Adverse Selection Problems without the Spence-Mirrlees Condition

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Abstract

This paper studies a class of one-dimensional screening problems where the agent’s utility function does not satisfy the Spence-Mirrlees condition (SMC). The strength of the SMC for hidden information problems is to provide a full characterization of implementable contracts using only the local incentive compatibility (IC) constraints. These constraints are equivalent to the monotonicity of the decision variable with respect to the agent’s unobservable one-dimensional parameter. When the SMC is violated the local IC constraints are no longer sufficient for implementability and additional (global) IC constraints have to be taken into account. In particular, implementable decisions may not be monotonic and discretely pooled types must have the same marginal utility of the decision (or equivalently, receive the same marginal tariff). Moreover, at the optimal decision, the principal must preserve the same trade-off between rent extraction and allocative distortion measured in the agent’s marginal rent unit. In a specific setting where non-monotone contracts may be optimal we fully characterize the solution.

Keywords: Spence-Mirrlees condition; global incentive compatibility; U-shaped condition; discrete pooling.
JEL Codes: H41, D82.
1 Introduction

The Spence-Mirrlees condition (SMC) has been until now the main assumption that allows us to fully characterize the solution of one-dimensional adverse selection or hidden information models. Specifically, implementable contracts (and a fortiori optimal contracts) of standard principal-agent models and separating equilibria of signaling games can be easily characterized under this condition.

We say that the SMC holds when the privately informed part has preferences that have increasing marginal rates of substitution between a decision variable and money with respect to a one-dimensional parameter or type (asymmetric information). From the Revelation Principle, implementable contracts are mechanisms (i.e., maps from types to decision and money) that induce the agent to truthfully reveal her type, i.e., they must satisfy the agent’s incentive compatibility (IC) constraints. Under the SMC, these constraints are equivalent to their local first- and second-order conditions and hence implementable decisions are non-decreasing in types. For principal-agent models, this is enough to provide an algorithm for computing the optimal contract (see [10], for instance). Therefore, the SMC transforms a two-level (complex) maximization problem into a much simpler optimal control program. The monotonicity of implementable (and optimal) contracts is a robust property in such

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2 In the literature of monotone methods for comparative statics there are some subtle differences among the various concepts of single-crossing. Milgrom and Shannon [16] define the order-theoretic single-crossing property for general lattice spaces. Edlin and Shannon [7] extend the analysis to strict monotone comparative statics (the strict SMC) by imposing a stronger differential restriction.
models.

Our main goal is to study a class of one-dimensional screening problems where the agent’s utility function does not satisfy the SMC. For a quite general framework we obtain necessary conditions for incentive compatibility which are now much more complex since they involve both the traditional local and some new global conditions. Next, we derive the necessary conditions for optimality. Finally, for a more special case we also obtain sufficient conditions for an optimum. Furthermore, for the reader not interested in going throughout all the mathematics of the paper we provide a practical algorithm for computing solutions which has an analogy with the SMC case described in the previous paragraph.

Assuming quasi-linearity, the SMC is equivalent to the constant sign of the cross derivative of the utility function with respect to decision and type. Our class of problems is characterized by the existence of a decreasing limit curve that separates two regions in the type versus decision plane where the sign of the cross derivative changes: the positive (above the limit curve) and negative (below the limit curve) single-crossing regions. The necessary local IC constraints imply that implementable decisions preserve monotonicity in each region, crossing or not the limit curve. In particular, when they cross, they must have a U-shaped form and a new necessary global IC condition emerges. Two types taking the same decision must get the same marginal tariff which leads to the same marginal utility of decision (Theorem 1). We will refer to it as the U-shaped condition. This indicates that the principal has to take care of a wider variety of strategic behavior from the agent when the SMC is violated, i.e., the local IC constraints no longer imply global truth-telling.

In general it is very hard to characterize the set of implementable decisions even in this class of problems. Following the relaxed approach of the literature, our strategy is to characterize the solution of a more relaxed program that takes into account, in addition to the first- and second-order conditions of the IC constraint, the U-shaped condition, called the U-shaped problem. A preliminary and important result is the necessary optimality condition for the binding U-shaped condition (Theorem 2). It says that pooling types must have the same trade-off between rent extraction and distortion measured in the agent’s marginal rent. The remaining of the paper provides a specific setup where these necessary conditions for implementability and optimality are sufficient (Theorem 3).

In one-dimensional single-crossing literature, Jullien’s work [11] is the most related to ours. He gives a complete analysis of the agent’s participation constraint and the possibility of countervailing incentives (see also [13] and [15]). Although the distortions of the optimal contract found by him are similar to ours, they come from the endogenous binding participation constraint whereas
ours are due to binding global IC constraints. Furthermore, he keeps the single-crossing assumption and, a fortiori, the monotonicity of the optimal contract and we in a complementary way avoid countervailing incentives due to the participation constraints (see Section 6 for more on this).

There are few examples in the literature that violate the SMC. Two of them are on signaling games and show that breaking down the SMC may lead to interesting phenomena. Bernheim [4] presents a model for a theory of conformity in which individuals care about status as well as intrinsic utility. When status is sufficiently more important than intrinsic utility, the SMC fails and many individuals conform to a single, homogeneous behavior (continuous pooling), despite the heterogenous underlying preferences. Bagwell and Bernheim [3] give a theory of conspicuous consumption where the Veblen effect (i.e., willingness to pay a higher price for a functionally equivalent good) occurs if and only if the SMC fails. However, in both applications the signaling equilibrium is always monotonic.

The paper is organized as follows. Section 2 presents the model and Section 3 relaxes the SMC. Section 4 provides new necessary conditions for implementability and optimality. Section 5 shows that these new necessary conditions are sufficient to characterize the optimal contract in a special case. In that section we also provide a practical guide for the reader not interested in going through the technical details in the Appendix. Section 6 gives the extensions and concluding remarks. Proofs and technical details are relegated to the Appendix.

2 The Model

The principal-agent relationship involves a one-dimensional decision (consumption, production, etc.) \( \xi \in [x, \tilde{x}] \equiv X \subset \mathbb{R} \) and a monetary transfer \( t \in \mathbb{R} \). The agent’s utility depends on a type parameter \( \theta \in \Theta = [\underline{\theta}, \overline{\theta}] \), known by her and unobservable to the principal, and is quasi-linear: \( v(\xi, \theta) + t \). The principal’s utility function is also quasi-linear and may depend on the agent’s type: \( u(\xi, \theta) - t \). The total surplus is then \( u(\xi, \theta) + v(\xi, \theta) \), maximal at the first-best level \( x^{FB}(\theta) \), for each \( \theta \). The principal has prior distribution over \( \Theta \) according to the cumulative distribution \( P \) with continuous density \( p : \Theta \to \mathbb{R}_{++} \). The functions \( u \) and \( v \) are three times continuously differentiable on \( X \times \Theta \).

Using the revelation principle, any general communication mechanism between the principal and the agent can be mimicked by a direct truthful one with no welfare loss to the principal. A direct mechanism (or contract) is a pair of functions \((x, t) : \Theta \to X \times \mathbb{R} \) which can be viewed as a procedure committing the principal to a rule relating the choice of \( x \) and \( t \) to messages sent by the
agent about their types.

A decision function \( x: \Theta \to X \) is implementable by a money transfer function \( t: \Theta \to \mathbb{R} \) if the incentive-compatibility constraint is satisfied:

\[
v(x(\theta), \theta) + t(\theta) \geq v(x(\tilde{\theta}), \theta) + t(\tilde{\theta}) \quad \forall \theta, \tilde{\theta} \in \Theta.
\]  

(1)

Equivalently, we say that the contract \((x, t)\) is implementable.

We assume that the agent’s reservation utility, \( v(0, \theta) \), is independent of his type and normalized to zero. A contract \((x, t)\) satisfies the individual-rationality constraint if

\[
v(x(\theta), \theta) + t(\theta) \geq 0, \quad \forall \theta \in \Theta.
\]  

(2)

An implementable contract that satisfies the IR constraint is called feasible.

We present the definition of the SMC that will be relaxed.

**Definition 1 (SMC)** The single-crossing or Spence-Mirrlees condition (SMC) is the constant sign of the cross partial derivative with respect to decision and type:\(^3\)

\[
v_{x\theta} > 0 \quad \text{on } X \times \Theta \quad \text{(CS\textsubscript{+})}
\]

or

\[
v_{x\theta} < 0 \quad \text{on } X \times \Theta. \quad \text{(CS\textsubscript{-})}
\]

Definition 1 implies that indifference curves of two different types cross at most once in the plane \((\xi, t)\). Under \text{CS\textsubscript{+}} \text{(CS\textsubscript{-})}, higher types are associated with higher (lower) marginal valuations of the decision. In what follows we also refer to \text{CS\textsubscript{+}} \text{(CS\textsubscript{-})} as the region where the sign of the cross derivative is positive (negative).

Below we present the standard necessary first- and second-order conditions of the IC constraint. Both the statement and proof are simple extensions to existing results in the literature (and do not depend on the SMC). The first part is the usual envelope theorem (see [17]).

