Percolation and isoperimetry on roughly transitive graphs

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February 16, 2016

Abstract

In this paper we study percolation on a roughly transitive graph $G$ with polynomial growth and isoperimetric dimension larger than one. For these graphs we are able to prove that $p_c < 1$, or in other words, that there exists a percolation phase. The main results of the article work for both dependent and independent percolation processes, since they are based on a quite robust renormalization technique. When $G$ is transitive, the fact that $p_c < 1$ was already known before. But even in that case our proof yields some new results and it is entirely probabilistic, not involving the use of Gromov’s theorem on groups of polynomial growth. We finish the paper giving some examples of dependent percolation for which our results apply.

1 Introduction

Since its introduction by Broadbent and Hammersley in [BH57], the model of independent percolation has received major attention from the physical and mathematical communities. From the perspective of applications, it has the potential to model several different systems, from the flow of fluids in porous media, to the transmission of information on networks or diseases on populations. On the theoretical side, this model has been source of challenging questions, and has given rise to beautiful theories. For a mathematical background of the model on $\mathbb{Z}^d$, see [Gri99] and [BR06] and the references therein.

Besides the classical independent model on $\mathbb{Z}^d$, this study has been generalized by both considering the model on more general graphs, see for instance [LP], [BS96], [HJ06] and [Pet08], or by adding dependence to the percolation configuration, see [LSS97], [BLM87], [Szn10] and [TW12] for some examples of such works.

In this article, we study vertex percolation on roughly transitive graphs, with or without dependence, showing the existence of a phase transition for the process as we vary the density of open vertices. Another important contribution of this work is to help develop multi-scale renormalization on roughly transitive graphs of polynomial growth. Renormalization is a powerful tool, which has been used to analyze several stochastic processes. However this technique has limitations that often restrict its use to the lattice $\mathbb{Z}^d$.

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1.1 Graphs under consideration

In this paper we consider both dependent and independent percolation on roughly transitive graphs. To define this concept precisely, we need to first introduce the notion of rough isometries.

Given graphs $G, G'$ and a constant $c_0 \geq 1$, a map $\phi : G \to G'$ is said to be a $c_0$-rough isometry if for any $x, y \in G$ we have

$$\frac{1}{c_0} d(x, y) - 1 < d(\phi(x), \phi(y)) \leq c_0 d(x, y)$$

(1.1)

and for any $y \in G'$, there exists some $x \in G$ such that

$$d(\phi(x), y) \leq c_0.$$ 

(1.2)

We say that a given graph $G$ is $c_0$-roughly transitive if for any $x, y \in G$ there exists a $c_0$-rough isometry $\phi$ satisfying $\phi(x) = y$.

**Remark 1.1.** There are other (equivalent) definitions of rough isometry, see e.g., Definition 3.7 of [Woe00]. In this work, it is convenient to use (1.1) together with (1.2) as used for example in [ET00].

In [BS96], Benjamini and Schramm suggested a connection between the existence of a phase transition for independent percolation on a given graph and its isoperimetric dimension.

**Definition 1.2.** We say that $G = (V, E)$ satisfies the isoperimetric inequality $I(c_i, d_i)$ if

for any finite set $A \subseteq V$, we have $|\partial A| \geq c_i |A|^{\frac{d_i - 1}{d_i}}.$

(1.3)

For example, it is not difficult to see that $\mathbb{Z}^d$ satisfies the $I(c, d)$ for some $c > 0$, see Theorem 6.37 of [LP], p. 210. In [BS96, Question 2], Benjamini and Schramm asked this:

**Question 1.3.** Is it true that if $G$ satisfies $I(c_i, d_i)$ for some $d_i > 1$ then $p_c(G) < 1$?

See the precise definition of $p_c(G)$ in (2.2) below. In this article we give a positive answer to the above question in the case of roughly transitive graphs of polynomial growth.

Isoperimetric conditions and independent percolation have been studied in various works. In [BS96], the authors proved that $p_c(G) < 1$ when $G$ has infinite isoperimetric dimension (meaning that (1.3) holds with $(d_i - 1)/d_i$ replaced by 1). In [Koz07], Kozma showed that $p_c(G) < 1$ when $G$ is a planar graph with isoperimetric dimension strictly larger than one, polynomial growth and no accumulation points. In [Tei14], a stronger version of (1.3) called local isoperimetric inequality was shown to imply $p_c(G) < 1$ for graphs with polynomial growth. Some arguments in this paper are very similar in spirit to those of [Tei14], the main novelty being that we can replace the stronger local isoperimetric inequality of [Tei14] by the classical (1.3) in the case of roughly transitive graphs.

In this paper we deal with graphs with polynomial growth, as specified in the following.

**Definition 1.4.** Given constants $c_u, d_u > 0$, we say that $G$ has polynomial growth (denoted by $V(c_u, d_u)$) if for every $r \geq 1$ and $x \in V$

$$|B(x, r)| \leq c_u r^{d_u}.$$ 

(1.4)
1.2 Main result

The first result we present here is the following.

**Theorem 1.5.** If $G$ is a roughly transitive graph of polynomial growth satisfying (1.3), then $p_c(G) < 1$. This gives a positive answer to Question 1.3 in this special case.

**Remark 1.6.** Let us note that whenever $p_c(G) < 1$ and $G$ has bounded degree, then the graph $G$ also undergoes a non-trivial phase transition for the Ising model, the Widom-Rowlinson model and the beach model. This follows from Theorems 1.1 and 1.2 of [Hāg00].

The above result is a consequence of our Theorem 1.8 below, which applies to both dependent and independent percolation processes. Roughly speaking, Theorem 1.8 states that, if the dependencies decay fast enough with the distance, then the percolation undergoes a non-trivial phase transition. To be more precise, we need to define what we mean by "decay of dependence".

**Definition 1.7.** We shall say that $P$ satisfies the decoupling inequality $D(\alpha, c)$ (where $\alpha > 0$ is a fix parameter) if for any $x \in V$, $r \geq 1$ and two events $G$ and $G'$ such that $G \in \sigma(Y_{z}, z \in B(x, r))$ and $G' \in \sigma(Y_{w}, w / \in B(x, 2r))$, we have

$$P(G \cap G') \leq (P(G) + c_\alpha r^{-\alpha}) P(G').$$

We are now in position to state the following.

**Theorem 1.8.** Let $G$ be a $c_0$-roughly transitive graph satisfying $V(c_u, d_u)$ and $I(c_i, d_i)$, with $d_i > 1$ and assume the law $P$ satisfies $D(\alpha, c_\alpha)$ with $\alpha > \alpha_*$ (see Remark 4.2 for the definition of $\alpha_*$). Then there exists a $p_* < 1$, depending only on $c_0, c_i, d_i, c_u$ and $d_u$, such that if $\inf_{x \in V} P[Y_x = 1] > p_*$, then $G$ contains almost surely a unique open infinite cluster. Moreover, fixed any value $\theta > 0$ if the marginal distributions of $Y_x$ are large enough,

$$\lim_{v \to \infty} v^\theta P[v < |C_o| < \infty] = 0,$$

where $C_o$ denotes the open connected component containing the origin.

We also prove a theorem establishing the existence of a non-trivial sub-critical phase. This result is simpler to prove but helps to establish a more complete picture of phase transition for dependent percolation on $G$.

**Theorem 1.9.** Let $G$ be a graph satisfying $V(c_u, d_u)$. Moreover, let $P$ be a probability measure that satisfies $D(\alpha, c_\alpha)$ with $\alpha > \alpha_*$, where $\alpha_* > 0$ is defined in Remark 3.6. Then there exists a $p_* > 0$, depending only on $c_u$ and $d_u$, such that if $\sup_{x \in V} P[Y_x = 1] < p_*$, then the graph contains almost surely no infinite open cluster. Moreover, fixed $\theta > 0$, if the marginal distributions of $Y_x$ is small enough,

$$\lim_{r \to \infty} r^\theta P[\text{diam}(C_x) > r] = 0,$$

where $C_x$ denotes the open connected component containing a fixed site $x \in V$. 

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Remark 1.10. a) Note that Theorem 1.9 does not require $G$ to be roughly transitive.

b) Moreover, this theorem does not follow from a simple path counting argument because of the dependence present in the law $\mathbb{P}$.

c) Given the above results, a natural question would be whether the condition $D(\alpha, c_{\alpha})$ on the decay of dependence of $\mathbb{P}$ could be weakened. Of course, the parameters $\alpha_{**}$ and $\alpha_{**}$ that appear above are not supposed to be sharp. However, let us observe that if the exponent $\alpha$ appearing in the decay of dependence of the law $\mathbb{P}$ is slow enough, then there are counterexamples showing that Theorem 1.8 does not hold, see Subsection 7.3.

1.3 Transitive graphs

We can specialize our main results to the special case of transitive graphs of polynomial growth. It is important to observe that the hypothesis (1.3) is not necessary in this case, since this can be deduced for instance from [BS92]. This yields to another consequence of our main result, which was already known before.

Corollary 1.11. Let $G = (V, E)$ be a transitive graph satisfying $V(c_u, d_u)$ having growth faster than linear, then $p_c(G) < 1$.

Although the above result was already known, as we discuss in detail in the next subsection, it is worth mentioning that our proof does not make use of Gromov’s theorem on groups of polynomial growth, relying instead on probabilistic tools only.

Previously known results Percolation on transitive graphs has been intensively studied in the last decades specially for the independent case. Let us now mention some of the works that more closely relate to the current article.

In [Lyo95], Russel Lyons proved that for independent percolation, $p_c(G) < 1$ if $G$ is a group of exponential growth (see also [LP, Chapter 7]). The case of Cayley graphs of finitely presented groups with one end has been dealt with in [BB99] also in the independent case. A similar question has also been considered on the Grigorchuk group, an example of group with intermediate growth (see [MP01]). In Corollary 3.2 of [APS12] it has been proved that $p_c(G) < 1$ for transitive graphs $G$ satisfying another isoperimetric inequality, see (2.4) and Definition 2.3 of [APS12]. If $G$ is a transitive amenable graph, it was proved in [BK89] that if for some $p$ there exists an infinite open cluster, then it is almost surely unique, see also Theorem 2.4 of [HJ06].

The most important relation between previously known results and our work comes at the intersection with Corollary 1.11, since transitive graphs can be associated with a group of automorphisms, benefiting therefore from important results on group theory.

More precisely, if $G$ is a transitive graph of polynomial growth, then $G$ is quasi-isometric to a Cayley graph of a nilpotent group (see [Tro84], [Los87], Theorem 4 of [Sab64] or [GIS+89, Theorem 2]). This yields two different proofs of Corollary 1.11. Let $G$ be the Cayley graph of a nilpotent group with super-linear growth. Then

a) We can use Theorem 7.19 of [LP] to conclude that there exists a subset of $G$ which is quasi-isometric to $\mathbb{Z}^2$, therefore $p_c(G) < 1$ as desired. This argument has the
advantage that it allows for duality arguments that can work even for dependent percolation.

b) Alternatively, we observe that $G$ is finitely presented (see Exercise 4.3 of [Pet15]) and use Theorem 9 of [BB99] to conclude that the number of cut-sets of size $n$ separating a fixed vertex from infinity is at most $c^n$. Then a simple Peierls-type argument can show that $p_c < 1$. The added benefit of this approach is that it gives an exponential bound on the probability (1.5) for Bernoulli percolation on transitive graphs.

Remark 1.12. In light of the above, let us emphasize some advantages of our approach.

a) For the case of transitive graphs, our proof does not make use of Gromov’s Theorem on groups of polynomial growth.

b) To the best of our knowledge, the bound in (1.5) does not seem to follow from the above arguments in the case of dependent percolation on transitive graphs.

c) Uniqueness of the infinite cluster obtained in Theorem 1.8 does not depend on the translation invariance of the law $\mathbb{P}$ as is the case with the argument in [BK89].

d) Note that being roughly isometric to each other defines an equivalence relation over the set of graphs. However, it is important to notice that the distortion constant $c_0$ worsens as we compose rough isometries. Therefore, for a given roughly transitive graph there is not necessarily an analogue of the group of isomorphisms that is fundamental in the case of transitive graphs.

e) We strongly believe that the techniques we develop here could be easily extended in order to work for weaker notions of transitivity, for example by weakening the notion of rough isometries. We however kept the current presentation in order to avoid an overly complicated exposition.

