Leaf conjugacies on the torus

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Hyperbolic Systems

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• $Tf$-invariant splitting:

$$TM = E^u \oplus E^s$$
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\[\lambda \quad 1 \quad \mu\]

\[s \quad u\]
Linear Example

\[ A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ on } \mathbb{R}^2/\mathbb{Z}^2 \]
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- \( A \) has eigenvalues \( \nu^{-1} < 1 < \nu \).

- Splitting \( T_xM = E^u(x) \oplus E^s(x) \) is the same for all points \( x \).
Invariant Foliations

- There are foliations $W^u$ and $W^s$ tangent to $E^u$ and $E^s$. 
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Invariant Foliations

• If \( x \) and \( y \) lie on the same stable leaf, then

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d_s(f(x), f(y)) < \lambda \, d_s(x, y)
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Here, \( d_s \) denotes distance along the leaf.
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- For $k \in \mathbb{Z}$,
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- Similarly, if $x$, $y$ lie on the same unstable leaf, then
  \[ \mu^k \ d_u(x, y) < d_u(f^k(x), f^k(y)) \]
Franks and Manning

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- Can take \( g \) to be the \textit{linearization}, defined by the action

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f_* : \pi_1(\mathbb{T}^d) \to \pi_1(\mathbb{T}^d).
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Non-differentiable Foliations

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• Partially hyperbolic diffeomorphisms are leaf conjugate if there is a homeomorphism $h : M \to M$ such that

$$hg(L) = fh(L)$$

for every center leaf $L$ of $g$. 
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for every center leaf \( L \) of \( g \).

• Is every partially hyperbolic \( f : \mathbb{T}^d \rightarrow \mathbb{T}^d \) leaf conjugate to its linearization?
The 3-torus

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\[ d_u(x, y) < Q \cdot \|x - y\|. \]
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and \( \dim E^c_f = 1 \),
Main Theorem

**Theorem.** If $f_0 : \mathbb{T}^d \to \mathbb{T}^d$ is partially hyperbolic with lifting $f : \mathbb{R}^d \to \mathbb{R}^d$,

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then $f_0$ is leaf conjugate to its linearization.
Main Theorem

**Theorem.** If $f_0 : \mathbb{T}^d \to \mathbb{T}^d$ is partially hyperbolic with lifting $f : \mathbb{R}^d \to \mathbb{R}^d$, $W_f^u$ and $W_f^s$ are quasi-isometric, and $\dim E_f^c = 1$, then $f_0$ is leaf conjugate to its linearization.

**Corollary.** Every partially hyperbolic diffeomorphism on $\mathbb{T}^3$ is leaf conjugate to its linearization.
Constructing $g$

- Use $f_0^* : \pi_1(\mathbb{T}^d) \to \pi_1(\mathbb{T}^d)$ to define a linear map $g : \mathbb{R}^d \to \mathbb{R}^d$. 

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- Using the constants $\lambda < \hat{\gamma} < 1 < \gamma < \mu$ for $f$, define a partially hyperbolic splitting for $g$. 

![Diagram]