**Lemma 1 (Necessary conditions for local IC constraints)** Suppose that \( x \) is an implementable bounded decision.

(i) (First-order condition) The agent’s rent function of \( x \), \( V^x \), is given by

\[
V^x(\theta) := v(x(\theta), \theta) + t(\theta) = V^x(\tilde{\theta}) - \int_{\tilde{\theta}}^{\theta} v_\theta(x(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta}, \quad \forall \theta \in \Theta.
\]  

(3)

\(^3\) The sub-index in the function denotes the partial derivative. The superior order derivative is denoted by a multi-index notation.
(ii) (Second-order condition) If \( x : \Theta \to X \) is càdlàg\(^4\), then

\[
x \text{ is non-decreasing (non-increasing) in } CS_+ (CS_-). \quad (M)
\]

By Lemma 1 (i), for each implementable \( x \) there exists a unique \( t \) (up to the constant \( V_x(\theta) \)) that implements it:

\[
t(\theta) = V_x(\theta) - v(x(\theta), \theta), \quad \forall \theta \in \Theta. \quad (4)
\]

The global incentive compatibility (1) requires that

\[
V_x(\theta) - v(x(\hat{\theta}), \theta) - t(\hat{\theta}) \geq 0.
\]

But, using (4), we get

\[
V_x(\theta) - v(x(\hat{\theta}), \theta) - t(\hat{\theta}) = V_x(\theta) - V_x(\hat{\theta}) + v(x(\hat{\theta}), \hat{\theta}) - v(x(\hat{\theta}), \theta)
\]

\[
= \int_{\theta}^{\hat{\theta}} [v_\theta(x(\theta), \theta) - v_\theta(x(\hat{\theta}), \hat{\theta})] d\hat{\theta}
\]

\[
= \int_{\theta}^{\hat{\theta}} \left[ \int_{x(\theta)}^{x(\hat{\theta})} v_{x\theta}(\tilde{x}, \theta) d\tilde{x} \right] d\hat{\theta},
\]

where the second equality results from (3) and the third is a consequence of the fundamental theorem of calculus.

We can then define the global incentive function (GIF) of \( x \) that satisfies (3):

\[
\Phi^x(\theta, \hat{\theta}) := \int_{\theta}^{\hat{\theta}} \left[ \int_{x(\theta)}^{x(\hat{\theta})} v_{x\theta}(\tilde{x}, \theta) d\tilde{x} \right] d\hat{\theta}. \quad (GIF)
\]

Hence, \( x \) is implementable if and only if \( \Phi^x \) is non-negative. For instance, if \( v_{x\theta} \) has a constant positive sign everywhere, then high types have higher marginal valuations for \( x \), i.e., their indifference curves in the space \((x, t)\) are less steep than those for low types. As a consequence, \( \Phi^x(\theta, \cdot) \) is non-negative everywhere if and only if \( x \) is non-decreasing. In other words, if the local incentive to cheat is avoidable in the report of true type, then the same is true for global reports. Therefore, the necessary monotonicity condition (M) joint with (3) are also sufficient for implementability under the SMC.

The principal’s hidden information or second-best problem is to choose a feasible contract with the highest expected payoff:

\[
\max_{x(\cdot), t(\cdot)} E[u(x(\cdot), \cdot) - t(\cdot)] \quad (SB)
\]

s.t. IC and IR constraints,

\(^4\) That is, right-continuous with left-hand side limit at each point of the domain.
where $E$ is the expectation operator with respect to the prior distribution $p$.

Plugging (4) into the objective function, we get

$$E[u(x(\cdot), \cdot) - t(\cdot)] = E[u(x(\cdot), \cdot) + v(x(\cdot), \cdot) - \mathcal{V}^x(\cdot)].$$

(5)

Integrating by parts the term $E[\mathcal{V}^x(\cdot)]$ and using (3), it becomes

$$E[f(x(\cdot), \cdot)] = \mathcal{V}^x(\bar{\theta})$$

where

$$f(\xi, \theta) = u(\xi, \theta) + v(\xi, \theta) + \frac{P(\theta)}{p(\theta)} v_\theta(\xi, \theta)$$

is the virtual surplus, i.e., the social surplus $(u + v)$ discounted by the agent’s informational rent $(-\mathcal{E} v_\theta)$.

If $v_\theta \leq 0$, then $\mathcal{V}^x$ given by (3) has a minimum at $\bar{\theta}$. Then, the IR constraint needs to be checked only at $\bar{\theta}$ and, since transfer is costly to the principal, at the optimal contract it must bind, i.e., the IR constraint can be replaced by $\mathcal{V}^x(\bar{\theta}) = 0$.

If $v_\theta$ changes its sign, then $\mathcal{V}^x$ can assume its minimum value in the interior of $\Theta$ (or even, at $\bar{\theta}$) depending on $x$. However, in order to simplify matters, we investigate the effect of breaking down the assumption on the cross derivative, not on $v_\theta$. From now on we assume that $v_\theta \leq 0$ (see the last section and Jullien [11] for more details on the general case).\footnotemark

\footnotetext{The case $v_\theta \geq 0$ would be completely analogous. The only difference is that the inverse of the hazard rate would be $(1 - P(\theta))/p(\theta)$ instead.}

If we only consider the (local) first-order condition of the IC constraint (3), then we can define the following relaxed problem:

$$\max_{x(\cdot)} E[f(x(\cdot), \cdot)]$$

and its solution, denoted by $x_1$, is called the relaxed solution. The relaxed problem reduces to a pointwise maximization of the virtual surplus whose first-order condition is

$$f_x(x_1(\theta), \theta) = 0,$$

(6)

for all $\theta \in \Theta$ such that $x_1(\theta) \in (x, \overline{x})$.

Under the assumption that follows (made throughout the paper), this pointwise first-order condition is sufficient for optimality.

A1. $f(\cdot, \theta)$ is concave, for all $\theta \in \Theta.$
If the solution of the relaxed problem (RP) is implementable, then it is the solution of Problem (SB). Therefore, under the SMC, \( x_1 \) is the solution of Problem (SB) if and only if it is monotonic. Otherwise, the monotonicity condition (M) is binding.

When (M) is binding, one has to perform the ironing principle\(^6\) on \( x_1 \). Using this principle Guesnerie and Laffont [10] managed to provide an algorithm to determine the solution under the SMC (for a complete analysis of the standard case see also [12]).

3 Relaxing the SMC

In this paper, we are concerned with a particular type of failure of the SMC in which there is a decreasing boundary \( x_0 \) that splits the space \( \Theta \times X \) into two single-crossing regions \( CS_+ \) above and \( CS_- \) below. More formally:\(^7\)

\[ A2. \ v_{x\theta}(\xi, \theta) = 0 \text{ defines implicitly a decreasing function } x_0 \text{ of } \theta \text{ on } \Theta \text{ such that } \xi \geq x_0(\theta) \text{ if and only if } v_{x\theta}(\xi, \theta) \geq 0 (CS_+), \text{ for all } \theta \in \Theta. \text{ Moreover, } v_{x\theta} > 0 \text{ holds on } X \times \Theta. \]

Figure 1 shows an example of such a boundary. Clearly, the SMC does not hold in Figure 1. To see how this can cause a problem, suppose that the solution to the standard relaxed problem is the function \( x_1 \) as shown. The relaxed solution lies entirely in \( CS_- \) and, since it is decreasing, it satisfies the local monotonicity. Nevertheless, the global IC condition GIF may fail. Consider the hatched area in Figure 1 given by \( \int_{\tilde{\theta}}^{\theta} \left[ \int_{x(\tilde{\theta})}^{x(\theta)} d\tilde{x} \right] d\tilde{\theta} \). If we weight by \( v_{x\theta} \), this area becomes the global incentive function \( \Phi^{x_1}(\theta, \tilde{\theta}) \) defined by (GIF). Thus, if the weighted hatched area above \( x_0 \) is greater in absolute terms than the

\(^6\) Mussa and Rosen [19] is the pioneering references of this procedure for adverse selection problems. Rochet and Choné [22] provide a generalization of the “ironing principle” for multidimensional screening models.

\(^7\) Athey [2] introduces “limited-complex” strategies which have a finite number of peaks in the context of screening equilibrium models like auctions (although she only shows the existence of pure strategy equilibria). In our case, A2 will imply that implementable decisions have at most one peak. Chassagnon and Chiappori [5] study a competitive insurance model with moral hazard and adverse selection where the SMC may fail, but in a two-type model.

\(^8\) By the implicit function theorem, a sufficient condition for the first part of A2, besides \( v_{x\theta} > 0 \), is \( v_{x\theta} > 0 \) on \( X \times \Theta \). Making the other possible sign changes for \( \xi \) and/or \( \theta \), we obtain other three equivalent cases.
weighted area below it, the global incentive compatibility condition is violated. More generally, when $x_0$ separates the space $\Theta \times X$ into two regions, the net effect on the global incentive function (GIF) may be positive or negative depending on the relative intensity of the local incentive effect in each region.

This indicates that other (global) IC constraints might be important. This is the subject of the next section.