We thank Yuval Peres, Gábor Pete, Russel Lyons and Itai Benjamini for bringing some of the above results to our attention.

1.4 Idea of the proofs

The proofs of Theorems 1.11 and 1.8 follow a renormalization scheme which allows us to bound the probability of certain “bad events” as the scale size grows. In this section we will focus on the case of Theorem 1.8 which is the more elaborate one.

For any $x \in V(G)$ and $L > 0$ set

$$S(x, L) = \text{“there exist two large connected sets in } B(x, 3L), \text{ which cannot be joined by an open path in } B(x, 3L^2)\text{”}.$$ 

This will play the role of the “bad event” in the proof of Theorem 1.8, see (4.1) for a precise definition.

The main advantage of the above event is that it plays two complementary roles. First, the events $S(x, L)$ are hierarchical (see the Cascading Lemma 4.1), therefore is possible to bound their probabilities using inductive arguments coming from a multi-scale renormalization procedure. Secondly, these events are rich enough that, once we show
that $\mathbb{P}[S(x, L)]$ decays fast as $L$ goes to infinity, we can derive the existence of a unique open infinite connected component, as desired (see Lemma 4.4).

For the inductive part of the argument, we need to introduce a rapidly growing sequence $(L_k)_{k \geq 1}$ of scales, see (3.2). As we mentioned above, our objective is to show that for large enough values of the percolation parameter $p$, the probabilities $p_k = \mathbb{P}[S(o, L_k)]$ of observing a separation event at scale $k$ go to zero fast as $k$ goes to infinity.

The proof of our main results can then be described through three steps:

a) We first show that $S(o, L_{k+1})$ implies the occurrence of $S(y_i, L_k)$ for several points $y_i \in B(o, 2L_{k+1}^2)$, see the Cascading Lemma 4.1. Note that the event $S(y_i, L_k)$ takes place in the smaller scale $L_k$.

b) Derive from the above a recursive inequality between $p_{k+1}$ and $p_k$, to show that if $p$ is close enough to 1, then $p_k$ goes to zero fast as $k$ goes to infinity, see Section 4.

c) Finally, in Lemma 4.4 we show that a fast decay of $p_k$ implies our main result.

Although all of the above steps are essential in establishing Theorems 1.11 and 1.8, we note that items b) and c) follow the same spirit to what has been done in [Tei14]. For the sake of completeness we also include their proofs in the current paper. However it is step a) that contains the main novelty of the current work, see the Cascading Lemma 4.1. It is this lemma that allows us to weaken the local isoperimetric inequality of [Tei14] to the canonical definition (1.3) for roughly transitive graphs of polynomial growth.

1.5 Sketch of the proof of the Cascading Lemma

The main new ingredient of this paper is the Cascading Lemma proved in Section 5. Setting up a renormalization scheme on a graph that is not $\mathbb{Z}^d$ requires a good understanding of the geometry of the graph in question and it is during the proof of Cascading Lemma that this difficulty is revealed. For this proof we make strong use of the isoperimetric inequality and rough transitivity of $G$.

The proof of the Cascading Lemma follows three main steps. Recall that we are assuming the occurrence of $S(o, L_{k+1})$, which provides us with two large sets $A^0, A^1 \subseteq B(x, L_{k+1})$ which cannot be connected by an open path. Our aim is to show the existence of such separation events in various balls of size $L_k$ inside $B(o, 2L_{k+1}^2)$.

i) The first step of the proof will be to reduce the quest of finding separation events $S(y_i, L_k)$ into simply connecting $A^0$ with $A^1$ through several paths. This is the content of Lemma 5.1.

ii) Therefore, we can assume by contradiction that there exists two sets $A^0$ and $A^1$ which cannot be connected by several paths as above. However, the isoperimetric inequality (1.3) guarantees the existence of several disjoint paths (not necessarily open) connecting $A^0$ to distance $L_{k+1}^2$ (similarly for $A^1$), see Lemma 5.3.

iii) Roughly speaking, in the last step we use the existence of $A^0$ and $A^1$ above in order to embed a binary tree into $G$, which would contradict the polynomial growth of this graph. We start with the ball $B(o, 3L_{k+1})$ (where the sets $A^0$ and $A^1$ reside) and two paths from the previous step as a building block. They will respectively
represent the root $\emptyset$ of the binary tree and the edges connecting $\emptyset$ to its descendants. Finally we use the rough transitivity of $G$ to replicate this pattern. Arguing in a recursive way we obtain the desired embedding, which leads to a contradiction on the polynomial growth of $G$.

Steps $i)$ and $iii)$ are illustrated in Figures 1 and 2 respectively.

This paper is organized as follows. In Section 2 we introduce some preliminary notation and prove an auxiliary result, followed by Section 3, where we show Theorem 1.9.

In Section 4 we define the separation events $S(x, L)$ and state two fundamental intermediate results (Lemmas 4.1 and 4.4). Then, assuming their validity, we prove Theorems 1.11 and 1.8, which corresponds to Step $b)$ in the outline of the proof of our main results.

Section 5 is devoted to proving the Cascading Lemma and is split into three subsections. Each of these subsections correspond to one step in the above sketch. Finally we show Lemma 4.4 in Section 6, and we conclude with Section 7, which is devoted to giving some examples of dependent percolation processes for which our results apply.

Acknowledgments

We are grateful to Yuval Peres, Gábor Pete, Russel Lyons and Itai Benjamini for bringing to our attention some fundamental references and suggestions. Thanks also to Mikhail Belolipetski for the fruitful discussions.

A.T. is grateful to CNPq for its financial contribution to this work through the grants 306348/2012-8 and 478577/2012-5. This work began during a visit of E.C. to IMPA, that she thanks for the support and hospitality.

2 Notation and auxiliary results

In this section we introduce some notation and prove some auxiliary results that will be useful throughout the paper.

2.1 Notation

For every finite set $A \subset V$ we denote by $|A|$ its cardinality, and by $\partial A$ its edge boundary:

$$\partial A := \{ \{x, y\} \in E : x \in A \text{ and } y \notin A \}.$$  

Analogously, its internal vertex boundary is denoted by

$$\partial_i A := \{ x \in A : \text{ there exists } y \in V \setminus A \text{ such that } \{x, y\} \in E \}.$$  

For any two vertices $x, y \in V$ we will denote by $d(x, y)$ the graph distance between $x$ and $y$, i.e., the minimum number of edges contained in a path that goes from $x$ to $y$. Analogously, for any two sets $A, B \subset V$ we set

$$d(A, B) := \min\{d(a, b) : a \in A, b \in B\}.$$  

By $B(x, R)$ we denote a ball centered at $x$ and radius $R \geq 0$ in the graph distance and let us define the growth function

$$\bar{v}_G(r) = \sup_{x \in G} |B(x, r)|,$$  

where we may omit the sub-index in $v_G$ if it is clear from the context.
Remark 2.1. Note that by (1.4) we have $\bar{v}_G(r) \leq c_a r^{d_a}$.

Independent percolation (sometimes called Bernoulli) can be described as follows. We associate for each vertex $x \in V$ an independent coin toss with success parameter $p \in [0,1]$, in case of success we say that the vertex is open otherwise we call it closed. This gives rise to a random sub-graph $G_p$ of $G$, induced by the set of open vertices.

One of the most interesting features of this model is that for several graphs it presents a phase transition at a critical value $p_c \in (0,1)$. To make the above statement more precise, we define the critical value $p_c = p_c(G)$ as follows

$$p_c := \sup\{p \in [0,1] : \mathbb{P}\{\text{there exists an infinite cluster on } G_p\} = 0\}.$$  \hspace{1cm} (2.2)

It follows that, for $p < p_c$, the induced sub-graph contains almost surely only finite connected components, while for $p > p_c$ it contains almost surely an infinite cluster. See [Gri99] for a proof that $p_c \in (0,1)$ for the case $V = \mathbb{Z}^d$, $d \geq 2$, endowed with edges connecting nearest neighbors vertices.

2.2 Some remarks about rough isometries

The results presented here follow the exposition of [ET00], to which the reader is referred for more details. Suppose that $\phi : G \to G'$ is a $c_0$-rough isometry. Then for any set $A \subseteq G$ we have

$$|\phi(A)| \geq \frac{|A|}{\bar{v}_G(c_0)}.$$ \hspace{1cm} (2.3)

In fact, if $d(x, y) \geq c_0$, then $\phi(x) \neq \phi(y)$ by (1.1). This implies that at most $\bar{v}_G(c_0)$ many points can share the same image under $\phi$ in $G'$.

Another interesting property of rough isometries is that they are almost invertible, in the following sense.

Given a $c_0$-rough isometry $\phi : G \to G'$, there is a $4c_0^2$-rough isometry

$$\psi : G' \to G \text{ such that } d(x, \phi \circ \psi(x)) \leq c_0 \text{ for any } x \in V.$$ \hspace{1cm} (2.4)

Indeed, let us define $\psi(x')$ as the point $x \in V$ such that $d(x', \phi(x))$ is minimized (choosing arbitrarily in case of ties). First of all, observe by (1.2) that $d(x', \phi \circ \psi(x')) \leq c_0$. We now show that $\psi$ is a $4c_0^2$-rough isometry and for this fix $x', y' \in G'$. We can assume that $x' \neq y'$ (the other case is trivial), then one estimates

$$\frac{1}{4c_0^2} d(x', y') - 1 \leq \frac{1}{4c_0^2} \left( d(x', \phi \circ \psi(x')) + d(y', \phi \circ \psi(y')) + d(\phi \circ \psi(x'), \phi \circ \psi(y')) \right) - 1 \leq \frac{1}{4c_0^2} d(\psi(x'), \psi(y')) + \frac{1}{c_0} - 1 \leq \frac{1}{4c_0^2} d(\psi(x'), \psi(y')) + \frac{1}{c_0} - 1 \leq \frac{c_0}{4} - 1 \leq d(x', y') \leq c_0 d(\phi \circ \psi(x'), \phi \circ \psi(y')) + c_0 \leq c_0 \left( d(x', \phi \circ \psi(x')) + d(\phi \circ \psi(y'), y') + d(x', y') \right) + c_0 \leq c_0 d(x', y') + 2c_0^2 + c_0 \leq 4c_0^2 d(x', y').$$

Also, if $x'$ belongs to the image of $\phi$, then $d(\phi(\psi(x')) , x') = 0$, so that $d(\psi(\phi(x)) , x) \leq c_0$, and consequently (1.2) also holds for $\psi$. This concludes the proof of (2.4)
Remark 2.2. It would be tempting to say that every roughly transitive graph is roughly isomorphic to a transitive one. This is however not the case, as shown in [ET00, Proposition 2].

We would like also to recall Open Question 2.3 of [Ben13]: “Is there an infinite \( c_0 \)-roughly transitive graph, which is not roughly-isometric to a homogeneous space, where a homogeneous space is a space with a transitive isometry group?”

On the other hand, recall from Remark 1.12 e) that the techniques presented here are believed to work beyond the case of roughly transitive graphs.

2.3 Paving

For the next lemma, we need also to introduce a lower bound on the volume growth of balls on \( G \).

Definition 2.3. Given constants \( c_l, d_l > 0 \), we say that \( G \) satisfies \( L(c_l, d_l) \) if for every \( r \geq 1 \) and \( x \in V \)

\[
|B(x, r)| \geq c_l r^{d_l}. \tag{2.5}
\]

Note that every infinite connected graph satisfies the above bound for \( d_l = 1 \) and we don’t need more than this for our proofs. However, if one knew in advance that the above condition holds for some \( d_l > 1 \), the final results will be improved through a smaller \( \alpha_* \) or \( \alpha_{**} \), see (4.5).