1

$s$ $c$ $u$
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```
\begin{tikzpicture}
  \draw[->] (0,0) -- (10,0);
  \draw[blue, ultra thick] (1,0) rectangle (3,0.5);
  \node at (1.5,0) {$\lambda$};
  \draw[green, ultra thick] (4,0) rectangle (6,0.5);
  \node at (5,0) {$\hat{\gamma}$};
  \draw[red, ultra thick] (7,0) rectangle (9,0.5);
  \node at (8,0) {$\mu$};
  \node at (2,0) {$s$};
  \node at (5,0) {$c$};
  \node at (8,0) {$u$};
\end{tikzpicture}
```
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\[
\begin{array}{c|c|c|c|c}
\lambda & \hat{\gamma} & 1 & \gamma & \mu \\
\hline
\end{array}
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![Diagram](image)

\[ \lambda_1 \quad \hat{\gamma} \quad 1 \quad \gamma \quad \mu \]
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[Diagram with eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\hat{\gamma}$]
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\end{array} \]

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\[\lambda_1 \quad \lambda \quad \hat{\gamma} \quad 1 \quad \lambda_2 \quad \gamma \quad \mu \quad \lambda_3\]

\[s \quad c \quad u\]
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Note, the splitting

\[ T_xM = E^u_g \oplus E^c_g \oplus E^s_g \]

is independent of $x \in M = \mathbb{T}^d$. 
Comparing Foliations

- Compare the foliations of $f$ to the flat, linear foliations of $g$. 
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• $W^s_f$ tangent to $E^s_f$:

$$\frac{x - y}{\|x - y\|} \rightarrow E^s_f \quad \text{as} \quad d_s(x, y) \rightarrow 0$$
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$$\frac{x - y}{\|x - y\|} \to E^s_f \quad \text{as} \quad d_s(x, y) \to 0$$

• What if $d_s(x, y) \to \infty$?
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• Compare the foliations of $f$ to the flat, linear foliations of $g$.

• $W^s_f$ tangent to $E^s_f$:

$\frac{x - y}{\|x - y\|} \to E^s_f$ as $d_s(x, y) \to 0$

• What if $d_s(x, y) \to \infty$?

$\frac{x - y}{\|x - y\|} \to E^s_g$
Three Scales
Three Scales

\[ E_s(x) \]
\[ E_c(x) \]
\[ E_u(x) \]
\[ W_s(x) \]
\[ W_c(x) \]
\[ W_u(x) \]
Shrinking Stable Segments
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- With quasi-isometry, the endpoints shrink together.
Shrinking Stable Segments

- With quasi-isometry, the endpoints shrink together.
Shrinking Stable Segments

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Shadowing

- Take $x$ and $y$ far apart on the same stable leaf.
Shadowing

• Take $x$ and $y$ far apart on the same stable leaf.

• $f^k(x) - f^k(y)$ shrinks exponentially as $k$ increases.
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• Since $f_{0*} = g_{0*}$, so must $g^k(x) - g^k(y)$. 
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\[
\begin{align*}
&f^k(x) - f^k(y) \\
&g^k(x) - g^k(y)
\end{align*}
\]
Shadowing

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Center Leaves

- Every center leaf of $f$ lies in a cylinder of radius $R_c$, 

\[
B_{R_c}(W^c_g(x)) \rightarrow \begin{aligned}
W^c_f(x) \\
W^c_g(x)
\end{aligned}
\]
Center Leaves

• Every center leaf of $f$ lies in a cylinder of radius $R_c$,

• but a priori the center leaf may be a circle.
Pseudoleaves

- The $us$-pseudoleaf of $f$ at $x$:

$$W_f^{us}(x) = \bigcup_{y \in W_f^u(x)} W_f^s(y).$$
Proper Embeddings

- The $us$-pseudoleaf is a properly embedded hyperplane.
Proper Embeddings

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Proper Embeddings

• The $us$-pseudoleaf is a properly embedded hyperplane.
Unique Intersection

- A center leaf intersects a pseudoleaf at most once.
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$W^c_f(x)$ is not a Circle

$W^c_f(x)$

$x$
$W^c_f(x)$ is not a Circle
$W^c_f(x)$ is Properly Embedded
$W_c^c(x)$ is Properly Embedded
Existence of Intersection