4 Necessary Conditions

4.1 Necessary conditions for implementability

For the purpose of this paper, let us consider a particular and interesting case of global binding IC constraint. Fix an implementable decision $x$ and suppose that it crosses continuously the limit curve $x_0$. Then, $x$ must cross $x_0$ in a U-shaped form due to condition (M) and there must be $\theta \neq \hat{\theta}$ in $(\underline{\theta}, \bar{\theta})$ such that $x(\theta) = x(\hat{\theta}) = \xi$ (discrete pooling). Figure 2 illustrates this situation.

The right graph shows a U-shaped decision such that $\theta$ and $\hat{\theta}$ are pooling and the left one is the correspondent contract in the plane $X \times \mathbb{R}$ of indirect mechanisms. Hence, if the contract is implementable, the indifference curves (of $\theta$ and $\hat{\theta}$) and the non-linear tariff are tangent at $\xi$. Equivalently, discretely pooled types must obtain the same marginal tariff (see subsection 5.1 for more on the indirect approach).
We then get our new global necessary incentive compatibility condition:

\[ v_x(\xi, \hat{\theta}) = v_x(\xi, \theta), \]  

i.e., the \textit{U-shaped condition}. Formally, we have:

\textbf{Theorem 1 (U-shaped condition)} Let \( x \) be an implementable decision such that \( x \) is continuous and strictly monotonic at \( \hat{\theta} \) and \( \theta \) such that \( \xi = x(\theta) = x(\hat{\theta}) \). Then, (UC) must hold.

Under the SMC, implementability is equivalent to checking the local upstream or downstream incentives, i.e., monotonicity. When it does not hold, the upstream and downstream incentives are necessary but not sufficient for implementability. At least an extra necessary global IC condition must be checked: the \textit{cross-stream incentives} (UC).

The SMC implies the quasi-convexity of \( \Phi^x(\theta, \cdot) \), for all \( \theta \), and the convexity of the space of implementable decisions (since monotonicity is preserved by convex combination). However, these are no longer true when the SMC is broken: \( \Phi^x(\theta, \cdot) \) may have two distinct minima (the pooling types) and the convex combination of implementable decisions that satisfy condition (UC) may not satisfy it.\(^9\)

\(^9\) This condition is equivalent to \( v_x \) being constant on every level set of an implementable decision. Rochet [21] presents the generalization of necessary and sufficient conditions for implementability. However, he does not use these conditions to treat a non-single crossing case and he does not characterize the associate optimality conditions (as we do here in a specific setup).

\(^{10}\) Araujo and Moreira [1] study moral hazard principal-agent problems relaxing the standard conditions that make the agent’s problem concave. They provided a generalization of the first-order approach. This paper follows the same route for adverse selection problems.
**Taxation Principle**

The “Taxation Principle” establishes that, given an implementable direct mechanism \((x(\cdot), t(\cdot))\), there is a non-linear tariff \(T(\cdot)\) that implements \(x(\cdot)\):

\[
T(\xi) = t(\theta),
\]

where \(\xi \in x(\theta)\). Reciprocally, given a continuous non-linear tariff \(T(\cdot)\), we can define the agent’s optimal decision such that

\[
x(\theta) \in \arg \max_{\xi \in X} \{v(\xi, \theta) + T(\xi)\}
\]

and transfer \(t(\theta) = T(x(\theta))\). In this case, we say that \(x(\cdot)\) is implemented by \(T(\cdot)\). See [9] for a complete exposition on the topic.

**Remark 1** By Lemma 1 (i), if \(T\) implements \(x\), then \(T\) is a.e. differentiable and

\[
v_x(\xi, \theta) + T'(\xi) = 0,
\]

for all \(\xi \in x(\theta)\), where \(T'(\xi)\) is the marginal tariff. Hence, pooling types must present identical marginal utility when choosing the same decision, which is the content of Theorem 1 (see the left graph of Figure 2).^{11}

### 4.2 Necessary conditions for optimality

The next theorem is one of our main results since it gives the necessary optimality condition associated to the condition (UC), i.e., it characterizes the critical U-shaped parts. Thus, let us define the **U-shaped problem** as the maximization of the expected virtual surplus subject to conditions (M) and (UC):

\[
\max_{x(\cdot)} E[f(x(\cdot), \cdot)] \quad \text{(UP)}
\]

\[
s.t. \ (M) \ \text{and} \ (UC).
\]

Let \(x^*\) be the optimal solution of Problem (UP).

**Theorem 2 (Critical U-shaped curve)** Suppose that \(x^*\) is continuous and strictly monotonic at \(\theta\) and \(\hat{\theta}\) such that \(\xi = x^*(\theta) = x^*(\hat{\theta})\). Then,

\[
\frac{f_x(\xi, \theta)}{v_{x\theta}(\xi, \theta)p(\theta)} = \frac{f_x(\xi, \hat{\theta})}{v_{x\hat{\theta}}(\xi, \theta)p(\hat{\theta})}.
\]

^{11} This is also called the demand-profile approach. See [23] for an exposition of the two approaches to dealing with screening problems: the parametric-utility and the demand-profile approaches.
To prove this theorem it is enough to make admissible perturbations $h$ around $\theta$ of the optimum solution $x^*$ to perform the variational calculus (see [14] for this type of approach). Let $[\theta_1, \theta_2]$ be a small interval around $\theta$ in the U-shaped part. Define the space of admissible perturbations by:

$$H = \{ h : [\theta_1, \theta_2] \to \mathbb{R}; h(\theta_1) = h(\theta_2) = 0 \text{ and } x = x^* + h \text{ satisfies (M) and (UC)} \}.$$ 

Figure 3 shows a typical perturbation $h \in H$ where $x_0$ is the vertical axis.

![U-shaped admissible perturbations](image)

Fig. 3. U-shaped admissible perturbations.

Take $x = x^* + h$. In order to be admissible, the function

$$U(\xi, \theta, \hat{\theta}) = v_{x}(\xi, \theta) - v_{x}(\xi, \hat{\theta})$$

should be identical to zero for $\xi = x(\theta) = x(\hat{\theta})$ with $\theta \in [\theta_1, \theta_2]$ and $\hat{\theta} \in [\hat{\theta}_2, \hat{\theta}_1]$. Taking the derivative with respect to $\hat{\theta}$, we see that $U_{\hat{\theta}}(\xi, \theta, \hat{\theta}) = -v_{x\theta}(\xi, \hat{\theta}) < 0$. By the implicit function theorem, for each $(\xi, \theta, \hat{\theta})$ such that $U(\xi, \theta, \hat{\theta}) = 0$, there exists a neighborhood of $(\xi, \theta)$ where the conjugate type $\hat{\theta}$ can be written as a continuously differentiable function of $(\xi, \theta)$: $\hat{\theta} = \varphi(\xi, \theta)$.

To simplify the exposition, let us suppose that $x^*$ and $h$ are differentiable on $[\theta_1, \theta_2]$ and denote the derivative with respect to $\theta$ by a dot over the function. Let us consider only the part of the principal’s objective functional affected by the admissible perturbation:
\[ G(h) = \int_0^{\theta_2} g(x(\theta), \theta) d\theta + \int_{\varphi(x(\theta_1), \theta_1)}^{\varphi(x(\theta_2), \theta_2)} g(x(\theta), \theta) d\theta \]
\[ = \int_0^{\theta_2} [g(x(\theta), \theta) - g(x(\theta), \varphi(x(\theta), \theta))(\varphi_x(x(\theta), \theta)\dot{x}(\theta) + \varphi_{\theta}(x(\theta), \theta))] d\theta \]

where \( g(\xi, \theta) = f(\xi, \theta)p(\theta) \) and the last equality is a simple change of variable.

Taking the Gateaux differential at 0 in the direction \( h \), we must have:
\[ \delta_h G(0) = \int_0^{\theta_2} [g_x h - \hat{g}((\varphi_{xx} \dot{x}^* + \varphi_{x\theta})h + \varphi_{\theta}^* h) - (\hat{g}_x + \hat{g}_{\theta} \varphi_x^*)(\varphi_{xx} \dot{x}^* + \varphi_{\theta}^*)h] d\theta = 0, \]

where we are omitting the arguments of the functions and the hat means computation at the conjugate type. Integrating by parts \( \int_{\theta_1}^{\theta_2} (\hat{g}\varphi_x) h d\theta \) and plugging it into the previous equation we get
\[ \int_{\theta_1}^{\theta_2} [g_x - \hat{g}_x \varphi_{\theta}] h d\theta = 0 \]

(see the Appendix A). Again, by the implicit function theorem \( \varphi_{\theta} = v_{x\theta}/\hat{v}_{x\theta} \) and since \( h \) is arbitrary we have the result.

Theorem 2 has a straightforward interpretation. Notice that the marginal change in the virtual surplus due to an increase of the slope of the type-\( \theta \) agent’s rent at a given decision \( \xi \) is exactly \( f_x(\xi, \theta)/v_{x\theta}(\xi, \theta) \) (i.e., the ratio between the allocative distortion and marginal rent). Hence, Theorem 2 simply says that at the optimal U-shaped part these marginal increases weighted by the density should be equalized across pooling types. Combing this condition and (UC), we can generally find a critical U-shaped part. See Appendix D for a derivation of Theorem 2 using the demand-profile approach.

