in a neighborhood of K_0, and $\|f - \tilde{f}\|_{C^1} < \epsilon$.*

Proof. We will consider the L^∞ metric in $\mathbb{R}^n$. Let $\theta > 0$ be such that the θ-neighborhood of K_0 is contained in W and f is C^r in it. We will now introduce a Whitney decomposition of U.

If $0 \leq m \leq n$, An m-cell x is some set of the form $\prod_{k=1}^n [2^{-t} a_k, 2^{-t}(a_k + b_k)]$ where $t \in \mathbb{Z}$, $a_k \in \mathbb{Z}$ and $b_k \in \{0,1\}$ with $\#\{b_k = 1\} = m$. For $m \geq 1$, we let its interior $\mathrm{int} x$ be $\prod_{k=1}^n (2^{-t} a_k, 2^{-t}(a_k + b_k))$, while for $m = 0$ we let $\mathrm{int} x = x$. Let ∂x be $x \smallsetminus \mathrm{int} x$.

We say that an n-cell x is ϵ-small if every n-cell of the same diameter as x which intersects x is contained in U. We say that a dyadic n-cell is ϵ-good if it is a maximal ϵ-small n-cell. We say that a dyadic m-cell, $0 \leq m \leq n-1$, is ϵ-good if it is the intersection of all ϵ-good n-cells that intersect its interior. We say that an ϵ-good m-cell x has rank $t = t(x)$ if the minimal diameter of the ϵ-good n-cells containing it is 2^{-t} (if $m > 0$, this is the same as the diameter of x). Notice that if x, y are ϵ-good cells and $x \cap y \neq \emptyset$ then $|t(x) - t(y)| \leq 1$. Each ϵ-good m-cell x is contained in 2^{n-m} n-cells of diameter $2^{-t(x)}$, called neighbors of x (which are not necessarily ϵ-good).

Fix some small $\epsilon > 0$. From now on, by m-cell we will understand an ϵ-good m-cell. Let N_m be the set of m-cells. There exists $C_0 = C_0(n)$ such that each m-cell contains at most C_0 k-cells, $0 \leq k \leq m$. Moreover, each $x \in N_m$ is the union of the interior of the k-cells, $0 \leq k \leq m$, contained in x.

For $x \in N_m$, let $D(x)$ be the $2^{-10(m+1)} 2^{-t(x)}$ neighborhood of x, and let $I(x) = \cup D(y)$ where the union is taken over all proper subcells $y \subset x$. Let $B(x) = D(x) \cup I(x)$ and $J(x) = D(x) \smallsetminus I(x)$. Notice that if x and y are distinct cells, $J(x)$ and $J(y)$ are disjoint. Let $R(x)$ be the interior of the union of all n-cells intersecting x. Notice that the $2^{-100(m+1)} 2^{-t(x)}$-neighborhood of $J(x)$ contained in the interior of the union of the neighbors of x.

For a cell x, let b be its baricenter, let $\lambda_x : \mathbb{R}^n \to \mathbb{R}^n$ be given by $\lambda_x(z) = b + 2^{-t(x)+1} z$, and let $H_x(z) = f(b) + Df(y)(z - b)$. We say that $h \in C^\infty_{\mathrm{vol}}(R(x), \mathbb{R}^n)$ is x-nice if $h - f$ is C^1-small, $(\lambda_x^{-1} \circ h \circ \lambda_x) - (\lambda_x^{-1} \circ H_x \circ \lambda_x)$ is C^∞ small, and for every neighbor y of x, $\mathrm{vol} f^{-1}(h(J(x))) \cap y = \mathrm{vol} J(x) \cap y$. Notice that this last condition implies that for every $y \in N_n$ containing x, $\mathrm{vol} f^{-1}(h(J(x))) \cap y = \mathrm{vol} J(x) \cap y$.

A family $\{h_x\}_{x\in N_m}$ is said to be nice if each h_x is x-nice and $\|h_x - f\|_{C^1} \to 0$ uniformly as $\operatorname{rank}(x) \to \infty$. Let $\rho = 2^{-100n}$. We will now construct inductively nice families $\{\tilde{h}_x\}_{x\in N_m}$, $0 \le m \le n$ such that $\tilde{h}_x = \tilde{h}_y$ in a $2^{-t(x)}\rho^{m+1}$-neighborhood of $B(y)$ whenever y is a subcell of x, and such that if x is $2^{-(m+1)}\theta$-close to K_0 then $\tilde{h}_x = f|R(x)$.

Let $x \in N_0$. If ϵ is small and x is $\theta/2$-close to K_0, then $\tilde{h}_x = f|R(x)$ is x-nice. Otherwise, if ϵ is sufficiently small, then by a small C^1 modification of H_x we obtain a map $\tilde{h}_x$ which is x-nice. The easiest way to see this is to first conjugate by λ_x, bringing things to unit scale. More precisely, we get into the setting of Lemma 3.1 (with $k = 0$) by putting $F = \lambda_x^{-1} \circ f^{-1} \circ H_x \circ \lambda_x$ and $W = \lambda_x^{-1}(J(x))$. Let s be the map given by the Lemma 3.1. Then $\tilde{h}_x = H_x \circ \lambda_x \circ s \circ \lambda_x^{-1}$ is x-nice. Moreover, $\{\tilde{h}_x\}_{x\in N_0}$ is a nice family since the estimates improve as the rank grows.