Proposition 2.4 below allows us to cover a large ball of radius \( r^2 \) with smaller balls of radius \( s \). This can be thought of as a replacement for paving arguments for renormalization procedures on the lattice \( \mathbb{Z}^d \).

Proposition 2.4. If \( G = (V, E) \) satisfies the volume growth estimates \( V(c_u, d_u) \) and \( L(c_l, d_l) \), then there is a constant \( c_1 = c_1(c_l, d_l, c_u, d_u) \) such that for every \( r \geq 1 \) and \( s \in [2, 2r^2] \), for every \( x \in V \), there exist \( K \subseteq B(x, 2r^2) \) such that

\[
|B(x, r^2)| \subseteq B(K, s) \quad \text{and} \quad |K| \leq c_1 r^{2d_u} \tag{2.6}
\]

Proof. Fix \( s \) in the range given in the hypothesis and take the set \( K \subseteq B(x, 2r^2) \) to be an arbitrary maximal set satisfying

\[
d(y, y') \geq s \quad \text{for every} \quad y, y' \in K. \tag{2.7}
\]

Since \( K \) is maximal, it also exists an \( s \)-net of \( B(x, 2r^2) \), or in other words \( B(x, 2r^2) \subseteq B(K, s) \).

By (2.7), all the balls \( \{B(y, s/2)\}_{y \in K} \) are disjoint. Therefore, by the lower bound \( L(c_l, d_l) \) we obtain

\[
|B(K, s/2)| = \sum_{y \in K} |B(y, s/2)| \geq |K| c_1 \frac{s^{d_u}}{2^{d_l}}.
\]

On the other hand, \( |B(K, s/2)| \leq |B(x, 2r^2 + s)| \leq c_u (2r^2 + s)^{d_u} \). By putting together these two facts, we obtain that there is a positive constant \( c_1 = c_1(c_l, d_l, c_u, d_u) \) such that

\[
|K| \leq c_1 r^{2d_u} s^{d_u}.
\]

The above argument implies that there exists a set \( K \subseteq B(x, 2r^2) \) such that the statement holds (for \( r \) large enough). By possibly increasing the constant \( c_1 \), we can assure that the statement holds for all \( r \geq 1 \).
2.4 Decoupling several events

Our next statement is a consequence of the decoupling inequality from Definition 1.7.

**Proposition 2.5.** Suppose that \( \mathbb{P} \) satisfies the decoupling inequality \( D(\alpha, c_\alpha) \) for some \( \alpha > 0 \). Now fix any value of \( r \geq 1 \), an integer \( J' \geq 2 \) and disjoint points \( y_1, y_2, \ldots, y_{J'} \in V \) such that

\[
\min_{1 \leq i < j \leq J'} d(y_i, y_j) \geq 3r.
\]

Then for any set of decreasing events \( G_1, \ldots, G_{J'} \) such that \( G_i \in \sigma(Y_z, z \in B(y_i, r)) \) we have

\[
\mathbb{P}(G_1 \cap \ldots \cap G_{J'}) \leq (\mathbb{P}(G_1) + c_\alpha r^{-\alpha}) \ldots (\mathbb{P}(G_{J'}) + c_\alpha r^{-\alpha}). \tag{2.8}
\]

**Proof.** The proof is immediate from Definition 1.7. In fact, setting \( G' = G_1 \cap \ldots \cap G_{J'-1} \),

\[
\mathbb{P}(G_1 \cap \ldots \cap G_{J'}) \leq (\mathbb{P}(G_{J'}) + c_\alpha r^{-\alpha}) \mathbb{P}(G_1 \cap \ldots \cap G_{J'-1}). \tag{2.9}
\]

By iterating this calculation, we obtain the statement. \( \square \)

3 Proof of Theorem 1.9

This proof is inspired to previous renormalization procedures that were developed for \( \mathbb{Z}^d \), see for instance [Szn10]. Here we adapt them to work on more general classes of graphs. Although Theorem 1.9 is not the central result of the current article, we present its proof before for two reasons. First, it is a warm-up to the proof of Theorem 1.8 and secondly, it includes some lemmas that will be useful later in the text.

Let us first define what we call the crossing event

\[
\mathcal{T}(x, L) = \left[ \text{there is an open path from } B(x, 3L) \text{ to } \partial B(x, 3L^2) \right]. \tag{3.1}
\]

Our main argument shows the decay of the probabilities of \( \mathcal{T}(x, L) \) following a renormalization scheme. This procedure relates the probabilities of the above events at different scales, that we now introduce.

Given some \( \gamma \geq 2 \), we set

\[
L_0 := 10000, \quad L_{k+1} = L_k^\gamma, \quad \text{for all } k \geq 0. \tag{3.2}
\]

**Remark 3.1.** We have not yet chosen \( \gamma \) because it will assume different values for the proofs of Theorems 1.8 and 1.9, see Remarks 3.6 and 4.2 below.

In the next definition we introduce the concept of a *cascading* family of events. Intuitively speaking, it means that if some event occurs at a given scale \( L_{k+1} \), then it must also occurs several times in the previous scale \( L_k \) in well separated regions.

**Definition 3.2.** We say that a family of events \( \{E(x, L_k)\}_{x \in V, k \geq 1} \) is *cascading* if for any \( J \geq 1 \) there exists \( c_2 = c_2(G, J, \gamma) \) for which the following holds. Fix any \( x \in V, k \geq c_2 \) and set \( K \subseteq B(x, 2L_{k+1}^2) \) such that \( B(K, L_k) \) covers \( B(x, L_{k+1}^2) \). Then, if the event \( E(x, L_{k+1}) \) occurs, there exists a sequence

\[
y_1, y_2, \ldots, y_J \in K \text{ with } d(y_j, y_l) \geq 9L_k^2 \text{ for all } j \neq l \tag{3.3}
\]

and such that \( E(y_j, L_k) \) occurs for all \( j \leq J \).
The importance of the above definition is that it allows us to relate the probabilities of events $\mathcal{E}$ at different scales using recursive inequalities together with the decoupling provided by $\mathcal{D}(\alpha, c_\alpha)$.

Lemma 3.3. The family of events $\{T(x, L)\}$ defined in (3.1) is cascading in the sense of Definition 3.2.

Proof. We first fix $J \geq 1$ and let $c_3 \geq 1$ be such that for all $k \geq c_3$ we have

$$3L_{k+1} + 30JL_k^2 \leq L_{k+1}^2, \tag{3.4}$$

which can be done by our choice of scales in (3.2).

To prove that the events $T(x, L)$ are cascading, let us pick $k \geq c_3$, $x \in V$ and assume that $T(x, L_{k+1})$ occurs, that is

there exists an open path $\sigma$ from $B(x, 3L_{k+1})$ to $\partial B(x, 3L_{k+1}^2)$. \tag{3.5}

Let us consider the concentric spheres $S_j = \partial B(x, 3L_{k+1} + (30j)L_k^2)$, for $j = 1, \ldots, J$. Note that all these spheres are contained in $B(x, L_{k+1}^2)$ by (3.4).

We now let $x_j$ be the first point of intersection of the path $\sigma$ to $S_j$. Given the set $K$ as in Definition 3.2 (or more precisely, such that $B(K, 3L_k)$ covers $B(x, L_{k+1})$) we can pick $y_j \in K \ (j \leq J)$ such that $x_j \in B(y_j, L_k)$.

We see that the distance between two distinct $y_j$’s is at least

$$d(y_j, y_{j'}) \geq d(x_j, x_{j'}) - 2L_k \geq 30L_k^2 - 2L_k \geq 9L_k^2 \tag{3.6}$$

as required in Definition 3.2. To finish the proof, observe that the open path $\sigma$ that guarantees the occurrence of $T(x, L_{k+1})$ can be split into pieces that show the occurrence of $T(y_j, L_k)$, for $j \leq J$. The piece corresponding to $j$ can be constructed for instance by picking the first time $\sigma$ touches $x_j$ until it first exits $B(y_j, 3L_k^2)$. This finishes the proof of the lemma. \qed

Remark 3.4. In the next section we will turn to the proof of Theorem 1.11 and for this we define another family of events (denoted by $\mathcal{S}(x, L)$) and prove a result which is analogous to Lemma 3.3, namely the Cascading Lemma. However, the proof of that the events $\mathcal{S}(x, L)$ are cascading will be more involved.

It is important to observe that some definitions and arguments in this section were written in a way that they can be used also during the proof of Theorem 1.8, instead of optimizing for brevity.

The importance of the definition of cascading events will become clear in the following bootstrapping result. Given a scale sequence $L_k$ as in a family of events $\mathcal{E}(x, L_k)$ let

$$p_k^\mathcal{E} = \sup_{x \in V} \mathbb{P}(\mathcal{E}(x, L_k)). \tag{3.7}$$

Lemma 3.5. Suppose that $G$ satisfies $\mathcal{V}(c_u, d_u)$ and $\mathcal{L}(c_1, d_1)$ and $\mathbb{P}$ has the decoupling inequality $\mathcal{D}(\alpha, c_\alpha)$, for $\alpha > 2\gamma d_u - d_1$. Moreover, let $\mathcal{E}(x, L_k)$ be a family of events which is cascading in the sense of Definition 3.2, then for any $\beta > 0$, there exists a constant $c_4 = c_4(\beta, c_1, d_1, c_u, d_u) \geq 1$ such that

if for some $k_0 \geq c_4$ we have $p_{k_0}^\mathcal{E} \leq L_{k_0}^{-\beta}$ then $p_k^\mathcal{E} \leq L_k^{-\beta}$ for all $k \geq k_0$. \tag{3.8}
Proof. Given $\beta > 0$, let us pick some $\beta' > \max\{\alpha, \beta\}$ and an integer $J'$ such that

$$J' \geq \max \left\{ 2, \frac{\gamma \beta'}{\alpha - (2\gamma d_u - d_l)} \right\}. $$

which is possible since $\alpha > 2\gamma d_u - d_l$.

By choosing $k$ large enough we can apply Proposition 2.4 and set $s := L_k$ and $r := L_{k+1}$, which gives us a set $K \subseteq B(o, 2L_{k+1}^2)$ such that

$$|K| \leq c_1 L_k^{2\gamma d_u - d_l} \quad \text{and} \quad B(o, L_{k+1}^2) \subseteq B(K, L_k).$$

Our purpose is to bound the probabilities $p_k^E$ using induction. In fact, using the fact that the events $E(x, L_k)$ are cascading, we have

$$p_{k+1}^E \leq \mathbb{P}\left[ \exists y_1, \ldots, y_{J'} \in K \text{ at mutual distance at least } 9L_k^2, \text{ s.t. } E(y_i, L_k) \text{ occurs } \forall i \leq J' \right] \leq \mathbb{P}\left[ \exists c_k L_k^{2\gamma d_u - d_l} \right]^{J'} \left( p_k^E + c_k L_k^{-\alpha} \right)^{J'}.$$

Assume as in (3.8) that for some $k_0$ large enough we have $p_{k_0}^E \leq L_{k_0}^{-\beta}$, we need to show that this condition holds for all $k \geq k_0$. In fact, by using the fact that

$$(p_k^E + c_k L_k^{-\alpha}) \leq (c_\alpha + 1)L_k^{-\min\{\alpha, \beta'\}},$$

as well as our assumption $p_{k_0}^E \leq L_{k_0}^{-\beta}$, we obtain:

$$\frac{p_{k_0+1}^E}{L_{k_0}^{-\beta}} \leq c_1 \left( c_\alpha + 1 \right)^{J'} L_{k_0}^{J'(2\gamma d_u - d_l) - \beta'\min\{\alpha, \beta'\} + \gamma \beta'} \leq c_1 \left( c_\alpha + 1 \right)^{J'} L_{k_0}^{-\beta'}.$$

By our choice of $J'$, the above is smaller than 1 for all $k$ large enough, proving (3.8). \qed

**Remark 3.6.** Recall that in Theorem 1.9 we have used the value $\alpha_{**}$ without giving its precise value. We can now introduce

$$\alpha_{**} := 4 d_u - d_l.$$

**Proof of Theorem 1.9.** We now fix $\gamma = 2$ and let the scale sequence $(L_k)_{k \geq 0}$ be defined as in (3.2). Observe also that for $\alpha > \alpha_{**}$ as in (3.9), we have $\alpha > 2\gamma d_u - d_l$ as required in Lemma 3.5.

