Compressive Sensing
Examples in Image Compression

Lecture 4, July 30, 2009

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Today’s Lecture

- Discuss applications of CS in image compression
- Evaluate CS efficiency
- Review important definitions and theorems
Software

- Matlab functions for CS imaging
- Optimization algorithms
- Data
- Demos

http://www.impa.br/~aschulz/CS/course.html
Experimental Setup

- 4 images of size $256 \times 256$

(a) Phantom

(b) Lena

(c) Camera man

(d) Text
Frequency Distributions

(a) Phantom

(b) Lena

(c) Camera man

(d) Text
Acquisition

- Images are stored as a matrix of pixels

\[ y_m = \langle \phi_m, x \rangle \]
Noiselets

- RIP
- orthogonal
- self-adjoint

\[ \Phi = \frac{1}{2} \cdot \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \]
Sparsity

\[ N = p \cdot q \]
Reconstruction

Acquisition:
\[ y = \Phi_{\Omega} x \]
**Reconstruction**

**Acquisition:**
\[ y = \Phi_{\Omega} x \]
Reconstruction

Acquisition:
\[ y = \Phi \Omega x \]

Reconstruction:
\[ y = \Theta \Omega s \]
\[ \tilde{s} = \min_s \|s\|_1 \quad \text{s. t.} \quad \begin{cases} y = \Theta s \\ \|y - \Theta s\|_2 \leq \epsilon \end{cases} \]
\[ \tilde{x} = \Psi^* \tilde{s} \]
**Acquisition:**

\[ y = \Phi_\Omega x \]

**Reconstruction:**

\[ y = \Theta_\Omega s \]

\[ \tilde{s} = \min_s \|s\|_1 \quad \text{s. t.} \quad \begin{cases} 
    y = \Theta s \\
    \|y - \Theta s\|_2 \leq \epsilon 
\end{cases} \]

\[ \tilde{x} = \psi^* \tilde{s} \]
Basic CS

- Images are sparse in the $\Psi$ domain
- No noise is added

Ordinary images are only *approximately* sparse:
Basic CS

- Images are sparse in the $\psi$ domain
- No noise is added

Ordinary images are only \textit{approximately} sparse:

Figure: Different visualizations of Lena’s DCT.
Forcing Sparsity

DCT Example:

(a) Original Image
(b) Original DCT
(c) Original DCT
(d) Result Image
(e) Result DCT
(f) Result DCT
First Experiment

- $x \xrightarrow{\psi} s \xrightarrow{\text{force sparsity}} s_S \xrightarrow{\psi^*} x_S$

- Acquisition: $y = \Phi_\Omega x_S$

- Reconstruction of $\tilde{x}$ via $l_1$ minimization

- Compare $\tilde{x}$ and $x_S$ using **PSNR** (Peak Signal to Noise Ratio)
Figure: Basic CS experiment: 10k-sparse Lena.

Theorem

Reconstruction is exact if

\[ M \gtrsim S \cdot \mu^2(\Phi, \Psi) \cdot \log N \]
Varying $S$

![Graph showing varying measurements and PSNR for different sparsity levels.](image)
DCT Linear Compression
DCT Linear Compression

We measure $M \ll N$ coefficients on the upper-left corner.
DCT Linear Compression

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DCT Linear Compression

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DCT Linear Compression

Reconstruction: \( \tilde{x} = \Psi^* \tilde{s} \)
DCT Linear Compression

Problem: we don't measure all the significant coefficients and measure some zero coefficients!
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CS: Linear + Noiselets Measurements

Reconstruction: \[ \tilde{s} = \min_s \|s\|_1 \quad \text{s. t.} \quad \Phi'_\Omega \psi^* s = y \]
\[ \tilde{x} = \psi^* \tilde{s} \]
Results: CS $\times$ Linear Compression

(a) 3.5K-Sparse

(b) 6K-Sparse

(c) 10K-Sparse

(d) 14K-Sparse
Coherence

**Before Boundary**: best to use linear compression  
**After Boundary**: best to use CS
Different Images
What happens when we consider the other images?

(a) Phantom  (b) Lena  (c) Camera man  (d) Text

(e) Linear Compression  (f) Compressive Sensing
Different Images
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Different Images
What happens when we consider the other images?

(a) Linear Compression  (b) Compressive Sensing

(a) Phantom  (b) Lena  (c) Camera man  (d) Text
Different Sparsity Domains

What happens when we consider different sparsity domains?

(a) DCT

(b) Block DCT

(c) Wavelets
Different Sparsity Domains

What happens when we consider different sparsity domains?

(a) DCT

(b) Block DCT

(c) Wavelets

PSNR Measurements ($\times 10^3$)

DCT\_l - 1 - N
B\_DCT\_l - 1 - N
DWT\_l - 1 - N
But this is not for real...

All this happens because we **FORCE** sparsity!