5 Sufficient Conditions

The aim of this section is to present a specific setup where the necessary conditions for implementability (Lemma 1 and Theorem 1) and optimality (condition (6) and Theorem 2) are indeed sufficient to characterize the second-best solution. Let us start by the definition of the space of contracts where we will provide this characterization.

12 In Appendix B we present another derivation of Theorem 2 through a Lagrangian approach that does not depend on the differentiability of the optimal decision and perturbation. In particular, we will show that these ratios are equal to the Lagrangian multiplier of the condition (UC) (see equation (12)).
From the SMC case (e.g., [19]) one would expect the optimal decision to be continuous. However, without single-crossing this might not be the case as we will see in what follows. Without the continuity property, a weaker requirement is the convex-valued correspondence property. By the maximum theorem, an implementable decision is a non-empty and upper semi-continuous compact valued correspondence. In this case, we can always select a subcorrespondence that is a càdlàg function. 13 Reciprocally, given a càdlàg decision \( x \) we define the extended version of \( x \) as the correspondence whose values are the intervals between the right- and left-hand limits of \( x \) at every \( \theta \), i.e., the extended \( x \) is a convex-valued correspondence.

To avoid further technical difficulties due to a richer space of possible deviations for the principal and the agent and, at the same time, to allow for discontinuities, we consider the following:

**Definition 2 (Space of decisions)** We call \( \mathcal{X} \) the space of extended version of càdlàg decisions.

Notice that \( \mathcal{X} \) is restrictive once it enlarges the strategic behavior of the agent who is now endowed with more options to mimic other types. The principal’s welfare would increase with more control over the out of contract realizations (i.e., implementable decisions without convex values). In Appendix D we provide an example with implementable decisions not in \( \mathcal{X} \) that improve the welfare by crossing (discontinuously) the limit curve \( x_0 \) many times.

### 5.1 A class of U-shaped problems

From the discussion on Figure 1, the necessary conditions may not be sufficient in general for implementability. Indeed, we argued that there may exist monotonic decisions that do not cross the limit curve \( x_0 \) (so trivially satisfy conditions (M) and (UC)) and that are not implementable. Therefore, to at the same time deal with a situation where global IC conditions matter and to have a tractable case we will assume two more conditions besides assumptions A1 and A2. First, we focus our analysis on a case where the solution \( x_1 \) of Problem (RP) has a unique minimum (unique peak).

---

13 Christensen [6] shows that any such correspondence has a minimal sub-correspondence with the same properties which is single-valued and continuous at a Gδ (i.e., a countable intersection of open sets - see [25]) and dense subset of the domain. Given that in our case this sub-correspondence has a closed graph and is locally monotonic at the points of continuity in the domain (see the proof of Lemma 1 (ii)), we can assume that \( x \) is a càdlàg function without loss of generality.

14 Notice that for the private value case (\( u_{x\theta} = 0 \)), it is easy to check that the first-best solution \( x^{FB} \) is U-shaped. Moreover, under A1 and A2, \( x^{FB} \) is inbetween
C1. The relaxed solution \( x_1 \) has a unique minimum and increasingly crosses \( x_0 \). Moreover, \( x_1(\hat{\theta}) < x_1(\bar{\theta}) \).

Let \( x_u : [\theta, \theta_u] \to X \) be the continuous decision function defined by equation (8), where \([\theta, \theta_u] \subset \Theta \) and \( x_u(\theta) = x_u(\theta_u) \). To complete our specific case, we assume that the following condition holds.

C2. The function \( x_u \) is the unique solution of equation (8) that satisfies condition (M).\(^{15}\)

Since the left and right hand sides of equation (8) must have the same sign, it is easy to see that, from assumptions A1-2, conditions C1-2 and Theorem 2, \( x_u \) must lie inbetween \( x_0 \) and \( x_1 \). In particular, \( x_1 \) does not satisfy condition (UC) and the problem raised in Figure 1 occurs for \( x_1 \).

Let \( x^* \) be the optimum for the U-shaped problem (UP) in the space \( X \). Appendix B\(^{16}\) provides necessary and sufficient conditions for \( x^* \) for the case where C1-2 hold. Let us only describe this solution here. As we saw in Section 4 standard variational calculus implies that: (i) \( x^* \) is the relaxed solution \( x_1 \) in every interval where conditions (M) and (UC) do not bind (from condition (6)); and (ii) \( x^* \) is the critical U-shaped curve \( x_u \) in every interval where only condition (UC) binds (from Theorem 2 and condition C2). Using this observation and the fact that \( x_u \) is inbetween \( x_0 \) and \( x_1 \), we have two natural cases to consider:

- \( x^*(\hat{\theta}) < x^*(\bar{\theta}) \). Then, \( x^* \) must be \( x_u \) in the first part of \( \Theta \) and jumps “vertically” to be \( x_1 \) in the second part of \( \Theta \). We then need to find the optimal jump point \( \theta_1 \) (vertical ironing procedure) to determine a solution. That is, we must find \( \theta_1 \) such that

\[
x^*(\theta) = \begin{cases} 
\tilde{x}_u(\theta, \theta_1), & \text{if } \theta < \tilde{\theta}_1 \\
x_u(\theta), & \text{if } \theta \in [\tilde{\theta}_1, \theta_1) \\
x_1(\theta), & \text{if } \theta \geq \theta_1,
\end{cases}
\]

where \( \tilde{\theta}_1 \) is discrete pooled with \( \theta_1 \) and \( \tilde{x}_u(\theta, \theta_1) \) is the curve discrete pooled

the curves \( x_0 \) and \( x_1 \) with no distortion at the bottom type \( \hat{\theta} \) and at the crossing point between \( x_0 \) and \( x_1 \). Therefore, condition C1 represents a natural case where \( x_1 \) crosses increasingly \( x_0 \).

\(^{15}\) In Appendix B we provide sufficient conditions on preferences and distribution of types to ensure this uniqueness.

\(^{16}\) There we use an approach based on a reformulation of the space of decisions proposed by Nödeke and Samuelson [20] and built on an insight of Goldman, Leland and Sibley [8]. The reformulation consists in writing decisions as type assignment functions (i.e., the inverse functions of the decisions) instead.
with the vertical line $\theta = \theta_1$ such that condition (UC) holds. Notice that the requirement of being in the space $X$ is important to define $x^*$ as $\tilde{x}_u(\cdot, \theta_1)$ in the interval $[\theta, \hat{\theta}_1]$. We show that $\theta_1$ must satisfy condition (VI) in Appendix B which is a generalization of Theorem 2. In the left graph of Figure 4 the thick line represents this solution candidate.

- $x^* (\theta) \geq x^* (\hat{\theta})$. Then, $x^*$ must be $x_1$ in the first part of $\Theta$ and jumps “horizontally” to be $x_u$ in the middle of $\Theta$. We then need to perform the optimal bunching (horizontal ironing procedure) to determine a solution. That is, we must find $\theta_2$ such that

\[
x^*(\theta) = \begin{cases} 
  x_1(\theta), & \text{if } \theta < \theta_2 \\
  \bar{\xi}, & \text{if } \theta \in [\theta_2, \hat{\theta}_3) \cup [\theta_3, \bar{\theta}]
\end{cases}
\]

where $\hat{\theta}_3 > \theta_2$ is discrete pooled with $\theta_3$ such that condition (UC) holds and $\bar{\xi} \geq x_1(\theta_2)$ is the bunching level. We show that the optimal bunching $\bar{\xi}$ (horizontal ironing procedure) must satisfy condition (HI) in Appendix B which is a generalization of the “ironing principle” of the standard literature. In the right graph of Figure 4 the thick line represents this solution candidate.\(^{17}\)

\[
\begin{align*}
\xi &
\end{align*}
\]

\(\theta \hat{\theta}_1 \theta_1 \theta_u \hat{\theta} \bar{\theta} \theta_3 \hat{\theta}_3 \theta_3 \theta_0 \theta \)

\(\xi \bar{\xi} x_u x_1 x_0 x_u \bar{\xi} x_u x_1 x_0 \)

\(9\)

Fig. 4. The solutions of Problem (UP).

The solution of Problem (UP) is the candidate that gives the highest expected payoff for the principal. Since Problem (UP) is evidently more relaxed than

\(^{17}\)There are two degenerate cases of the right graph of Figure 4: (i) $\theta_2 = \theta$ and $\bar{\xi} > x_1(\hat{\theta})$ such that $x^*$ is the composition of bunching intervals and the U-shaped part; (ii) $\theta_3 = \hat{\theta}_3$ such that $x^*$ is the composition of the relaxed solution and a unique bunching interval.
the second-best problem (SB) in \( \mathcal{X} \), to show that \( x^* \) is the second-best solution it is enough to show that \( x^* \) is implementable. The following theorem, whose proof is in Appendix B, concludes that, under conditions C1 and C2, this is the case.

**Theorem 3 (Second best decision)** Assume that A1-2 and C1-2 hold. The solution of Problem (UP) is the solution of Problem (SB) when decisions are restricted to the space \( \mathcal{X} \).