Therefore, we are in position to apply Lemma 3.5 for some arbitrarily chosen $\beta > 2\theta$. In order to show that $p_k \leq L_k^{-\beta}$ for large enough $k$, we have simply to show that $p_{k_0} \leq L_{k_0}^{-\beta}$ for some $k_0 \geq c_4$.

But by a simple union bound,

$$p_{k_0}^T = \sup_{x \in V} \mathbb{P}(T(x, L_{k_0})) \leq \mathbb{P}[Y_z = 1 \text{ for some } z \in B(x, L_{k_0})] \leq c_\alpha L_{k_0}^{2\gamma d_u} \sup_{x \in V} \mathbb{P}[Y_x = 1].$$

Therefore, as soon as

$$\sup_{x \in V} \mathbb{P}[Y_x = 1] \leq \frac{1}{c_\alpha L_{k_0}^{2\gamma d_u}},$$

we have $p_{k_0}^T \leq L_{k_0}^{-\beta}$ as desired and therefore $p_k \leq L_k^{-\beta}$ for all $k \geq k_0$.

To finish, given a large enough $r \geq 1$, take $k$ such that $L_k \leq r < L_{k+1}$. Then,

$$r^\theta \mathbb{P}[\text{diam}(C_0) > r] \leq L_k^\theta p_k^T \leq L_{k+1}^{2\theta} \leq L_2^{2\theta}$$

The proof now follows from the fact that $\beta > 2\theta$. \qed
4 Proof of Theorem 1.8

The proof of Theorem 1.8 follows the same lines of the previous section. We are going to define a family of events $S(x, L)$ and then show that they are cascading in the sense of Definition 3.2. This task will however be much more involved than in the previous section.

We now define what we call a separation event. This will play the role of a “bad” event whose probability we intend to bound from above. Roughly speaking, the separation event says that inside a big ball one can find two large and separated clusters (which are not necessarily open).

Denoting by $\text{diam}(Y)$ the diameter of the set $Y$, for every $x \in V$ and $L \in \mathbb{R}_+$, the separation event $S(x, L)$ is defined as follows:

$$S(x, L) := \left\{ \text{ there are } A^0, A^1 \subseteq B(x, 3L) \text{ with } d(A^0, A^1) > 1, \text{diam}(A^i) \geq L/100, \text{ s.t. there is no open path in } B(x, 3L^2) \text{ connecting } A \text{ with } A'. \right\}.$$  

(4.1)

See Figure 1 for an illustration of the above event.

Recall $d_i$ and $d_u$ from (1.3) and (1.4) respectively, and consider a fixed constant $\gamma \geq 2$ such that

$$\gamma \left( \frac{d_i - 1}{d_i} \right) > 2d_u,$$

(4.2)

and as above we set

$$L_0 := 10000, \text{ and } L_{k+1} = L_k^\gamma, \text{ for all } k \geq 0.$$  

(4.3)

By $p_k$ we denote the probability to observe a separation event at scale $k$, i.e., set

$$p_k := \inf_{x \in V} \mathbb{P}[S(x, L_k)].$$  

(4.4)

In the above definition we use the infimum over $x \in V$, as we are not necessarily assuming that $G$ is transient of that $\mathbb{P}$ is translation invariant.

A fundamental step in the proof of Theorem 1.8 is to show that for values of $p$ close enough to one, the probabilities $p_k$ decay to zero very fast as $k$ increases.

In this section we assume that $G$ satisfies the extra Condition 1 below and prove Theorem 1.8. This condition will later be proved to hold true for roughly transitive graphs satisfying $\mathcal{V}(c_u, d_u)$ and $\mathcal{I}(c_i, d_i)$ with $d_i > 1$, see the Cascading Lemma 4.1.

Roughly speaking Condition 1 states that if $S(o, L_{k+1})$ occurs for some $k+1$, then we can find various separation events at the smaller scale $k$.

**Condition 1.** We say that a given graph satisfies Condition 1 for some $\gamma \geq 2$ and $L_k$ as in (4.3), if the collection of events $\{S(x, L_k)\}_{x \in V, L_k \geq 1}$ is cascading in the sense of Definition 3.2.

Before proceeding, let us briefly recall how a statement similar to the above was derived in [Tei14] and the main challenges that we face in our context. On that paper, a stronger hypothesis on the underlying graph was assumed, namely that $G$ verifies certain local isoperimetric inequalities.

In the current work, we only make use of the standard isoperimetric inequality (1.3), together with the hypothesis that $G$ is roughly transitive and has polynomial growth (see also Remark 2.3 (c) in [Tei14]). In particular, the next lemma will guarantee that Condition 1 is implied by $\mathcal{V}(c_u, d_u)$ and $\mathcal{I}(c_i, d_i)$, with $d_i > 1$. This will be an important novelty of this work and we will postpone its proof to Section 5.
Figure 1: The six balls $B(x,L_{k+1}), \ldots, B(x,3L_{k+1})$ and $B(x,L_{k+1}^2), \ldots, B(x,3L_{k+1}^2)$. The sets $A^0$ and $A^1$ from the definition of $S(x,L_{k+1})$ are pictured, together with a solid path connecting them. According to the definition of $S(x,L_{k+1})$, this solid path must pass through a closed vertex. The gray dots in the picture represent the set $K$ from Proposition 2.4. We also indicate the occurrence of the event $S(y,L_k)$ as in Lemma 5.1.

**Lemma 4.1** (Cascading Lemma). Let $G$ be $c_0$-roughly transitive, satisfying the conditions $V(c_u,d_u)$, $(1.3)$ with $d_i > 1$, and let $\gamma$ be as in $(4.2)$, then $G$ satisfies Condition 1. Moreover, the constant $c_2$ appearing in Definition 3.2 depends only on $c_0,J,c_i,d_i,c_u$ and $d_u$.

We will now give a proof of Theorem 1.8, assuming the validity of the Cascading Lemma above, which will be proved in Section 5.

**Remark 4.2.** In Theorem 1.8, we assumed that $\alpha > \alpha_*$, which still had to be defined. We can now introduce

$$\alpha_* := 2 \left( \frac{2d_u d_i}{d_i - 1} \right) d_u - d_i,$$

(4.5)

Note that for $\alpha > \alpha_*$

we can find $\gamma$ as in $(4.2)$ and such that $\alpha > 2\gamma d_u - d_i$ as in Lemma 3.5. (4.6)

Recall the definitions of $p_k$ from (4.4). We first show the decay of $p_k$ for large enough $p$ in the following lemma.

**Lemma 4.3.** Suppose that $G$ is a roughly transitive graph satisfying $V(c_u,d_u)$, $\mathcal{L}(c_i,d_i)$ and $\mathcal{I}(c_i,d_i)$ with $d_i > 1$, $\gamma$ satisfies $(4.2)$ and fix $\beta$ fulfilling $D(\alpha,c_u)$. Then, given any $\beta > 0$, there exists $p_* = p_*(\beta,\gamma,c_i,d_i,c_1,d_1,c_u,d_u) < 1$ such that for any $p > p_*$ we have

$$p_k \leq L_k^{-\beta}, \text{ for every } k \geq 1.$$  

(4.7)
Proof. Since $\gamma$ satisfies (4.2), we can use the Cascading Lemma to conclude that the events $S(x, L)$ are cascading. Hence Lemma 3.5 implies that if for some large value $k_0$ we have $p_{k_0} \leq L_{k_0}^{-\beta}$, then this relation holds for all $k \geq k_0$.

We now observe that as the percolation parameter $p$ converges to one, then the probability of $S(o, L_{k})$ (for some fixed $k \geq c_7$) converges to zero, since balls will likely be completely open. This implies that, if $\inf_{x \in V} \mathbb{P}[Y_x = 1] \geq p_*$ as in the statement of the theorem, we will have $p_k \leq L_k^{-\beta}$ for all $k \geq c_7$. By possibly increasing $p_*$ we can make sure that the above holds for all $k \geq 1$.

The statement of Theorem 1.8 now follows from Lemma 4.3 above and the following lemma, whose proof is deferred to Section 6.

Lemma 4.4. Fix an arbitrary value $\theta > 0$, $\gamma \geq 2$ satisfying (4.2) and $\beta > \gamma(2 + \theta)$. Then, if $p_k \leq L_k^{-\beta}$ for all $k$ large enough,

$$\mathbb{P}[\text{there is a unique infinite connected open cluster } C_{\infty}] = 1$$

(4.8)

and moreover, for every $k$ large enough we have

$$\sup_{x \in V} \mathbb{P}[L < |C_x| < \infty] \leq L^{-\theta},$$

(4.9)

where $C_x$ stands for the open connected component containing $x$.

Note the similarity between the above result and Lemma 4.1 of [Tei14]. It is worth mentioning that despite this similarity, a new proof of the above lemma is required since the definitions of $S(o, L_k)$ and consequently of $p_k$ are different.

5 Proof of the Cascading Lemma

As we mentioned above, the most innovative step in proving Theorem 1.11 was the intermediate Cascading Lemma, that we now prove. The argument is split into three main steps that can be informally described as follows.

Step 1. Suppose we have two sets $A^0, A^1$ which are separated as in the definition of $S(o, L_{k+1})$. We first show that paths connecting $A^0$ to $A^1$ necessarily cross a separation event at the smaller scale $L_k$. This is explained in Subsection 5.1.

Step 2. Therefore our task is now reduced to showing that there are several paths connecting these two sets inside $B(o, 3L_{k+1}^2)$. This is not an immediate consequence of the isoperimetric inequality (1.3). However, this inequality shows that there must be several disjoint paths connecting $A^0$ to $\partial B(o, 3L_{k+1}^2)$ (same for $A^1$), see Subsection 5.2.

Step 3. Finally, we will show that indeed there exist several paths connecting $A^0$ to $A^1$ and this is done by contradiction. More precisely, assuming that there are only few paths connecting these sets, we have a type of “local bottleneck” in our graph. This, together with rough transitivity will allow us to replicate this local bottleneck in different parts of the graph and they act as a branching point for paths of the graph. Therefore, we are able (under this contradiction assumption) to embed a chunk of a binary tree inside $G$, see Figure 2. This will contradict the polynomial growth that we assumed in first place, concluding the proof of the Cascading Lemma. This final argument can be found in Subsection 5.3.
5.1 Using paths to find separation events

The first step in the proof of the Cascading Lemma is to reduce the task of finding separation events at the finer scale $k$ to simply finding paths between the separated sets $A^0$ and $A^1$ at scale $L_{k+1}$.

First, we need to choose a large enough constant $c_8 > 0$ such that, for $k \geq c_8$ we have

$$L_k \leq \frac{L_{k+1}}{2000}, \tag{5.1}$$

which is legitimate, given the inductive definition of $L_k$ in (3.2).

The next lemma helps us obtaining separation events from paths connecting $A^0$ to $A^1$.

**Lemma 5.1.** For some $x \in V$ and $k \geq c_8$, consider a set $K \subseteq B(x, 2L_{k+1}^2)$ such that $B(K, L_k)$ covers $B(x, L_{k+1}^2)$ and a pair of sets $A^0, A^1$ as in the definition of the event $S(x, L_{k+1})$. In other words, assume that

(a) $A^0$ and $A^1$ are connected and contained in $B(x, 3L_{k+1})$

(b) their diameters are at least $L_{k+1}/100$ and

(c) no open path inside $B(x, 3L_{k+1}^2)$ connects $A^0$ and $A^1$.

Then, for every path $\sigma$ in $B(x, L_{k+1}^2)$ connecting $A^0$ to $A^1$ there exists $y \in K$ such that

(i) $\sigma$ intersects $B(y, L_k)$ and

(ii) the event $S(y, L_k)$ holds.