• Each pseudoleaf lies roughly in the $E_g^u \oplus E_g^s$ direction.
Existence of Intersection

- Each pseudoleaf lies roughly in the $E^u_g \oplus E^s_g$ direction.
- Each center leaf lies in the $E^c_g$ direction.
Existence of Intersection

• Each pseudoleaf lies roughly in the $E_g^u \oplus E_g^s$ direction.

• Each center leaf lies in the $E_g^c$ direction.

• As these subspaces are transverse, every pseudoleaf intersects every center leaf exactly once.
Franks-Manning

Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be hyperbolic with linearization $g : \mathbb{R}^d \to \mathbb{R}^d$. 
Franks-Manning

- Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be hyperbolic with linearization $g : \mathbb{R}^d \to \mathbb{R}^d$.
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Franks-Manning Reloaded

$L$
Franks-Manning Reloaded

\[ g^n(\mathcal{L}) \]
Franks-Manning Reloaded
Franks-Manning Reloaded
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\[ f_n(g^{-n}(\mathcal{L})) \]

\[ g^n(\mathcal{L}) \]
Franks-Manning Reloaded

\[ g^{-n}(\mathcal{L}) \]

\[ g^n(\mathcal{L}) \]

\[ f^{-n} \]
Franks-Manning Reloaded
A Conjugacy of Leaf Spaces

• Let $C_f$ be the quotient space of center leaves of $f$.

  $C_g$ the quotient space of center leaves of $g$. 
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A Conjugacy of Leaf Spaces

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$$H g(\mathcal{L}) = f H(\mathcal{L}) \quad \text{and}$$
A Conjugacy of Leaf Spaces

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  $C_g$ the quotient space of center leaves of $g$.
- $H : C_g \rightarrow C_f$ is the unique map such that
  \[
  Hg(\mathcal{L}) = fH(\mathcal{L}) \quad \text{and} \quad H(\mathcal{L} + z) = H(\mathcal{L}) + z \quad \text{for} \quad z \in \mathbb{Z}^d.
  \]
Characterizing $f$

- We know the topology of the leaves, and the directions in which they lie.
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- We know how the leaves intersect.
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• We know the topology of the leaves, and the directions in which they lie.

• We know how the leaves intersect.

• $H : C_g \rightarrow C_f$ shows that $f$ acts on center leaves as a hyperbolic linear map.

• Now construct a leaf conjugacy $h_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ from $H$. 

Mapping Slab to Slab

To construct a leaf conjugacy $h_0 : \mathbb{R}^d \to \mathbb{R}^d$, first map one solid slab to another.

$x_0$
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Mapping Slab to Slab

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- How to define the topological hyperplane $\sigma$?
Sections

**Definition.** A *section* is a continuous map $\sigma : C_f \to \mathbb{R}^d$ such that
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$$\sigma(\mathcal{L}) \text{ is on the leaf } \mathcal{L} \text{ for } \mathcal{L} \in C_f.$$
Sections

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$$\sigma(\mathcal{L}) \text{ is on the leaf } \mathcal{L} \text{ for } \mathcal{L} \in C_f.$$ 

Example. The pseudoleaf $W_f^{us}(x)$ defines a section

$$\sigma : C_f \to \mathbb{R}^d, \quad \mathcal{L} \mapsto \mathcal{L} \cap W_f^{us}(x).$$
Desirable Sections

- Unfortunately, pseudoleaves don’t make very good sections.
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• Need a section which is bounded in the $E^c_g$ direction
Desirable Sections

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• Need a section which is bounded in the $E^c_g$ direction and uniformly continuous.
Desirable Sections

• Unfortunately, pseudoleaves don’t make very good sections.

• Need a section which is bounded in the $E_g^c$ direction and uniformly continuous.

• These properties hold for compact subsets of psuedoleaves.
Stitching