**Theorem**

*Reconstruction is exact if*

\[ M \gtrsim S \cdot \mu^2(\Phi, \Psi) \cdot \log N \]

**Important Parameters:**
- \(\mu\) - Noiselets are highly incoherent with all considered domains
- \(S\) - We are tempering with it! - Lets stop and see what happens...
When $\Phi$ are Noiselet measurements the RIP holds and we can use:

**Theorem**

Let $s_S$ be the best $S$-sparse approximation of $s$. If the RIP holds, the solution $\tilde{s}$ to

$$\tilde{s} = \min_s \|s\|_1 \quad \text{subject to} \quad \Theta \Omega s = y$$

obeys

$$\|\tilde{s} - s\|_2 \lesssim \frac{1}{\sqrt{S}} \cdot \|s - s_S\|_1$$
Let’s verify this theorem empirically.
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Test image *Lena*

<table>
<thead>
<tr>
<th>Measurements</th>
<th>Sparsity is forced</th>
<th>Sparsity is not forced</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 20k$</td>
<td>$S = 3.5k$</td>
<td>PSNR = 28.8</td>
</tr>
<tr>
<td>$M = 25k$</td>
<td>$S = 6k$</td>
<td>PSNR = 30.7</td>
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<td>$M = 35k$</td>
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Different Sparsity Domains

![Graphs showing PSNR with varying measurements for (b) Phantom, (c) Lena, (d) Camera man, and (e) Text.](image)
Different Sparsity Domains

(b) Phantom

(c) Lena

(d) Camera man

(e) Text
Different Sparsity Domains

(b) Phantom

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Different Sparsity Domains

Total Variation (TV)

$$\min_x \|x\|_{TV} \quad \text{subject to} \quad y = \Phi \Omega x$$

- $l_1$-norm of the (appropriately discretized) gradient.
Different Sparsity Domains

(b) Phantom

(c) Lena

(d) Camera man

(e) Text
Different Sparsity Domains

Singular Value Decomposition (SVD)

\[ X = U S V \]

- Calculated for each image!
- Upper bounds - insights into performance limitations.
Different Sparsity Domains

![Graphs showing PSNR against Measurements for different sparsity domains.](image)

- **(b) Phantom**
- **(c) Lena**
- **(d) Camera man**
- **(e) Text**
Measurement Errors

**Measurements:** \( y = \Phi_\Omega x + n \)

- \( n \) is a random variable with normal distribution

**Theorem**

*If the RIP holds and \( \|n\|_2 \leq \epsilon \), the solution \( \tilde{s} \) to*

\[
\tilde{s} = \min_{s} \|s\|_1 \quad s.t. \quad \|\Theta_\Omega s - y\|_2 \leq \epsilon
\]

*obeys*

\[
\|\tilde{s} - s\|_2 \lesssim \frac{1}{\sqrt{S}} \cdot \|s - s_S\|_1 + \epsilon
\]
Gaussian Errors

Measurements ($\times 10^3$) vs. PSNR for different variances:
- var = 0.1
- var = 0.5
- var = 1
- var = 2
- var = 3
- var = 4
- var = 5
- var = 10

The graph shows the relationship between the number of measurements and the peak signal-to-noise ratio (PSNR) for Gaussian errors with varying variances.
Compression

Number of Measurements $\times$ Number of Bits

- To compress we have to quantize the acquired data!
Each measurement may assume $K$ values.
Quantization

- Larger quantization steps: less bits
- Smaller quantization steps: less errors
Rate

Rate = number of bits per pixel

Original Image:
- \( N \) pixels
- 8 bits/pixel

Acquired Measurements:
- \( M \ll N \) coefficients
- \( H_y \) bits/coefficients

\[
\text{Rate} = \frac{M}{N} H_y \text{ bits/pixel}
\]

If Rate < 8 data is being compressed!
The Rate \times Distortion Criteria

\[ \text{PSNR (fidelity)} \]

\[ \Rightarrow \]

\[ \text{Rate (bits per pixel)} \]
The Rate $\times$ Distortion Criteria

$\Rightarrow$ which is the best compression scheme?
Rate × Distortion

Measurements $\times 10^3$

PSNR

step = 0.01
step = 0.02
step = 0.05
step = 0.1
step = 0.2
step = 0.5
step = 1
step = 2
step = 3
step = 4
step = 5
step = 10
step = 20
step = 50
step = 100
Rate $\times$ Distortion
Compressive Sensing

Rate × Distortion

![Graph showing the relationship between rate and distortion for different step sizes. The x-axis represents rate, and the y-axis represents PSNR. The graph includes multiple lines, each representing a different step size. The step sizes are color-coded and range from 0.01 to 100.](image-url)
Rate × Distortion

Quantization Step = 2
Measurements = 33k

Quantization Step = 0.2
Measurements = 20k

Quantization Step = 20
Measurements = 55k
Rate × Distortion Efficiency

Rate × Distortion

PSNR

Rate

Legend:
- red: step = 0.01
- dark red: step = 0.02
- orange: step = 0.05
- yellow: step = 0.1
- green: step = 0.2
- light green: step = 0.5
- blue: step = 1
- light blue: step = 2
- dark blue: step = 3
- purple: step = 4
- magenta: step = 5
- gray: step = 10
- dark gray: step = 20
- light gray: step = 50
- black: step = 100
Both Sparsity and Measurement Errors

A real example: real image and quantized measurements.

Recovery Strategies:
- $\psi = \text{DCT}$
- $\psi = \text{block DCT}$
- $\psi = \text{DWT}$
- TV- minimization
- $\psi = \text{SVD} - \text{of each image!}$
Both Sparsity and Measurement Errors

![Graphs of PSNR vs. Rate for different images](image)

- (b) Phantom
- (c) Lena
- (d) Camera man
- (e) Text
Sparsity $\times$ Quantization Errors

$$\|\tilde{x} - x\|_2 \lesssim \left( \epsilon_q + \frac{1}{\sqrt{S}} \|x_S - x\|_1 \right),$$

The reconstruction error in CS is of the order of the maximum of the quantization ($\epsilon_q$) and sparsity errors ($\epsilon_s$).
Figure: Rate × PSNR of image Lena
Figure: Measurements × PSNR of image Lena
Let's wrap it up!

CS Applications

Image acquisition and compression
  • Interesting, but
  • Limitations ...  

Tomorrow

Applications in graphics and vision
  • One Pixel Camera
  • Dual Photography