See also Figure 1.

**Proof.** The proof of this lemma essentially follows the steps of the proof of Lemma 3.2 in [Tei14]. Therefore, we will not repeat the entire argument here. Instead, we just indicate what substitutions should be done to make that proof match exactly the context of the present article. First, replace each occurrence of $B(y, jL_k/6)$, for $j = 1, 2, 3$, by $B(y, jL_k)$. Then replace the balls $B(y, jL_k/6)$, for $j = 4, 5$ and 6 with $B(y, (j - 3)L_k^2)$.

The above lemma will allow us to reduce Condition 1 to the following simpler condition, which only concerns the geometry of $G$, not the realization of the percolation process.

**Condition 2.** We say that a graph $G$ satisfies Condition 2 if for any $J \geq 1$ there exists a constant $c_9 = c_9(G, J, \gamma)$ for which the following holds. Given $x \in V$, a scale $k \geq c_9$, connected sets $A^0, A^1 \subseteq B(x, 3L_{k+1})$ with diameters at least $L_{k+1}/100$ and any collection $y_1, \ldots, y_{J-1} \in B(x, 2L_{k+1}^2)$, there exists a path $\sigma$ contained in $B(x, L_{k+1}^2)$, connecting $A^0$ with $A^1$ while avoiding the set of balls $\bigcup_{j \leq J-1} B(y_j, 12L_k^2)$.

**Lemma 5.2.** Condition 2 implies Condition 1.

**Proof.** In order to establish Condition 1, we first fix $x \in V$, a set $K \subseteq B(x, 2L_{k+1}^2)$ such that $B(K, L_k)$ covers $B(x, L_{k+1}^2)$ and assume that the event $S(x, L_{k+1})$ holds. We now need to show that the events $S(y_j, L_k)$ occur for several points $y_1, \ldots, y_J$, which will be done using induction in $j = 1, \ldots, J$. 

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The occurrence of $S(x, L_{k+1})$ implies the existence of sets $A^0$ and $A^1$ in $B(x, 3L_{k+1})$ as in (4.1). To start the induction, we use the fact that $B(x, 3L_{k+1})$ is connected to obtain a path between $A^0$ and $A^1$ and employing Lemma 5.1 we obtain the point $y_1 \in K$ satisfying $S(y_1, L_k)$. Then, supposing that we have already found the sequence $y_1, \ldots, y_J \in K$ for $J' < J$ as above, we use Condition 2 to obtain a path from $A^0$ to $A^1$ that avoids $\cup_{j \leq J'} B(y_j, 12L_k^2)$. Therefore we can use Lemma 5.1 again in order to obtain a new vertex $y_{J'+1} \in K$ within distance at least $9L_k^2$ from all the previous $y_1, \ldots, y_J$ and for which $S(y_{J'+1}, L_k)$ holds. We can now continue inductively until we get Condition 1. \hfill \Box

5.2 Finding disjoint paths

The next lemma uses the Max-Flow-Min-Cut Theorem to show that we can find several disjoint paths connecting a large set $A \subset B(x, 3L_{k+1})$ to the (internal) boundary of the ball $B(x, 2L_{k+1}^2)$. By possibly trimming some of these paths, we are able to find one that avoids several balls in the previous scale.

This lemma carries some similarities with Condition 2, however the path that one obtains is not connecting $A^0$ to $A^1$, but rather $A^0$ to far away. This difference is in the heart of distinction between the isoperimetric condition (1.3) and the local isoperimetric inequality of [Tel14].

**Lemma 5.3.** Suppose that a given graph $G = (V, E)$ satisfies (1.3) and let $v \geq 1$ be fixed. Then there is a constant $c_{10} := c_{10}(J, \gamma, v, c_i, d_i, c_o, d_o)$ such that the following holds. Fixed any $x \in V$, $k \geq c_{10}$, a collection $z_1, \ldots, z_n \in B(x, L_k^2)$ with $n \leq J \log^2(L_k)$ and any set $A \subseteq B(x, 2L_{k+1}^2)$ with $|A| \geq L_{k+1}/(100v)$, we can find a path from $A$ to $\partial_i B(x, 2L_{k+1}^2)$ that does not touch the union $\bigcup_{i \leq n} B(z_i, 20c_oL_k^2)$.

For every finite connected set $A$ define

$$N(A) := c_i |A|^{d_i-1} \delta_i^{-1},$$

with $c_i > 0$. Note the resemblance with (1.3).

**Proof.** We start by showing that when $k$ is large enough, there are at least $N(A)$ disjoint paths connecting $A$ to $\partial_i B(x, 2L_{k+1}^2)$. In fact, suppose by contradiction that this is not verified. Then, by the Max-Flow Min-Cut Theorem, there exists a set of edges $C_A$ inside the ball $B(x, 2L_{k+1}^2)$ which disconnects $A$ from $\partial_i B(x, 2L_{k+1}^2)$ and such that $|C_A| < N(A)$. Then we have

$$|C_A| < N(A) = c_i |A|^{d_i-1} \delta_i^{-1}.$$

But then, this implies that there is a set $\tilde{A}$ (containing $A$) of points that can be reached from $A$ without using edges in $C_A$ that has to satisfy

$$|\partial \tilde{A}| = |C_A| < c_i |A|^{d_i-1} \delta_i^{-1} \leq c_j |\tilde{A}|^{d_j-1} \delta_j^{-1},$$

contradicting condition (1.3) and hence proving the first step.
Now we use this fact in order to a path that satisfies the statement of the lemma. In fact, using the above we have

\[
\text{# disjoint paths from } A \text{ to } \partial_i B(x, 2L_{k+1}^2) \geq N(A) = c_i |A| \frac{d_i-1}{4k} \\
\geq c_i \left( \frac{L_{k+1}}{100v} \right)^{d_i-1} \geq c L_k \gamma \frac{d_i-1}{4k} 
\]

(4.2), \(k \) large

\[
> 3J \log^2 (L_k) c u (20 c_0 L_k) 2d_a \\
\geq 3J \log^2 (L_k) |B(x, 20 c_0 L_k)|. 
\]

This bound shows that if we remove all those paths connecting \( A \) to \( \partial_i B(x, 2L_{k+1}^2) \) which happen to intersect \( \bigcup \) \( i \leq m \) \( B(z_i, 20 c_0 L_k^2) \), we are still left with several paths.

4.3 Embedding a tree into \( G \)

In this section we will assume that Condition 2 fails, since we have already proved Theorem 1.11 assuming Condition 1, which follows from Condition 2.

Negating Condition 2 is equivalent to saying that there exist

some number \( J \geq 1 \), (5.4)

a sequence of points \( x_l \in V \), (5.5)

a diverging sequence \( k_l \rightarrow \infty \) of scales, (5.6)

sets \( A_0^l, A_1^l \subseteq B(x_l, 3L_{k+1}) \) satisfying \( d(A_0^l, A_1^l) > 1 \), \( \text{diam}(A_i^l) \geq L_{k+1}/100 \) (5.7)

and for each \( l \geq 1 \) a collection \( y_1^l, \ldots, y_{J-1}^l \in B(x_l, 2L_{k+1}^2) \) (5.8)

such that

every path in \( B(x_l, L_{k+1}^2) \) connecting \( A_0^l \) to \( A_1^l \)

touches the set \( \bigcup_{j \leq J-1} B(y_j^l, 12L_{k+1}^2) \). (5.9)

We are now in position to start embedding a binary tree inside \( G \), which will ultimately lead to a contradiction on the polynomial growth that we assumed on \( G \). The nodes of this tree will simply be vertices of \( G \), however two adjacent vertices in the tree will not be mapped to neighbors in \( G \). Instead, they will be mapped into reasonably far apart points as we describe in detail soon.

The nodes of our binary tree are indexed by words in the alphabet \{0, 1\}. For every such a word \( \omega \in \Gamma \), we denote by \( |\omega| \) its length and by \( \omega' \omega \) the word obtaining by appending \( \omega \) to the right of \( \omega' \). In this case, we say that \( \omega' \) is a prefix of \( \omega' \omega \). This prefix is said to be proper if \( \omega \) is non-empty.

We denote the bad set

\[
H_l = \bigcup_{j \leq J-1} B(y_j^l, 20 c_0 L_{k+1}^2). 
\]

Note that the balls used to define \( H_l \) have radius \( 20 c_0 L_{k+1}^2 \), which is larger than the ones appearing in (5.9). This difference will be important later once we start playing with rough isomorphisms, since we want paths avoiding \( H_l \) to be mapped to paths that do not touch \( \bigcup_{j \leq J-1} B(y_j^l, 12L_{k+1}^2) \).
**Remark 5.4.** In the next lemma, given some \( l \geq 1 \) and any word \( \omega \in \Gamma \) such that \(|\omega| \leq \log^2(L_{k_1})\), we will construct a \( c_0 \)-rough isometry \( \phi_{\omega} \) of \( G \). Given such a map, we can define

\[
\begin{align*}
\phi_{\omega}(x_l) &:= \phi_{\omega}(x_l), \\
A_i(\omega) &:= \phi_{\omega}(A_i), \quad i = 0, 1, \\
B_l(\omega) &:= \mathcal{B}(x_l(\omega), 3L_{k_1} + 1) \quad \text{and} \\
H_l(\omega) &:= \phi_{\omega}(H_l).
\end{align*}
\]  

(5.11)

Note that \( \phi_{\omega} \) will be the identity map on \( G \).

Therefore, we can think of \( x_l, A_i^0 \) and \( A_i^1 \) as \( x_l(\emptyset) \), \( A_i^0(\emptyset) \) and \( A_i^1(\emptyset) \) respectively. In the same way, we have that \( y_j^l = y_j^l(\emptyset) \) for all \( j \) and \( l \) as above.

The next lemma constructs an embedding of a binary tree into \( G \) satisfying a list of requirements. Later we will use this together with (5.9) to show that all leafs of the constructed tree have to be disjoint, contradicting the polynomial growth of the graph \( G \), see Lemma 5.6.

**Lemma 5.5.** Let \( G \) be a \( c_0 \)-roughly transitive graph satisfying (1.3) and assume the existence of a collection \((J, (x_l), (k_l)), (A_i^0), (A_i^1), (y_j^l)_{l \geq 1}\) as in (5.4)–(5.8). Then, there exists \( c_{11} = c_{11}(J, \gamma, \gamma, c_1, d_i, d_n, c_{10}) \) such that for \( k_1 \geq c_{11} \) and for each \( \omega \) such that \( 1 \leq |\omega| \leq \log^2(L_{k_1}) \), we can construct

1. a \( c_0 \)-rough isometry \( (\phi_{\omega}) \) and

2. a path \( \gamma_{\omega} \),

in such a way that the following holds. For \( \omega \) such that \(|\omega| \leq \log^2(L_{k_1}) - 1\),

\[
\text{if } \omega = \omega^i \text{ for } i = 0, 1, \text{ then } \gamma_{\omega} \subseteq \mathcal{B}(x_l(\omega), L_{k_1}^{3/2}),
\]

(5.12)

for any \( \omega \) proper prefix of \( \omega \), \( B_l(\omega) \) is disjoint from \( H_l(\omega') \),

(5.13)

if \( \omega = \omega^i \) with \( i = 0, 1 \), then \( \gamma_{\omega} \) connects \( A_i^l(\omega') \) to \( x_l(\omega) \) and

(5.14)

if \( \omega \) is a proper prefix of \( \omega \), then the path \( \gamma_{\omega} \) is disjoint from \( H_l(\omega') \).

(5.15)

See Figure 2 for an illustration of the above.

**Proof.** We first choose the constant \( c_{11}(c_1, d_i, c_n, d_n) \geq c_{10} \) in such a way that for \( k_1 \geq c_{11} \),

\[
\log^2(L_{k_1}) > J
\]

(5.16)

\[
2(J - 1)(3L_{k_1} + 1 + 20c_0 L_{k_1}^2) \log^2 L_{k_1} < L_{k_1}^{3/2} - 3L_{k_1 + 1}
\]

(5.17)

which can be done by our choice of the scales \( L_k \) in (4.2). Since we are assuming (1.3) and that \( c_{11} \geq c_{10} \), the conclusion of Lemma 5.3 is at our disposal (at each scale \( k_1 \geq c_{11} \)).