Let now $1 \le m \le n-1$ and assume that for every $k \le m-1$ we have defined a nice family $\{\tilde{h}_x\}_{x\in N_k}$ with the required compatibilities.

If $x \in N_m$ is $2^{-(m+1)}\theta$-close to K_0 just take $\tilde{h}_x = f|R(x)$ as definition and it will satisfy the other compatibility by hypothesis. Otherwise, let Q be the open ρ^m-neighborhood of $B(y)$ and define a map $h_x \in C^\infty_{\mathrm{vol}}(Q, \mathbb{R}^n)$ such that $h_x = \tilde{h}_y$ in the ρ^m-neighborhood of $B(y)$ for every subcell $y \subset x$. Restricting h_x to the $\rho^m/2$ neighborhood of $I(x)$, which is a full compact set (that is, it does not disconnect $\mathbb{R}^n$), since $m \le n-1$, and extending it to $R(x)$ using Corollary 4, we get $h_x^{(1)} \in C^\infty_{\mathrm{vol}}(R(x), \mathbb{R}^n)$ which is C^r close to H_x after rescaling by λ_x. By a small C^1 modification of $h^{(1)}(x)$ outside the ρ-neighborhood of $I(x)$, we can obtain a nice family $\{\tilde{h}_x\}_{x\in N_m}$. This is an application of Lemma 3.1 analogous to the one described before.

By induction, we can construct the nice families as above for $0 \le m \le n-1$. Let now $x \in N_n$. As before, when x is close to K_0 the definition is forced and there is no problem of compatibility by hypothesis. Otherwise, let Q be the open ρ^n-neighborhood of $I(x)$. As before, define a map $h_x : Q \to \mathbb{R}^n$ by gluing the definitions of $\tilde{h}_y$ for subcells of x. Notice that $\mathbb{R}^n \smallsetminus I(x)$ has two connected components, and the bounded one is contained in x. By construction, $I(x)$ is the disjoint union of the $J(y)$ contained in it. Thus the volumes of $h_x(I(x)) \cap f(x)$ and $I(x) \cap x$ are equal. This implies that the bounded component of $\mathbb{R}^n \smallsetminus h_x(I(x))$ has the same volume as the bounded component of $I(x)$. We can restrict $h(x)$ to the $\rho^n/2$ neighborhood of $I(x)$ and extend it to a map $\tilde{h}_x \in C^\infty_{\mathrm{vol}}(R(x), \mathbb{R}^n)$ which is x-nice using Corollary 4 (after rescaling by λ_x and then rescaling back). Thus we obtain a nice family $\{\tilde{h}_x\}_{x\in N_n}$ with all the compatibilities.

The nice family $\{\tilde{h}_x\}_{x\in N_n}$ is such that whenever two n-cells x and y intersect we have $\tilde{h}_x = \tilde{h}_y$ in a neighborhood of the intersection. Let $\tilde{f} : W \to \mathbb{R}^n$ be defined by $\tilde{f}(z) = f(z)$, $z \notin U$ and $\tilde{f}(z) = \tilde{h}_x(z)$ for every $z \in x$, $x \in N_n$. Then $\tilde{f} \in C^1_{\mathrm{vol}}(W, \mathbb{R}^n)$, since near $\partial U \cap W$ the rank of a n-cell x is big and hence $\|\tilde{h}_x - f|R(x)\|_{C^1}$ is small. Moreover $\|\tilde{f} - f\|_{C^1}$ is small everywhere, and $\tilde{f} = f$ in a neighborhood of K_0 by construction. □

4.2. **Manifolds.** We now conclude the proof of Theorem 1 by a triangulation argument. Triangulate M so that for every simplex D there are smooth charts $g_i : W_i \to \mathbb{R}^n$, $\tilde{g}_i : \tilde{W}_i \to \mathbb{R}^n$ such that $f(W_i) \subset \tilde{W}_i$ and D is precompact in W_i. Such charts may be assumed to be volume preserving by [18].

Enumerate the vertices. Apply Theorem 5 in charts to smoothen f in a neighborhood of the first vertex without changing f in a neighborhood of simplices that do not contain this vertex. Repeat with the subsequent vertices. Now suppose we have already smoothed f in a neighborhood of m-simplices, for some $0 \leq m \leq n-1$. Enumerate the $m+1$-simplices and apply Theorem 5 in charts to smoothen it in a neighborhood of the first $m+1$-simplex, without changing it in a neighborhood of simplices that do not contain it (in particular we do not change it near its boundary). Repeat with the subsequent $m+1$-simplices. After n steps we will have smoothed f on the whole M.

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