**Algorithm.** Figure 4 suggests also an algorithm to compute the second-best decision:

- First, draw the relaxed solution \( x_1 \) (satisfying condition C1) and the critical U-shaped solution \( x_u \) (satisfying condition C2).
- Second, find the best ironing procedure such that conditions (M) and (UC) are satisfied. There are two possibilities: the vertical ironing principle shown in the left graph of Figure 4 or the horizontal ironing principle shown in the right graph of Figure 4.
- Third, compute the principal’s payoff at each of these solutions to obtain the second-best solution.

Summarizing, one has to adjust optimally first the level of the vertical ironing \((\theta_1)\) and then the level of horizontal ironing \((\zeta)\) on the curves \( x_1 \) and \( x_u \) such that conditions (M) and (UC) hold in order to find the second-best solution. If \( \zeta = \pi \) we get the solution presented in the left graph of Figure 4. Otherwise, we get the solution presented in the right graph of Figure 4.

**Economic interpretation.** The relaxed solution \( x_1 \) represents the best trade-off between rent extraction and distortion when one takes into account only the local first-order condition of the IC constraint. When \( x_1 \) is not implementable, we have to characterize the least distorted implementable decision compared to \( x_1 \). Under our assumptions, there are two extreme cases to consider. In the first distortions of high types are relatively more costly which favors no further distortion for them and new distortions for low types. These distortions are related to the binding monotonicity and U-shaped conditions. Since the monotonicity condition implies that local IC constraints are upward (downward) binding in \( \mathcal{CS}_- \) (\( \mathcal{CS}_+ \)), the critical U-shaped curve, \( x_u \), is upward (downward) distorted when compared to \( x_1 \) in \( \mathcal{CS}_- \) (\( \mathcal{CS}_+ \)). Theorem 2 implies that the trade-off between rent extraction and distortion measured in the agent’s marginal rent unit are equalized between pooling types. This provides our first optimal decision (the left graph of Figure 4).

On the other hand, if the distortion costs for low types are relatively high, there is no further distortion for these types. However, high types may be partially or completely pooled (the right graph of Figure 4). In the former case, some intermediate types may be discretely pooled, i.e., they may present partial
separation. The interpretation is similar to the classical “ironing principle” if we add the new U-shaped feature: full separation for high (low) types may induce costly pooling for low (high) types.\textsuperscript{18}

One of the new feature of the optimal decision is the possibility of discrete pooling, i.e., isolated types taking the same decision. In the standard literature there are only two possibilities: separation or continuous pooling. In the former the agent’s type is known \textit{ex-post} by the principal and in the last the principal knows a range of types where the agent belongs to. When the SMC fails, discrete pooling may arise and the principal may not know the true type between two types or two ranges of types (i.e., disconnected continuous pooling sets). Moreover, middle types are assigned to lowest decision levels, but enjoy positive rents. More interestingly, when there is discrete pooling, there is no distortion for the type where $x_1$ crosses $x_0$.

Unlike the single-crossing case where continuous pooling implies discontinuity only for the derivative of the optimal decision function with respect to type, it may also be discontinuous as well. The reason is that the hazard rate of types jumps upward at the transition point from the discrete pooling region to the separation region (see the discussion of the demand-profile approach in Appendix D). That is, the trade-off between rent extraction and distortion changes discontinuously from discrete pooling to separation.

6 Extension and Concluding Remarks

Before giving the final concluding remarks, let us present a quick discussion on the countervailing incentives extension. The model studied in this paper rules out exclusion of types, i.e., the agent’s reserve utility is given by

$$\nabla(\theta) = v(0, \theta),$$

(RU)

assumed to be constant and equal to zero. However, even when it is not constant, the IR constraint is equivalent to checking it only at one of the extreme

\textsuperscript{18}In Appendix C, we provide a monopolistic labor market example. In the model workers have two productivity parameters that are mixed through a verifiable signal such that the SMC is violated. We perform some numerical examples consistent with the solutions characterized in this paper.
types under the SMC.\footnote{For instance, if $v_{x\theta} < 0$ then Lemma 1 (i) implies that}

Jullien [11] studied a more general case where $\bar{V}(\theta)$ does not necessarily satisfy (RU). His main insight is that, in the presence of countervailing incentives, upward and downward distortions compared to the first-best may happen. Under $\text{CS}_-$ ($v_{x\theta} < 0$), upward distortions better prevent low types to overstate their types, because they have higher marginal benefit from the decision. When the reserve utility is large for low types, the contracts offered to them may become attractive for high types. If the reserve utility is large enough, then downward distortions better prevent understatement of high types.

However, countervailing incentives can also occur when the SMC does not hold even if the agent’s reserve utility does satisfy (RU). Like in [11], the derivative of the difference between the marginal rent and the marginal reserve utility with respect to type can change sign when the SMC fails.

Under countervailing incentives, the necessary conditions of Theorem 2 remain the same when the Lagrangian multiplier of the IR constraint, $\gamma$, is constant in $\theta$. Otherwise, there may be a stronger conflict between rent extraction and efficiency. This is because the equalization of the marginal virtual benefit of increasing the rent profile between pooling types is affected by the shadow value associated with a reduction in the participation level ($\gamma$). [11] made assumptions (potential separation and homogeneity) to avoid this extra difficulty under single-crossing. This potential conflict is related to the local IC constraints in his case whereas in ours is to the global IC constraints.

This paper provides a first treatment of one-dimensional screening problems without the SMC. Although it leads to a non-concave principal-agent problem, we were able to deal with it mathematically and gave an economically meaningful solution. We first derive new necessary global implementability conditions which are translated into the U-shaped conditions joint with their necessary optimality conditions. These and the local IC conditions are sufficient for implementability and optimality in a specific set up.

There are several directions we may pursue. One is to cover other possible shapes for $x_0$ and $x_1$. The first interesting case is when $x_1$ is decreasing contained in $\text{CS}_-$, but the global IC does not hold (see Figure 1). Although, this seems to be a natural case to study, it leads to extra difficulties avoided in this paper. Indeed, besides condition (UC), there is at least one extra class of
global IC constraints that can bind: the marginal rent identity. Theorem 1A in the Appendix A derives this extra necessary condition. Another direction is to explore these new necessary implementability and optimality conditions for multidimensional screening problems. Finally, one may want to investigate the consequences of the lack of the monotonicity property of implementable (and optimal) contracts for the empirical literature on hidden information models.
Appendix A. Proofs (necessity)

Proof of Lemma 1. (i) The IC constraint (1) implies that, for $\theta > \hat{\theta}$,

$$\frac{v(x(\theta), \theta) - v(x(\theta), \hat{\theta})}{\theta - \hat{\theta}} \geq \frac{V^x(\theta) - V^x(\hat{\theta})}{\theta - \hat{\theta}} \geq \frac{v(x(\hat{\theta}), \theta) - v(x(\hat{\theta}), \hat{\theta})}{\theta - \hat{\theta}}.$$

Since $v$ is $C^3$ and $x$ is bounded, the above inequality shows that $V^x$ is a Lipschitz function. Then, $V^x$ is differentiable a.e. and

$$\frac{d}{d\theta} V^x(\theta) = v_{\theta}(x(\theta), \theta)$$

a.e. The fundamental theorem of calculus for absolutely continuous functions gives (i).

(ii) By (GIF), we know that the global IC constraint for $x$ is equivalent to

$$\Phi^x(\theta, \hat{\theta}) = \int_{\theta}^{\hat{\theta}} \left[ \int_{x(\tilde{\theta})}^{x(\hat{\theta})} v_{x\tilde{\theta}}(\tilde{x}, \tilde{\theta}) d\tilde{x} \right] d\tilde{\theta} \geq 0,$$

for all $\theta, \hat{\theta} \in \Theta$. Let $I \subset \Theta$ be an interval such that $v_{x\tilde{\theta}}(x(\theta), \theta) > 0$ for all $\theta \in I$. We claim that $x$ is non-decreasing in $I$. Otherwise, there exists $\theta \in I$ such that the set $I_{\theta} = \{ \hat{\theta} \in I; \hat{\theta} > \theta$ and $x(\hat{\theta}) < x(\theta) \}$ is a non-empty set. By the right continuity, $I_{\theta}$ contains a non-empty interval $J$. In this case, it is easy to see that $\Phi^x(\theta, \hat{\theta}) < 0$ for all $\hat{\theta} \in J$, which is a contradiction. The other case is analogous.

Proof of Theorem 1. We will prove a more general result than Theorem 1. In particular, item (ii) of Theorem 1A below is a kind of dual necessary implementability condition of item (i) not explored in this paper.

Theorem 1A. Let $x$ be an implementable decision and $\theta, \hat{\theta} \in (\underline{\theta}, \overline{\theta})$ be such that $\Phi^x(\theta, \hat{\theta}) = 0$.