In order to construct the maps \( \phi_{\omega} \), we follow an induction argument on the length of the word \( \omega \). The only word of length zero is \( \emptyset \) and we have already defined \( \phi_{\emptyset} \) as the identity map. Assume that for \( n < \log^2(L_{k_1}) - 1 \) we have already constructed the maps \( (\phi_{\omega})_{|\omega| \leq n} \) and paths \( (\gamma_{\omega})_{1 \leq |\omega| \leq n} \), satisfying (5.13)–(5.15). Then, given any word \( \omega \) with \(|\omega| = n \), our task now is to define \( \phi_{\omega} \) and \( \gamma_{\omega} \) with help of Lemma 5.3.

To apply Lemma 5.3, we need to choose the points \( z_1, \ldots, z_m \) to be avoided, which roughly speaking will correspond to the points \( \{y_j^l(\omega')\} \), for each \( \omega' \) prefix of \( \omega \). More precisely, we denote by \( \omega_k \) the unique prefix of \( \omega \) with \(|\omega_k| = k\) and set

\[
z_k(j - 1) + j - 1 = y_j^l(\omega_k), \quad \text{with } k = 0, \ldots, |\omega| \text{ and } j = 1, \ldots, J - 1.
\]

(5.18)
Recall that $|\omega| \leq \log^2(L_k)$, so that the number of $z_i$'s is no larger than $2J \log^2(L_{k_i})$. Using (2.3), we conclude that

$$|A_i^0(\omega)| \geq \frac{|A_i^0|}{c_u(c_0)^d_u} = \frac{L_{k_i+1}}{100v},$$

where $v = c_u(c_0)^d_u$ (cf. Remark 2.1). The same also being true for $A_i^1(\omega)$. We are now in position to apply Lemma 5.3, which provides us with paths $\gamma_{\omega,0}$ from $A_i^0(\omega)$ to $\partial_iB(x_i(\omega), 2L_{k_i+1}^2)$ and $\gamma_{\omega,1}$ from $A_i^1(\omega)$ to $\partial_iB(x_i(\omega), 2L_{k_i+1}^2)$ that satisfy (5.15). More precisely, these paths are such that

$$\gamma_{\omega,0} \text{ and } \gamma_{\omega,1} \text{ do not touch the union of the balls } B(z_i, 12L_{k_i}^2), \text{ for any } i. \quad (5.19)$$

These paths will give rise to the two children of $\omega$ ($\omega_0$ and $\omega_1$). Recall that these paths go quite far, reaching distance $2L_{k_i+1}^2$ from $x_i(\omega)$, however we are going to truncate these paths earlier in such a way that (5.12) holds and moreover

the end points of the paths $\gamma_{\omega,0}$ and $\gamma_{\omega,1}$ lie within distance at least $3L_{k_i+1}$ from $H_l(\omega')$ for any $\omega'$ prefix of $\omega$. \quad (5.20)

Before proving the above, let us briefly see why this would finish the proof of the lemma. We call these end-points $x_i(\omega_0)$ and $x_i(\omega_1)$ respectively and using the rough transitivity of the graph $G$ we can find two $c_0$-rough isometries, satisfying $\phi_{\omega,0}(x_i(\omega_0)) = x_i(\omega_0)$ and $\phi_{\omega,1}(x_i(\omega)) = x_i(\omega_1)$. We can now define $A_i^1, B_i$ and $H_i$ as in (5.11), obtaining another layer
of the tree. The fact that these satisfy (5.13)–(5.15) is a consequence of their construction, (5.19) and (5.20).

We still need to prove that we can stop the paths \( \gamma_0 \) and \( \gamma_1 \) in such a way that they satisfy (5.12) and (5.20). First observe that a point \( x \) being within distance at least \( 3L_{k_1+1} \) from the sets \( H_1(\omega') \) (for \( \omega' \) prefix of \( \omega \)) is equivalent to \( x \) being within distance \( 3L_{k_1+1} + 20c_9L_{k_1}^2 \) from the collection of points \( K = \{y_j(\omega') : \omega' \text{ prefix of } \omega \text{ and } j \leq J - 1 \} \). We first stop these paths as soon as they reach \( L_{k_1+1}^{3/2} \) (recall that they reach \( \partial B(x_l(\omega), 2L_{k_1+1}^{2}) \)) therefore \( \gamma_0 \) and \( \gamma_1 \) will automatically satisfy (5.12).

Even after this truncation, the range of these paths still have diameter at least \( 3L_{k_1+1} - 3L_{k_1+1} \). Therefore, by (5.17) they cannot be covered by \((J-1)\log^2 L_k\) balls of radius \( 3L_{k_1+1} + 20c_9L_{k_1}^2 \). This proves that we can stop the paths \( \gamma_0 \) and \( \gamma_1 \) in a way that their endpoints satisfy (5.20), finishing the proof of the lemma.

In order to conclude the proof of the Cascading Lemma we will show that under the current assumptions all the points \( (x_l(\omega))_{|\omega| = \lceil \log^2(L_k) \rceil} \) are disjoint, contradicting the polynomial growth that we have assumed on the graph \( G \).

**Lemma 5.6.** There exists a constant \( c_{12} = c_{12}(\gamma, c_9) \) such that for all \( k_1 \geq c_{12} \) we have the following. Let \( n_l = \lceil \log^2(L_k) \rceil \) and fix the construction of the \( c_9 \)-rough isometries \( \phi^j_{\omega} \), for \( |\omega| \leq n_l \) as in Lemma 5.5. Then for all pair of words \( \omega, \omega' \) such that \( |\omega| = |\omega'| = n_l \), the points \( x_l(\omega) \) and \( x_l(\omega') \) are distinct.

**Proof.** Suppose that there are two words \( \omega \) and \( \omega' \), both of length \( n_l \), for which

\[ x_l(\omega) = x_l(\omega') \]  

(5.21)

and let \( \hat{\omega} \) be their closest common ancestor (in other words, \( \hat{\omega} \) is the longest common prefix of \( \omega \) and \( \omega' \)).

We first fix \( c_{12} \) large enough so that for \( k \geq c_{12} \), one has

\[ 8c_9L_{k_1}^2 > 4c_9^2(4c_9^2 + c_9 + 1). \]  

(5.22)

This specific choice will become clear later.

Our aim is to build a path between \( A^j_l \) and \( A^j_l \), which is contained in \( B(x_l, L_{k_1+1}^2) \) and avoids the set \( \bigcup_j \bigcup_{j \leq J-1} B(y_j, L_{k_1}^2) \). This will lead to a contradiction to (5.9), which we have obtained from negating Condition 2.

As a first step, we will construct a path \( \sigma \) such that

\[ \sigma \text{ is contained in } B(x_l, n_lL_{k_1+1}^{3/2}), \text{ connects } A^0_l(\hat{\omega}) \text{ to } A^1_l(\hat{\omega}), \text{ and avoids the set } H_l(\hat{\omega}). \]  

(5.23)

Then we will use the rough inverse of \( \hat{\phi}^j_{\omega} \) to map \( \sigma \) to the desired path.

Before building \( \sigma \), we first construct a path from \( A^0_l(\hat{\omega}) \) to \( x_\omega \). In order to do this, we first write \( \omega_0, \ldots, \omega_n \) to be the sequence of prefixes of \( \omega \), obtained by setting \( \omega_0 = \hat{\omega} \) and adding one letter at a time until \( \omega_n = \omega \).

We start by observing that \( A^0_l(\hat{\omega}) \) can be connected to \( x_l(\omega_1) \) by the path \( \gamma_\omega \) which avoids \( H_l(\hat{\omega}) \) by (5.14) and (5.15). Supposing by induction that we have already reached \( x_l(\omega_j) \) for some \( j < n \) by a path that avoids \( H_l(\hat{\omega}) \), we are now going to extend this path until \( x_l(\omega_{j+1}) \). We know by (5.14) and (5.15) that if \( \omega_{j+1} = \omega_{j+i} \) (\( i = 0, 1 \)), then the path
\( \gamma_{\omega_{j+1}} \) connects \( A_1^j(\omega_j) \) to \( x_l(\omega_{j+1}) \) while avoiding \( H_l(\hat{\omega}) \), therefore this is a good candidate for the extension we need.

The obstacle to perform this extension comes from the fact that this path does not necessarily start at \( x_l(\omega_j) \), in fact its starting point \( \gamma_{\omega_{j+1}}(0) \) is somewhere in \( A_1^j(\omega_j) \subseteq B(x_l(\omega_j), 3L_{k_{j+1}}) \), see Figure 2. But using the fact that this ball is connected and disjoint from \( H_l(\hat{\omega}) \) (by (5.13)), we can connect \( x_l(\omega_j) \) to \( \gamma_{\omega_{j+1}}(0) \) and finally to \( x_l(\omega_{j+1}) \).

Proceeding with this induction, we can construct the required path from \( A_1^0(\hat{\omega}) \) to \( x_l(\omega) \) which avoids \( H_l(\hat{\omega}) \). We can also build a similar path from \( A_1^j(\hat{\omega}) \) to \( x_j \) and by concatenating these two we have proved (5.23).

We now use the path \( \sigma \) obtained in (5.23) to derive a contradiction to (5.9), finishing the proof of the lemma. For this, pick a \( 4c_0^2 \)-rough isometry \( \psi \) which is the rough inverse of \( \phi_{\omega}^j \) as in (2.4). We now consider the image of the path \( \sigma \) under the map \( \psi \), obtaining a sequence of vertices \( x_1, \ldots, x_M \), for some suitable \( M \geq 1 \).

This sequence does not necessarily constitute a path, however, by (1.1) we have

\[
d(x_m, x_{m+1}) \leq 4c_0^2 \text{ for every } m = 1, \ldots, M - 1.
\]

Recall that the path \( \sigma \) connects \( A_1^0(\omega) \) to \( A_1^1(\omega) \), which are images of \( A_1^0(\omega) \) and \( A_1^1(\omega) \) under \( \phi_{\omega}^j \). Therefore, the point \( x_1 \) (which is the image of the first point of \( \sigma \)) is within distance at most \( 4c_0^2 \) from \( A_1^0(\omega) \) (and similarly for \( x_M \) and \( A_1^1(\omega) \)). So we can add points \( x_0 \in A_1^0 \) and \( x_{M+1} \in A_1^1 \) to the sequence, without violating (5.24).

We now use (5.24) and the above property of \( x_0 \) and \( x_{M+1} \) to turn the sequence \( (x_m)_{m=0}^{M+1} \) into a path by connecting \( x_m \) to \( x_{m+1} \), one by one, while using no more than \( 4c_0^2 \) intermediate points to join each pair. This gives rise to a path \( \sigma' \) for which we need to verify:

a) \( \sigma' \) connects \( A_1^0 \) to \( A_1^1 \),

b) \( \sigma' \) is contained in \( B(x_l, c_0 n_l L_{k_{j+1}}^{2/3}) \subseteq B(x_l, L_{k_{j+1}}^2) \),

c) \( \sigma' \) does not intersect the set \( \bigcup_{j \leq J-1} B(y_j^l, 12L_{k_j}^2) \).

In fact, a) is a consequence of the construction of the path. The statement b) follows since \( \psi \) is a \( 4c_0^2 \)-rough isometry. Finally, to show c), we fix \( y_j^l \) and \( x \in B(y_j^l, 12L_{k_j}^2) \) and, observing that

\[
\phi_{\omega}^j(x) \in B(y_j^l(\hat{\omega}), 12c_0 L_{k_j}^2)
\]
we estimate

\[
d(\sigma', x) \geq d(\sigma', \psi(\phi_{\omega}^j(x))) - d(\psi(\phi_{\omega}^j(x)), x) \geq d(\psi(\sigma), \psi(\phi_{\omega}^j(x))) - 4c_0^2 - c_0
\]
\[
\geq \frac{1}{4c_0^2} d(\sigma, \phi_{\omega}^j(x)) - 1 - 4c_0^2 - c_0 \overset{(5.25)}{\geq} \frac{1}{4c_0^2} (20 - 12) c_0 L_{k_j}^2 - 4c_0^2 - c_0 - 1 > 0.
\]

This finishes the proof that \( \sigma' \) indeed contradicts (5.9), yielding the lemma.