• Idea: Stitch together plaques of pseudoleaves to produce a global section.
Stitching

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• Perform stitching by a weighted average along each center leaf.

\[ \mathcal{L} \in C_f \]
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Constructing a Global Section

• Take part of a pseudoleaf at a lattice point. Translate the plaques to other lattice points.
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Constructing a Global Section

• Take part of a pseudoleaf at a lattice point. Translate the plaques to other lattice points.

• Carefully average to yield a uniformly continuous global section.
Mapping Slab to Slab

\[ x_0 \]
Mapping Slab to Slab

$W_{us}(x_0)$

$x_0$
Mapping Slab to Slab

$W_{\text{us}}(x_0)$

$x_0$

$\sigma$
Mapping Slab to Slab

\[ W_{gs}(x_0) \]

\[ x_0 + v \]

\[ \sigma \]
Mapping Slab to Slab

\[ W^\text{us}_g(x_0 + v) \]

\[ W^\text{us}_g(x_0) \]

\[ x_0 + v \]

\[ x_0 \]

\[ \sigma \]
Mapping Slab to Slab

\[ W_{g,x_0}(x_0 + v) \]

\[ W_{g,x_0}(x_0) \]

\[ x_0 \]

\[ x_0 + v \]
Mapping Slab to Slab

\[ W_{us}^g(x_0 + v) \]

\[ W_{us}^g(x_0) \]

\[ x_0 + v \]

\[ x_0 \]

\[ \sigma + v \]

\[ \sigma \]
Mapping Slab to Slab
Mapping Slab to Slab

\[ W_g^u(x_0 + v) \]

\[ W_g^c(y) \]

\[ x_0 \]

\[ y \]

\[ \sigma + v \]

\[ \sigma \]
Mapping Slab to Slab

\[ W_{\text{us}}(x_0 + v) \]

\[ W_{g}(x_0) \]

\[ x_0 + v \]

\[ x_0 \]

\[ y \]

\[ W_{c}(y) \]

\[ H(W_{c}(y)) \]

\[ \sigma + v \]

\[ \sigma \]
Mapping Slab to Slab

\[ W_g(x_0 + v) \]

\[ W_{us}^c(y) \]

\[ W_g(y) \]

\[ H(W_g^c(y)) \]

\[ \sigma + v \]

\[ \sigma \]
Mapping Slab to Slab

\[ W^u(x_0 + v) \]

\[ W^c(y) \]

\[ x_0 + v \]

\[ x_0 \]

\[ y \]

\[ W^c(y) \]

\[ 
\sigma + v \\
\sigma \\
H(W^c(y)) 
\]
Mapping Slab to Slab

$$W_{us}(x_0 + v)$$

$$W_{us}(x_0)$$

$$W^c_g(y)$$

$$h_0(y)$$

$$H(W^c_g(y))$$

$$\sigma$$

$$\sigma + v$$

$$x_0 + v$$

$$x_0$$

$$y$$

$$h_0$$
A Leaf Conjugacy on $\mathbb{R}^d$

- Extend $h_0$ to all of $\mathbb{R}^d$ by translating by multiples of $v$. 
A Leaf Conjugacy on $\mathbb{T}^d$

- $h_0$ is a leaf conjugacy on $\mathbb{R}^d$, 
A Leaf Conjugacy on $\mathbb{T}^d$

- $h_0$ is a leaf conjugacy on $\mathbb{R}^d$,

but there is no reason to think it descends to $\mathbb{T}^d$. 
A Leaf Conjugacy on $\mathbb{T}^d$

- $h_0$ is a leaf conjugacy on $\mathbb{R}^d$,
  but there is no reason to think it descends to $\mathbb{T}^d$.
- For $z \in \mathbb{Z}^d$, define $h_z : \mathbb{R}^d \to \mathbb{R}^d$ as a translate of $h_0$:
  \[ h_z(x) = h_0(x - z) + z. \]
Averaging

- Average over all $h_z$ along the center leaves.
Averaging

- Average over all $h_z$ along the center leaves.

\[ W_g^c(x) \quad \Downarrow \quad H(W_g^c(x)) \]
Averaging

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Averaging

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- Average over all $h_z$ along the center leaves.

$$W^c_g(x) \rightarrow H(W^c_g(x))$$
Averaging

- Average over all $h_z$ along the center leaves.

- Find a $\mathbb{Z}^d$-invariant leaf conjugacy $h : \mathbb{R}^d \to \mathbb{R}^d$ that descends to $\mathbb{T}^d$. 
Further Questions

- Franks-Manning extends to nilmanifolds and infranilmanifolds.
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• What is the partially hyperbolic extension?
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• Is quasi-isometry necessary? Is it redundant for tori?
Further Questions

• Franks-Manning extends to nilmanifolds and infranilmanifolds.

• What is the partially hyperbolic extension?

• Is \( \dim E^c_f = 1 \) necessary?

• Is quasi-isometry necessary? Is it redundant for tori?

• Can we classify all partially hyperbolic diffeomorphisms on 3-manifolds?
The End
Asymptote: 2D & 3D Vector Graphics Language

http://asymptote.sf.net

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