(i) If $x$ is strictly monotonic and continuous at $\hat{\theta}$, then

$$v_x(x(\hat{\theta}), \hat{\theta}) = v_x(x(\hat{\theta}), \theta).$$

(ii) If $x$ is continuous at $\theta$, then

$$v_{\theta}(x(\hat{\theta}), \theta) = v_{\theta}(x(\theta), \theta).$$

Proof of Theorem 1A. (i) Suppose that $\hat{\theta} < \theta$ and $x(\hat{\theta}) > x(\theta) = \min_{\hat{\theta} \in [\theta, \theta]_{\hat{\theta}}} x(\hat{\theta})$
(the other cases are analogous). Define the local inverse of $x$ at $\hat{\theta}$ by $\psi$. Applying Fubini’s theorem, (GIF) can be written, with some abuse of notation, as

$$\Phi^x(\theta, \xi) = \int_{\xi}^{x(\theta)} \int_{\psi(\xi)}^{\theta} v_x(\xi, \hat{\theta}) d\hat{\theta} d\xi.$$  

Since $\xi = x(\hat{\theta})$ is a minimum for $\Phi^x(\theta, \cdot)$, taking the derivative with respect to $\xi$ at $x(\hat{\theta})$ and equating to zero, we obtain the result.

(ii) Observe that if we fix $\hat{\theta}$, $\theta$ is a minimum for $\Phi^x(\cdot, \hat{\theta})$. Thus, the result is again a direct consequence of the first-order condition.

Turning back to the proof of Theorem 1, suppose that $x(\theta) = x(\hat{\theta})$. Interchanging the parameters and using the definition of $\Phi^x$ in (GIF), we have that $\Phi^x(\theta, \hat{\theta}) = -\Phi^x(\hat{\theta}, \theta)$. The implementability of $x$ implies that $\Phi^x$ is non-negative and hence $\Phi^x(\theta, \hat{\theta}) = 0$. Therefore, Theorem 1 follows from Theorem 1A (i).

Proof of Theorem 2. The only remaining part is the integration by parts:

$$\int_{\theta_1}^{\theta_2} (\hat{g} \varphi_x) h d\theta = \int_{\theta_1}^{\theta_2} (\hat{g} \varphi_x) h d\theta$$

$$= \int_{\theta_1}^{\theta_2} [(\hat{g} \varphi_x) + \dot{\hat{g}}(\varphi_x \dot{x}^* + \varphi_{x})] d\theta.$$

Plugging this last equation into the first-order condition $\delta_h G(0) = 0$ in the text, for all direction $h$,

$$\int_{\theta_1}^{\theta_2} [g_x - \dot{\hat{g}} \varphi_x] h d\theta = 0.$$

Appendix B. Proofs (sufficiency)

As in the standard problem, we will construct a relaxed problem and iron it. First, we will introduce a reformulation of the space of decisions proposed by Nöldeke and Samuelson [20]. The reformulation consists in writing decisions as type assignment functions (i.e., the inverse functions of the decisions) instead.

Under A2, decision functions in $\mathcal{X}$ can have more than one inverse. Figure 5 shows how to define these inverse functions in this case. The left graph shows a typical $x \in \mathcal{X}$ and the right one gives their two respective inverses, $\psi_b$ and $\psi_s$. Figure 5 suggests that we define them as extended càdlàg versions in the whole domain $[\underline{x}, \bar{x}]$ (as in Definition 2):

As we will see below, it turns out that this technique is helpful for the non-single crossing case because it avoids the difficulties of the Hamiltonian approach to deal with the global IC constraints. More precisely, it transforms
the principal’s problem into a pointwise maximization problem.

**The reformulation of the problem.** Our goal now is to rewrite Problem (UP) in terms of type assignment functions. Let \( x \in \mathcal{X} \), implemented by \( T(\cdot) \), and \( \psi_b \) and \( \psi_s \) their inverses. By Remark 1, \( T'(\xi) = -v_x(\xi, \psi_b(\xi)) \) when \( \psi_b(\xi) > \theta \) and \( T'(\xi) = -v_x(\xi, \psi_s(\xi)) \) when \( \psi_s(\xi) < \theta \), a.e. (Using the terminology of [20], the pair \((T, \psi_i)\) is consistent for each \( i = b, s \)).

Using Fubini’s theorem, the principal’s objective function (5) at \( x \) in terms of type assignment function becomes

\[
E[u(x(\cdot), \cdot) - T(x(\cdot))] = E[u(\pi, \cdot)] - T(\pi) - E\left[ \int_{x(\cdot)}^\pi (u_x(\xi, \cdot) - T'(\xi))d\xi \right] \\
= E[u(\pi, \cdot)] - T(\pi) + \int_\pi \left[ F(\xi, \psi_b(\xi)) - F(\xi, \psi_s(\xi)) \right]d\xi,
\]

where

\[
F(\xi, \theta) = \int_\theta^\xi [u_x(\xi, \tilde{\theta}) + v_x(\xi, \theta)]p(\tilde{\theta})d\tilde{\theta}
\]

is the cumulative marginal virtual surplus.

Let us rewrite the monotonicity and U-shaped conditions in terms of type assignment functions. The monotonicity condition (M) is trivially equivalent to:

\[
\psi_b \text{ and } \psi_s \text{ are non-increasing and non-decreasing, respectively.} \quad (\text{M}')
\]

The following lemma says that condition (UC) as a constraint in Problem (UP) is equivalent to (UC’) below in the type assignment approach.\(^{20}\)

\(^{20}\)Roughly speaking we are writing a version of the U-shaped condition valid where it binds. This approach resembles the “double relaxed program” of Rogerson [24]. Although studying a moral hazard problem, his idea was to transform an “equality constraint” into an “inequality constraint”. 

23
Lemma 2 (U'-shaped condition) Assume A2. Let \( x \in X \) be an implementable decision with type assignments \( \psi_b \) and \( \psi_s \). If \( \xi \in X \) is a point of continuity for both \( \psi_b \) and \( \psi_s \), then

\[
\psi_s(\xi) < \overline{\theta} \implies v_x(\xi, \psi_b(\xi)) \leq v_x(\xi, \psi_s(\xi)).
\]  

(UC')

Proof. If \( \psi_s(\xi) < \overline{\theta} \), then condition (UC') holds with equality when \( \psi_b(\xi) > \overline{\theta} \) due to Theorem 1. Suppose that \( \psi_b(\xi) = \overline{\theta} \). First, take \( \xi \) the infimum of \( \xi \) such that \( \psi_b(\xi) = \overline{\theta} \) and \( \psi_s(\xi) < \overline{\theta} \). Thus, either \( \psi_b(\xi) = \psi_s(\xi) = \overline{\theta} \) (i.e., the graph of \( x \) belongs to \( CS_+ \)) and then (UC') trivially holds with equality. Or \( \xi \) is the point where \( \psi_b(\xi) \) starts to be the constant function \( \theta \) (i.e., \( \xi = \overline{x}(\overline{\theta}) \)). Notice that condition (M') implies that \( \psi_b(\xi) = \theta \), for all \( \xi > \xi \). By continuity, condition (UC') holds with equality for all \( \xi \) below \( \xi \) such that \( \psi_b \) and \( \psi_s \) are continuous at \( \xi \). A2 and Remark 2 below imply that the function

\[
\gamma(\xi) = v_x(\xi, \psi_s(\xi)) - v_x(\xi, \overline{\theta})
\]

is increasing. Since \( \lim_{\xi \uparrow \xi} \gamma(\xi) = 0 \), \( \gamma(\xi) > 0 \) for all \( \xi > \xi \). This is equivalent to say that condition (UC') holds with strict inequality for all \( \xi > \xi \). \( \blacksquare \)

Observe that, since \( v_{\theta} \leq 0 \), the IR constraint is equivalent to \( T(\overline{\pi}) + v(\overline{\pi}, \overline{\theta}) = 0 \). We can then define the following relaxed version of the U-shaped problem (with some abuse of notation we use the same label for both problems, the original and this relaxed version):

\[
\max_{\psi_b, \psi_s} \int_{\xi} [F(\xi, \psi_b(\xi)) - F(\xi, \psi_s(\xi))]d\xi
\]

s.t. (UC') and (M').

(UP)

Let \( \psi_b(\cdot) \) and \( \psi_s(\cdot) \) be any feasible pair of type assignment functions for Problem (UP). Let \( \xi \in X \) the level that \( \psi_s(\cdot) \) hits \( \overline{\theta} \), i.e., the U-shaped part ends. By conditions (M') and (UC'), we must then have

\[
\begin{align*}
& v_x(\xi, \psi_b(\xi)) \leq v_x(\xi, \psi_s(\xi)), \text{ for all } \xi \leq \xi, \\
& \psi_s(\xi) = \overline{\theta}, \text{ for all } \xi \geq \xi.
\end{align*}
\]

Therefore, we can parametrize the pair of feasible type assignment functions of Problem (UP) by \( \xi \). Using this observation, we now proceed analogously to the standard literature by relaxing the constraint (M') from Problem (UP). For each \( \xi \in X \) we then define the problems that constitute the relaxed version of Problem (UP), the pointwise relaxed U-shaped problems (RUP):

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(i) for $\xi \leq \xi$, $(\psi_b, \psi_s) \in [\underline{\theta}, x_{\theta}^{-1}(\xi)] \times [x_{\theta}^{-1}(\xi), \overline{\theta}]$ solves
\[
\max F(\xi, \psi_b) - F(\xi, \psi_s) \\
\text{s.t. } v_x(\xi, \psi_b) \leq v_x(\xi, \psi_s)
\] (RUP)

(ii) for $\xi \geq \xi$, $\psi_s = \underline{\theta}$ and $\psi_b \in [\underline{\theta}, x_{\theta}^{-1}(\xi)]$ solves
\[
\max F(\xi, \psi_b) - F(\xi, \underline{\theta}).
\] (RUP)

Observe that formally this relaxed version of the U-shaped problem depends on $\xi$. With some abuse of notation we do not make this dependence explicit in what follows. Notice also that compactness and continuity trivially imply that Problem (RUP) always has a solution. Let $(\psi_b^{\xi}, \psi_s^{\xi})$ be the solution of Problem (RUP).