It is now very easy to finish the proof of the Cascading Lemma.

**Proof of the Cascading Lemma.** Supposing that \( G \) does not satisfy Condition 1, we know by Lemma 5.2 that it does not satisfy Condition 2 either. This provides us with a sequence \( (J, (k_j), (A_1^j), (A_1), (y_j^l)) \) satisfying (5.4)–(5.8). Employing Lemma 5.5, we can construct for each \( l \geq 1 \) the rough isometry \( \phi_{\omega}^j \), for \(|\omega| \leq n_l := |\log^2(L_{k_j})| \) satisfying (5.12)–(5.15).
Lemma 5.6 now claims that the points \((x_l(\omega))_{|\omega| \leq n_l}\) obtained in the above construction are disjoint. However, there are \(2^{n_l}\) such points and by (5.12) they are all contained in the ball \(B(o, n_l L_{k_l+1}^{1/2})\). This contradicts the polynomial growth of \(G\) assumed in (1.4), finishing the proof of the Cascading Lemma.

6 Proof of Lemma 4.4

To conclude the proof of Theorem 1.11 we still need to show that Lemma 4.4 holds. The main ideas of the proof are taken from [Tei14, Lemma 4.1], which we report here for seek of clarity. We split the proof into several auxiliary results, in order to make it more clear.

Choose \(\beta > \max\{2\gamma d_u - d_l, \gamma (2 + \chi)\}\) for some arbitrary value \(\chi > 0\) and suppose that \(G\) is a roughly transitive graph satisfying \(V(c_u, d_u)\) and \(I(c_i, d_i)\) with \(d_i > 1\). Then we are in the condition to apply Lemma 4.3, obtaining that \(p_k \leq L_k^{-\beta}\) for all \(k \geq 1\).

Start by fixing a path \(\sigma : \mathbb{N} \to V\) that satisfies the following property:

\[d(\sigma(i), \sigma(j)) = |i - j|, \text{ for all } i, j \in \mathbb{N}.\]

In particular, \(d(x, y)\) denotes the graph distance between vertices \(x\) and \(y\). The existence of such paths will not be discussed here, but the interested reader is referred to [Wat86]. (More precisely, this result holds whenever the graph \(G\) is infinite, locally finite, simple and connected.) Now, given \(\sigma\), define the following collection of points:

\[x_{k,i} := \sigma(i L_k^2/10), \text{ for } k \geq 1 \text{ and } i = 0, \ldots, L_{k+1}^2/L_k^2,\]

and, for some fixed \(k_0 \geq 1\) define the following event:

\[\mathcal{G}_0 := \bigcap_{k \geq k_0} \bigcap_{i=1}^{L_{k+1}^2/L_k^2} S(x_{k,i}, L_k)^c.\]

The next claim shows that \(\mathcal{G}_0\) occurs with high probability.

Claim 6.1. The event \(\mathcal{G}_0\) occurs with probability bounded from below by \(1 - c L_{k_0}^{-\gamma \chi}\), where \(c > 0\) is a constant, \(\gamma\) is as in (4.2) and \(\chi > 0\) is as in the statement of Lemma 4.4.

Proof. We show that \(\mathbb{P}(\mathcal{G}_0) \leq c L_{k_0}^{-\gamma \chi}\). In fact,

\[
\mathbb{P}(\mathcal{G}_0) = \mathbb{P}\left( \bigcup_{k \geq k_0} \bigcup_{i=1}^{L_{k+1}^2/L_k^2} S(x_{k,i}, L_k) \right) \leq \sum_{k \geq k_0} \sum_{i=1}^{L_{k+1}^2/L_k^2} p_k \leq \sum_{k \geq k_0} L_k^2 \gamma^{-2} L_k^{-\beta}
\]

\[
\leq \sum_{k \geq k_0} L_k^{-\chi \gamma} = L_{k_0}^{-\chi \gamma} \sum_{k \geq 0} (L_0^{-\chi \gamma})^{\gamma k_0 (\gamma^k - 1)}.\]

Now, since \(\gamma^{k_0} > 1\) and \(L_0^{-\chi \gamma} < 1\), the sum \(\sum_{k \geq 0} (L_0^{-\chi \gamma})^{\gamma k_0 (\gamma^k - 1)}\) converges, leading to the claim.

The next auxiliary result shows that conditional on \(\mathcal{G}_0\) being realized, we can find several open paths which can be connected, discovering an infinite (open) connected component.
Lemma 6.2. Condition on $G_0$ being realized. Then for all $k \geq k_0$ there is an infinite connected component that intersects the ball $B(x_{k,0}, L^2_k/100)$.

Proof. Fix $k \geq k_0$, the argument proceeds inductively on $k$. For all $i = 0, \ldots, L^2_{k+1}/L^2_k - 1$, we use the fact that by hypothesis (i.e., conditioning on $G_0$) we have that $S(x_{k,i}, L_k)^c$ is realized. Hence, we can find an open path $\sigma_{k,i} \subseteq B(x_{k,i}, 3L^2_{k+1}/2)$ that connects the ball $B(x_{k,i}, L^2_k/100)$ to $B(x_{k,i+1}, L^2_k/100)$.

The next step consists in joining all such open paths $\sigma_{k,i}$’s into one (open) connected component. Therefore, we first estimate the diameters of such paths:

$$\text{diam}(\sigma_{k,i}) \geq d(B(x_{k,i}, L^2_k/100), B(x_{k,i+1}, L^2_k/100)) \geq d(x_{k,i}, x_{k,i+1}) - 2L^2_k/100 \geq L^2_k/50.$$  

The last inequality follows from our choice of $\sigma$, in fact $d(x_{k,i}, x_{k,i+1}) \geq L^2_k/10$. Such bound implies that before exiting the ball $B(x_{k,i}, L^2_k)$, the path $\sigma_{k,i}$ has diameter at least $L^2_k/100$. At this point, since we are under the assumption that $G_0$ holds, we can find again open paths $\gamma_{k,i}$ that join $\sigma_{k,i}$ with $\sigma_{k,i+1}$ (for all $i = 0, \ldots, L^2_{k+1}/L^2_k - 2$) that are contained inside the ball $B(x_{k,i}, 3L^2_{k+1})$.

Our next step is to join the $\sigma_{k,i}$’s and the $\gamma_{k,i}$’s in order to obtain longer open paths. Note that the paths $\gamma_{k,i}$ are necessary to avoid any issue coming from the fact that the balls $B(x_{k,i}, L^2_k/100)$ are not necessarily open. In order to join such open paths, we define a new open path $\sigma_k$ that goes through $\sigma_{k,i}$ and $\gamma_{k,i}$, for all values of $i = 0, \ldots, L^2_{k+1}/L^2_k - 1$.

Now, by construction, we have

$$\text{diam}(\sigma_k) \geq d(x_{k,0}, x_{k,L^2_{k+1}/L^2_k-1}) - L^2_k/100 \geq \left(\frac{L^2_{k+1}}{L^2_k} - 1\right) \left(\frac{L^2_k}{10}\right) - \frac{L^2_k}{100} \geq \frac{L^2_{k+1}}{100}.$$  

Now observe that since $G_0$ is realized, for all $k \geq k_0$ the paths $\sigma_k$ and $\sigma_{k+1}$ must be on the same (open) connected component. In fact, since we are assuming $S(x_{k,0}, L_{k+1})^c$, before $\sigma_k$ and $\sigma_{k+1}$ can find “a way out” from the ball $B(x_{k,0}, L_{k+1})$, they will have already gained a diameter of at least $L^2_{k+1}/100$.

Proving inductively on $k \geq k_0$ we get the statement. \qed

The next result gives a sufficient condition that will imply Lemma 4.4.

Claim 6.3. Assuming that $G_0$ is realized, there exists a unique infinite (open) connected component $C_\infty$. Moreover, denoting by $C_x$ the connected component containing any vertex $x \in V$, either $C_x \equiv C_\infty$, or $\text{diam}(C_x) \leq L^2_{k_0}$.

Proof. First of all, observe that the infinite cluster has to be unique due to $G_0$, since the existence of two or more infinite components would imply that $S(x_{k,0}, L_k)$ holds for all but finitely many $k$’s.

Furthermore, the fact that either $C_x \equiv C_\infty$ or $\text{diam}(C_x) \leq L^2_{k_0}$ is a consequence of the following observation. If $\text{diam}(C_x) > L^2_{k_0}$, but $C_x \neq C_\infty$, we would find two separated components intersecting the ball $B(x_{k,0}, L^2_{k_0}/100)$, for some $k_0 \geq 1$. But this fact would contradict Lemma 6.2, hence the statement is proven. \qed

Finally we have everything in place to prove Lemma 4.4.

Proof of Lemma 4.4. By putting together Claims 6.1 and 6.3 we obtain the first half of the Lemma. More precisely, we obtain that if $p_* < p < 1$ (with $p_*$ found in Lemma 4.3), and hence $p_k \leq L^{-\beta}_{k}$ for all $k \geq 1$, then there is a unique infinite cluster $C_\infty$ almost surely.

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Regarding the second part, we observe that for every value \( k_0 \) large enough we have

\[
\mathbb{P}(L_{k_0}^2 < \text{diam}(C_x) < \infty) \leq cL_{k_0}^{-\gamma x} < cL_{k_0}^{-\chi}.
\]

This concludes the proof of Lemma 4.4. 

\[\square\]

7 Examples

This section is devoted to giving some examples of dependent percolation processes for which our results apply. These examples include loop soups, germ-grain models and divide and color percolation.

7.1 Loop soups

The model of loop soups was informally introduced by Symanzik in [Sym69] and was rigorously defined in [LW04] in the context of Brownian loops. The model has been intensively studied, see for example [LJ12] and [LJL13], displaying some very interesting percolation features, see [CS14].

To properly define this model, we start by introducing a space of closed loops on \( G \)

\[
W = \{(x_0, \ldots, x_{k-1}) \in V^k; k \geq 1, x_0 = x_{k-1} \text{ and } \{x_i, x_{i-1}\} \in E \text{ for all } i < k\}.
\]

We now fix a parameter \( \kappa > 0 \) and endow the countable space \( W \) with the measure

\[
\mu(w) = \frac{1}{k^k} \left(\frac{1}{\Delta(1+\kappa)}\right)^k, \tag{7.1}
\]

where \( k \) gives the length of the loop \( w \) and \( \Delta \) is the degree of any given vertex in \( G \).

We define an equivalence relation on \( W \), where we identify two loops (denoting this by \( w \sim w' \)) if they have the same path length \( k \) and \( w(i) = w'(i+j) \) for some \( j \geq 1 \), where the sum is taken on \( \mathbb{Z}/(k\mathbb{Z}) \).

Given the equivalence relation \( \sim \), we define the space of unmarked loops \( W^* \) as \( W/\sim \) and define the push forward \( \mu^* \) of \( \mu \) under the canonical projection from \( W \) to \( W^* \). The process we are interested in is a Poisson Point Process \( \omega^\beta \) on \( W^* \) with intensity \( \beta \mu^* \), where \( \beta > 0 \) is an parameter controlling the amount of loops that enter the picture.

We will be interested in both the occupied and vacant set left by the loop soup, or more precisely: \( L^\beta = \cup_{w \in \text{supp}(\omega^\beta)} \text{Range}(w) \) and \( V^\beta = V \setminus L^\beta \).

Let us state a decoupling inequality for this model, inspired by the (2.15) of [Szn10].