The strategy to characterize the solution of Problem (UP) is based on three steps. First, for each fixed $\xi$ we characterize the solutions of Problem (RUP). Second, we deal with the possible lack of monotonicity of these solutions. Third, we compute the optimal $\xi$ to find the solution of Problem (UP).

**Characterization of the solution of Problem (RUP).** From now on we use the convention: $x_{\theta}^{-1}(\xi) = \underline{\theta}$ for all $\xi \geq x_0(\underline{\theta})$ and $x_{\theta}^{-1}(\xi) = \overline{\theta}$ for all $\xi \leq x_0(\overline{\theta})$.

(1) $\xi \leq \xi$. Dropping the Lagrangian multipliers of the boundary conditions for simplicity, we define the Lagrangian of Problem (RUP) as:

$$L(\xi, \psi_b, \psi_s, \mu) = F(\xi, \psi_b) - F(\xi, \psi_s) - \mu(v_x(\xi, \psi_b) - v_x(\xi, \psi_s)),$$

where $\mu \geq 0$ is the Lagrangian multiplier of condition (UC').

Using the fundamental theorem of calculus and the definition of the cumulative marginal virtual surplus (10) we get

$$L(\xi, \psi_b, \psi_s, \mu) = \int_{\psi_b}^{\psi_s} [f_x(\xi, \tilde{\theta})p(\tilde{\theta}) - \mu v_{x\theta}(\xi, \tilde{\theta})]d\tilde{\theta}. \quad \text{(L)}$$

Taking the derivative with respect to $\psi_i$ we have the following necessary optimality conditions:

$$0 \geq f_x(\xi, \psi_i)p(\psi_i) - \mu v_{x\theta}(\xi, \psi_i) \quad \text{(11)}$$

with equality for $\psi_b \in (\underline{\theta}, x_{\theta}^{-1}(\xi))$ or $\psi_s \in (x_{\theta}^{-1}(\xi), \overline{\theta})$ and the usual complementary slackness condition: $0 = \mu(v_x(\xi, \psi_b) - v_x(\xi, \psi_s))$.

In particular, we obtain the already known necessary optimality conditions:
• $\mu > 0$, $\psi^b_\xi \in (\theta, x_0^{-1}(\xi))$ and $\psi^s_\xi \in (x_0^{-1}(\xi), \theta) \implies$

$$\frac{f_x(\xi, \psi^b_\xi)}{v_x(\xi, \psi^b_\xi)p(\psi^b_\xi)} = \mu = \frac{f_x(\xi, \psi^s_\xi)}{v_x(\xi, \psi^s_\xi)p(\psi^s_\xi)}$$  \hspace{1cm} (12)

and (UC) by the complementary slackness condition, which are exactly the content of Theorem 2.

• $\mu = 0$ and $\psi^b_\xi = \psi^s_\xi \implies f_x(\xi, \psi^b_\xi) = 0$, which is exactly equation (6) (analogously for $\psi^s_\xi \in (x_0^{-1}(\xi), \theta)$).

• $\mu > 0$ and $\psi^b_\xi = \theta \implies \psi^s_\xi = \psi(\xi)$, where $\psi$ be the conjugate of the vertical line $\theta = \theta$, i.e., the curve in $CS_+$ defined by

$$v_x(\xi, \psi(\xi)) = v_x(\xi, \theta),$$

for all $\xi \in X$. Analogously, $\mu > 0$ and $\psi^s_\xi = \theta \implies \psi^b_\xi = \psi(\xi)$, where $\psi$ is the conjugate of the vertical line $\theta = \theta$.

The following remark shows that conjugations of vertical lines are decreasing under A2.

**Remark 2** $A2$ implies that, for each fixed $\theta \in \Theta$, the implicit solution $\psi(\xi) \neq \theta$ of

$$v_x(\xi, \psi(\xi)) - v_x(\xi, \theta) = 0$$

is always decreasing. Indeed, by the implicit function theorem and $v_x > 0$,

$$\frac{d\psi}{d\xi}(\xi) = \frac{v_{xx}(\xi, \theta) - v_{xx}(\xi, \psi(\xi))}{v_{x\theta}(\xi, \psi(\xi))} < 0.$$ 

By Remark 2, $\overline{\psi}(\xi)$ and $\psi(\xi)$ are decreasing functions, and eventually hit the vertical lines $\theta = \theta$ and $\theta = \theta$ when $\xi$ goes to 0 and to $\theta$, respectively. Notice that these hitting points are the same and denoted by $\xi \in X$. Let us extend $\overline{\psi}(\xi)$ (resp. $\overline{\psi}(\xi)$) as $\theta$ (resp. $\theta$), for all $\xi \leq (\text{resp.} \geq) \theta$.

(2) $\xi \geq \xi$. Here the analysis is more simple here. It is equivalent to make $\psi = \theta$ and $\mu = 0$ in the above Lagrangian (L). Therefore, $\psi^s_\xi$ is characterized by (6).

The following lemma gives a complete characterization of the solution of Problem (RUP). We use the following conventions for this lemma. Let us assume that $\underline{x} = \min_{\theta \in \Theta} x_1(\theta)$ and $\overline{x} = \max_{\theta \in \Theta} x_1(\theta)$ without loss of generality. Let $\psi^b_\xi$ and $\psi^s_\xi$ be the lowest and highest inverse functions of $x_1$ if they are in $CS_-$ and $CS_+$, respectively. Notice that condition C1 implies that they are, respectively, decreasing and increasing. Let $\psi^b_\xi$ and $\psi^s_\xi$ be the type assignment functions of
Lemma 3 (Solution of Problem (RUP)) Assume A1-2 and C1-2. If $\bar{\xi} = \bar{x}$, then the optimal type assignment are:\footnote{To be consistent with the definition of type assignment functions, on the interval $[x, \xi_0]$ they would be $\psi_b = \psi_s = x_0^{-1}(\xi_0)$. Notice, however, that this alternative definition does not modify the principal’s pointwise payoff in this interval.}

(i) $\psi_b^\bar{\xi} = \psi_s^\bar{\xi} = x_0^{-1}(\xi), \forall \xi \in [\bar{x}, \xi_0]$.

(ii) $\psi_j^\bar{\xi} = \psi_j^\bar{\xi}(\xi), j = b, s, \forall \xi \in [\xi_0, \xi_1]$.

(iii) $\psi_b^\bar{\xi} = \max\{\psi_b^1(\xi), \psi_s(\xi)\} \text{ and } \psi_s^\bar{\xi} = \max\{\psi_s^1(\xi), \psi_s(\xi)\}, \forall \xi \in (\xi_1, \bar{x}]$.

If $\bar{\xi} \in X$, $\psi_b^\bar{\xi}$ and $\psi_s^\bar{\xi}$ are characterized by (i)-(iii), $\forall \xi \in [\bar{x}, \bar{\xi}]$, and:

(iv) $\psi_b^\bar{\xi} = \psi_b^1(\xi)$ and $\psi_s^\bar{\xi} = \bar{\xi}, \forall \xi \in [\bar{x}, \bar{\xi}]$.

Proof. (i) Let $V(\xi)$ be the value function of Problem (RUP). By the envelope theorem (see [17]) and the Lagrangian (L), we have that

$$V'(\xi) = \int_{V_1(\xi)}^{\psi_s^\xi(\xi)} [f_{xx}(\xi, \tilde{\theta}) - \mu(\xi)u_{x\theta}(\xi, \tilde{\theta})]d\tilde{\theta}. \quad (10)$$

By A1 and A2, $V'(\xi) \geq 0$. Notice that $V(\xi_0) = 0$ because $\psi_b^\bar{\xi}(\xi_0) = \psi_s^\bar{\xi}(\xi_0)$ (see item (ii) below). Therefore, $V(\xi) = 0$ and $\psi_b^\bar{\xi} = \psi_s^\bar{\xi} = x_0^{-1}(\xi)$, for all $\xi \in [\bar{x}, \xi_0]$.

(ii) Let $\xi \in [\xi_0, \xi_1]$. Suppose that constraint (UC') is not binding for Problem (RUP) at the solution. Then, condition (11) implies that $\psi_j^\bar{\xi} = \psi_j^1, j = b, s$. Since $x_u$ is inbetween $x_0$ and $x_1$, A2 implies that (UC') is violated in this case, a contradiction. Then, constraint (UC') must bind for Problem (RUP) at the solution. By the uniqueness of $x_u$, $\psi_b^\bar{\xi}$ and $\psi_s^\bar{\xi}$ should be the type assignment functions of $x_u$, for all $\xi \in [\xi_0, \xi_1]$.