**Lemma 7.1.** Fix \( \beta, \kappa > 0 \). Then, for \( r \geq 1, J \geq 2, \) points \( x_1, x_2, \ldots, x_J \in V \) satisfying

\[
\min_{1 \leq i < j \leq J} d(x_i, x_j) \geq 3r
\]

and for events \( \mathcal{G}_1, \ldots, \mathcal{G}_J \) such that \( \mathcal{G}_i \in \sigma(Y_z, z \in B(x_i, r)) \) we have

\[
\mathbb{P}(\mathcal{G}_1 \cap \cdots \cap \mathcal{G}_J) - \mathbb{P}(\mathcal{G}_1) \cdots \mathbb{P}(\mathcal{G}_J) \leq 2\beta J \exp\{-c(\kappa)r\}. \tag{7.2}
\]
Proof. Let us first define the sets
\[ W_i = \{ w \in W; \text{Range}(w) \subseteq B(x_i, 3r/2) \}. \] 
We denote by \( \omega_i^\beta \) the Poisson point process \( \omega^\beta \) restricted to \( W_i \), for \( i = 1, \ldots, J \). Note that the \( \omega_i^\beta \)'s are independent, since their supports \( W_i \) sets are disjoint.

Writing \( G'_i \) for the event \( G_i \) evaluated for the trimmed point process \( \omega_i^\beta \), we can estimate
\[
\left| P(G_1 \cap \cdots \cap G_J) - P(G'_1 \cap \cdots \cap G'_J) \right| 
\leq \left| P(G_1 \cap \cdots \cap G_J) - P(G'_1 \cap \cdots \cap G'_J) \right| + \left| P(G'_1 \cap \cdots \cap G'_J) - P(G'_1 \cap \cdots \cap G'_J) \right| 
\leq 2J \sup_{i} \mathbb{P} [ G_i \Delta G'_i ] \leq 2J \sup_{i} \mathbb{P} \left[ \text{there is } w \in \text{supp}(\omega^\beta) \text{ intersecting both } B(x_i, r) \text{ and } B(x_i, 3r/2) \right].
\] 
(7.4)

In order to bound the last term in the above equation we make use of the definition of the intensity measure in (7.1), finishing the proof of the lemma.

We are now in position to state the first application of our main result.

**Theorem 7.2.** Given a \( c_0 \)-roughly transitive graph satisfying \( \mathcal{V}(c_u, d_u) \) and \( \mathcal{I}(c_i, d_i) \) for some \( d_i > 1 \) and fix \( \kappa > 0 \), define the Poisson Point Processes on \( G \) as above. Then,

a) For \( \beta > 0 \) small enough, almost surely the set \( \mathcal{L}^\beta \) contains no infinite connected component, while \( \mathcal{V}^\beta \) contains a unique one.

b) On the other hand, if \( \beta > 0 \) is large enough, almost surely there exists a unique infinite cluster in \( \mathcal{L}^\beta \), but \( \mathcal{V}^\beta \) is composed solely of finite components.

This result allows us to define two critical values corresponding to the appearance of infinite clusters in \( \mathcal{L}^\beta \) and \( \mathcal{V}^\beta \).

**Remark 7.3.** Note that the above arguments are more general than for Loop Soups only, in fact it should work for any *Germ-Grain* model. These models are defined as a decorated Poisson Point process, where each point gets associated with a random object to be inserted in the graph. Under conditions that the random objects have sufficiently light tails (for instance, exponentially bounded), then the above proof should work equally well for such models.

### 7.2 Divide and color

The divide and color model was introduced by H"aggstr"om in \cite{H"a01}, and it is a process that is governed by two parameters \( (p, q) \in [0, 1] \) and evolves in two steps. In this section we will follow the description in \cite{BBT13}, to which the reader is referred for more details and further results.

1. Firstly we perform a Bernoulli percolation on the edges of \( G \), i.e. each edge of the given graph is retained with probability \( p \), independently of each other. This partitions the vertices of \( G \) into clusters, corresponding to the connected components induced by open edges.
2. Secondly we color the resulting connected components either black or white with probability \( q \) or \( 1-q \) respectively, independently for distinct components. All vertices of a component take the same color, which induces dependence in this site percolation model.

The rest of this section is devoted to proving that the decoupling condition (2.8) holds true for this model under some conditions on the parameter \( p \). In order to do so, we need to introduce some further notation.

We start by defining a Bernoulli percolation by associating at each edge \( e \) a random variable \( \eta \in \{0,1\} \) that takes value \( \eta(e) = 1 \) with probability \( p \) and \( \eta(e) = 0 \) with probability \( 1-p \). Given such an assignment, the vertices of \( G \) can be split into clusters and we associate a random variable \( \xi \in \{0,1\} \) to each connected component \( C \) determined above. The variables \( \xi(C) \) are i.i.d. and satisfy \( P(\xi = \text{black}) = q \) and \( P(\xi = \text{white}) = 1-q \).

Finally, we re-open all edges (essentially forgetting the variables \( \eta(e) \)) and we ask ourselves whether there exists an infinite cluster of black sites in the above coloring.

Let \( \mu_{p,q} \) denote the measure governing the site-percolation process as described above. Then [Häg01, Proposition 2.5] assures that for any graph \( G \) and any \( p \in [0,1] \) there exists a critical value \( q^*_G(p) \in [0,1] \) such that

\[
\mu_{p,q}(\text{there exists an infinite black } q\text{-cluster}) = \begin{cases} 
0 & \text{if } q < q^*_G(p), \\
> 0 & \text{if } q > q^*_G(p).
\end{cases}
\]

It is clear that if \( p > p_c(G) \), or in other words if the first stage of the process can lead to an infinite cluster, then \( q^*_G(p) = 0 \), since for every positive \( q \) there is a chance that the cluster containing the origin is infinite and is painted black. Therefore, one can focus on the subcritical and critical phases \( p \leq p_c(G) \).

On the subcritical phase, there is a strong belief that the size of a typical cluster should have exponential tails. To make this more precise, let us define the critical value for “strongly subcriticality”.

\[
p_\ast = \sup \{ p \in [0,1]; \text{ for some } \theta > 0, \ P_p[\text{diam}(C_o) \geq n] \leq \exp\{\theta n\}\}.
\]

(7.5)

It is clear that \( p_\ast \) is smaller than \( p_c \) and it is commonly believed that \( p_c = p_\ast \) for a large variety of graphs. This equality has been proved for the \( d \)-dimensional lattice in [AB87] and [Men86] and later extended to transitive graphs in [AV08] and [TDC].

Another important observation is that for any graph \( G \) with degrees bounded by \( \Delta \), we have \( p_\ast \geq 1/\Delta \), as one can easily prove by a counting path argument.

Intuitively speaking, once \( p < p_\ast \), then the clusters are small and the dependence of the divide and color model should be short-ranged. This is made precise in the following proposition.

**Proposition 7.4.** Fix a graph \( G \) and let \( p < p_\ast(G) \). Then, for any \( \alpha > 0 \) there exists a constant \( c_\alpha = c_\alpha(p,\alpha,J) \) for which the condition (2.8) holds for the divide and color model on \( G \) for any \( q \in [0,1] \).

As a consequence, if \( G \) is roughly transitive, has polynomial growth and isoperimetric dimension larger than one, then \( q^*_G(p) < 1 \).

**Proof.** Given \( x_1, \ldots, x_J \in V \) and \( r \geq 1 \), we are going to construct a simple decoupling of what happens in the various regions \( B(x_j,r) \). For this, we define \( J+1 \) independent
percolation measures on $G$. More precisely, let $(\eta_j(e))_{e \in E}$ be independent Bernoulli variables (all of them i.i.d. with parameter $p$), for $j = 0, 1, \ldots, J$. We also define a mixed configuration $\eta_{\text{mix}}$ which is given by

$$
\eta_{\text{mix}}(e) = \begin{cases} 
\eta_j(e), & \text{if } e \in B(x_j, 3r/2) \text{ and } \\
\eta_0(e), & \text{otherwise.}
\end{cases}
$$

(7.6)

We now use the above configurations to construct $J + 1$ instances of the divide and color model, which will be denoted by $(Y^j_x)_{x \in V}$, $j = 1, \ldots, J$ and $j = \text{mix}$. Obviously, they use the clusters determined by their respective edge configuration $\eta_j$ defined above. Moreover, we add the restriction that if a given cluster of $\eta_j$ is contained in $B(x_j, 3r/2)$ (in which case it coincides with that of $\eta_{\text{mix}}$), then both $Y^j$ and $Y^\text{mix}$ will assign the same color to this cluster during the coloring stage.

Note that $Y^\text{mix}$ has the correct law of the model, and we are now in position to prove that it satisfies (2.8). For this, fix events $\mathcal{G}_1, \ldots, \mathcal{G}_J$ as in Proposition 2.5 and estimate

$$
\mathbb{P}(\mathcal{G}_1(Y^\text{mix}) \cap \cdots \cap \mathcal{G}_J(Y^\text{mix})) = \mathbb{P}(\mathcal{G}_1(Y^1)) \cdots \mathbb{P}(\mathcal{G}_J(Y^J))
\leq \mathbb{P}(Y^\text{mix}_x \neq Y^j_x, \text{ for some } j \leq J, x \in B(x_j, r))
\leq \mathbb{P}(\text{for some } j, \text{ an open path in } \eta_j \text{ connects } B(x_j, r) \text{ to } B(x_j, 3r/2))
\leq \exp\{-\theta r\}.
$$

This finishes the proof of the proposition by properly choosing the constant $c_\alpha$. \hfill \Box

### 7.3 Slow decay of dependence

Let us briefly comment on the decay of correlation that we have assumed on the law $\mathbb{P}$. It has been proved in [BLPS99], Theorem 1.1 that if $G$ is an amenable Cayley graph, then for any $p < 1$ there exists some invariant percolation law $\mathbb{P}^*$ on $G$ such that $\mathbb{P}^*[Y_o = 1] > p$ but the set $\{x; Y_x = 1\}$ does not percolate. In contrast with this statement, Theorem 1.8 states the existence of an absolute value $p^*$ above which every percolation law satisfying $\mathcal{D}(\alpha, c_\alpha)$ admits a unique infinite open cluster. This distinction is clearly a consequence of the quantitative decay of correlations that we have assumed through $\mathcal{D}(\alpha, c_\alpha)$.

A natural question at this point is about the sharpness of Theorem 1.8. For instance, is it true that Theorem 1.8 still holds true if we replace the polynomial decay assumption by some slower one? To shed some light into this question, let us mention an example from [TW12]. It consists of a family of dependent percolation measures $(\mathbb{P}^u)_{u>0}$ that satisfy a polynomial decay of correlations. However, the exponent $\alpha$ appearing in the decay is not sufficiently high, so that for all $u > 0$, there is $\mathbb{P}^u$-a.s. no percolation for $\{x; Y_x = 1\}$, despite the fact that $\mathbb{P}^u[Y_o = 1]$ converges to one as $u$ tends to zero.

More precisely, in [TW12] the authors define a Poisson process on $\mathbb{R}^d$ which determines a set of lines passing through the space. The intensity of this process is given by a non-trivial Haar measure on the space of lines, which invariant under translations and rotations, unique up to scaling.

Having defined this process of lines, one removes from $\mathbb{R}^d$ the cylinders of radius one and axis centered in these lines. The resulting set is called $\mathcal{V}$. By varying the intensity of the Poisson process, a phase transition in the percolation of $\mathcal{V}$ occurs for all $d \geq 3$, see [TW12, Theorems 4.1 and 5.1] and [HST15].
In our setting we look at the intersection of $\mathcal{V}$ and $\mathbb{R}^2$, where $\mathcal{V} \subset \mathbb{R}^3$. In this case, the cylinders intersected with the plane consist of ellipses with random major axis size. In Proposition 5.6 of [TW12], they show that, for every intensity $u > 0$ of the Poisson process, there is no infinite component in $\mathcal{V} \cap \mathbb{R}^2$. On the other hand, the model satisfies a condition very similar to $D(\alpha, c_\alpha)$ with $\alpha = 2$, see Lemma 3.3 of [TW12].

References