(iii) For all $\xi \in [\xi_1, \bar{x}]$ the solution of Problem (RUP) is at the boundary. A1 and condition (11) directly imply that the maximum of Problem (RUP) is attained at

$$\psi_b^\bar{\xi} = \begin{cases} \psi_b^1(\xi), & \text{if (UC') is slack} \\ \psi_s(\xi), & \text{if (UC') is binding} \end{cases} \text{ and } \psi_s^\bar{\xi} = \begin{cases} \psi_s^1(\xi), & \text{if (UC') is slack} \\ \psi(\xi), & \text{if (UC') is binding} \end{cases},$$

for all $\xi \in [\xi_1, \bar{x}]$. Since $\psi_b^1(\xi) \geq \psi_s(\xi)$ if and only if (UC') is slack, we can write
\( \bar{\psi}_b = \max\{\psi_b^1(\xi), \bar{\psi}(\xi)\} \). Analogously, \( \psi_s^\ast = \max\{\psi_s^1(\xi), \bar{\psi}(\xi)\} \).

(iv) Now take any \( \tilde{\xi} \in X \). For each \( \xi \leq \tilde{\xi} \), by definition of Problem (RUP), the solution is characterized by (i), (ii) and (iii) above. For all \( \xi \geq \tilde{\xi} \), \( \psi_s^\ast = \bar{\theta} \) by the definition of \( \tilde{\xi} \). A1 and condition (11) imply that the maximum of Problem (RUP) is attained at \( \psi_b^\ast = \psi_b^1(\xi) \). □

From Remark 2 and Lemma 3 (iii), \( \psi_b^\ast \) does not satisfy condition (M') when \( \psi_b^\ast = \bar{\theta} \). Thus, the solution of Problem (RUP) may display non-monotonicities and to get a solution of Problem (UP) from the solution of Problem (RUP) may involve ironing. Therefore, we determine the optimal jump \( (\theta_1) \) on the solution of Problem (RUP) for each \( \tilde{\xi} \) and then find the optimal bunching \( (\bar{\xi}) \).

**Optimal jump (dealing with the lack of monotonicity).** Fix the bunching level \( \tilde{\xi} \) and consider Problem (RUP) with the constraint (M'). Let \( (\psi_b^\ast, \psi_s^\ast) \) be its optimal solution.\(^{22}\) The Hamiltonian is:

\[
H(\xi, \psi_b, \psi_s, \mu, \psi_b', \psi_s', \delta_b, \delta_s) = L(\xi, \psi_b, \psi_s, \mu) + \delta_b \psi_b' - \delta_s \psi_s',
\]

where \( \psi_i' \) is the derivative of \( \psi_i \) and \( \delta_i \geq 0 \) is the multiplier of (M') for \( i = b, s \).

The Pontryagin principle implies

\[
\dot{\delta}_b = -\partial_{\psi_b} H \quad \text{and} \quad \dot{\delta}_s = \partial_{\psi_s} H
\]

\[
0 \leq \partial_{\psi_b} H = \delta_b \quad \text{and} \quad 0 \leq -\partial_{\psi_s} H = \delta_s
\]

\[
0 = \delta_i \psi_i', \quad i = b, s.
\]

For intervals where condition (M') does not bind we have the solution of Problem (RUP). If \( [\tilde{\xi}_1, \tilde{\xi}_2] \) is a maximum bunching interval for \( \psi_s^\ast \) then, from the above optimality conditions and the Lagrangian \( (L) \), we must have \( \theta_1 = \psi_s^\ast(\xi) \),

\[
v_x(\xi, \psi_b^\ast(\xi, \theta_1)) = v_x(\xi, \theta_1)
\]

and

\[
0 \leq \delta_i(\xi) = -\int_{\tilde{\xi}_1}^{\xi} [f_x(\tilde{\xi}, \theta_1)p(\theta_1) - \mu(\tilde{\xi})v_x\theta(\tilde{\xi}, \theta_1)]d\tilde{\xi}, \quad (13)
\]

for all \( \xi \in [\tilde{\xi}_1, \tilde{\xi}_2] \), with an equality for \( \xi = \tilde{\xi}_2 \) and \( \theta_1 < \bar{\theta} \). By Remark 2, \( \psi_b^\ast(\xi, \theta_1) \) is decreasing on \( [\tilde{\xi}_1, \tilde{\xi}_2] \) and condition (M') is then slack for \( \psi_b^\ast \).

\(^{22}\) We are making some abuse of notation here because we use star for the solution of Problem (UP) and we are not making explicit the dependence of this solution on \( \tilde{\xi} \). Notice, however, that when \( \tilde{\xi} \) is chosen to be optimal, this is exactly the solution of Problem (UP).
Therefore, from condition (11),

\[ \delta_b(\xi) = 0 \text{ and } \mu(\xi) \leq \frac{f_x(\xi, \psi^*_b(\xi, \theta_1))}{v_{x\theta}(\xi, \psi^*_b(\xi, \theta_1))} p(\psi^*_b(\xi, \theta_1)), \]

for all \( \xi \in [\widetilde{\xi}_1, \widetilde{\xi}_2] \), with an equality when \( \psi^*_b(\xi, \theta_1) > \theta_1 \) and \( \mu(\xi) = 0 \) otherwise.

If \( \theta_u < \overline{\theta}_1 \), then Lemma 4 (iii) below shows that there exists \( \theta_1 \in [\overline{\theta}_1, \theta_u] \) such that \( \xi_1 = x_u(\theta_1), \overline{\xi}_2 = x_1(\theta_1) \wedge \overline{\xi} \) and condition (13) at \( \xi = \overline{\xi}_2 \) becomes

\[ -[f(x_1(\theta_1) \wedge \xi, \theta_1) - f(x_u(\theta_1), \theta_1)] p(\theta_1) + \int_{\theta_u(\theta_1)}^{\overline{\theta}_1} f_x(\xi, \psi^*_b(\xi, \theta_1)) \frac{v_{x\theta}(\xi, \psi^*_b(\xi, \theta_1))}{v_{x\theta}(\xi, \psi^*_b(\xi, \theta_1))} p(\psi^*_b(\xi, \theta_1)) d\xi = 0. \]

Using \( \psi^*_b(\xi, \theta_1) \) to change variable \( \xi \) for \( \theta \) in the previous integral we get the jump condition (or vertical ironing principle):\(^2\)

\[ \int_{\theta}^{\overline{\theta}_1} f_x(\overline{x}_u(\theta_1), \theta) v_{x\theta}(\overline{x}_u(\theta_1), \theta_1) p(\theta) d\theta = [f(x_1(\theta_1), \theta_1) - f(x_u(\theta_1), \theta_1)] p(\theta_1), \]

when \( \overline{\xi} = \overline{\pi} \), for each feasible vertical bunching level \( \theta_1 \in [\overline{\theta}_1, \theta_u] \), let \( \overline{x}_u(\cdot, \theta_1) \) be its associate decision function which is defined by

\[ v_x(\overline{x}_u(\theta, \theta_1), \theta) = v_x(\overline{x}_u(\theta_1), \theta_1), \]

for all \( \theta \in [\theta, \overline{\theta}_1] \), where \( \overline{\theta}_1 \) satisfies \( \overline{x}_u(\overline{\theta}_1, \theta_1) = x_u(\theta_1) = x_1(\theta_1) \).

If \( \theta_u = \overline{\theta}_1 \), Lemma 4 (iv) below shows that \( \theta_1 = \overline{\theta}_1 \) is the optimal bunching if and only if the equality in (14) becomes the inequality “greater or equal”.

**Lemma 4 (Characterization of vertical ironing)** For each \( \theta_1 \in [\overline{\theta}_1, \theta_u] \), let \( \delta_s : [\widetilde{\xi}_1, \widetilde{\xi}_2] \to \mathbb{R} \) be the function defined by (13), where \( \widetilde{\xi}_1 = x_u(\theta_1) \) and \( \widetilde{\xi}_2 = x_1(\theta_1) \wedge \overline{\xi} \). Then:

(i) \( \exists \xi^* \in [\widetilde{\xi}_1, \widetilde{\xi}_2] \) such that \( \delta_s|_{[\xi_1, \xi^*]} \) is positive and increasing; and \( \delta_s|_{[\xi^*, \xi_2]} \) is decreasing and convex;

(ii) if \( \theta_1 = \overline{\theta}_1 \), then \( \xi^* = \widetilde{\xi}_2 \) and \( \delta_s \) is always positive; if \( \theta_1 = \theta_u < \overline{\theta}_1 \), then \( \xi^* = \xi_1 \) and \( \delta_s \) is always negative;

(iii) if \( \theta_u < \overline{\theta}_1 \), then there exists \( \theta_1 \in (\overline{\theta}_1, \theta_u) \) for which equation (14) holds;

\(^2\) Formally, \( \theta_u \) is the maximal parameter where \( x_u \) is defined and \( \overline{\theta}_1 \) is parameter where the curves \( \psi^*_1 \) and \( \overline{\psi} \) cross.

\(^2\) Although this kind of phenomenon is not usual in the standard adverse selection literature, Wilson [27] mentions the possibility of “vertical ironing” (see pp. 179). Notice that condition (VI) is a natural generalization of Theorem 2.
(iv) if \( \theta_u = \overline{\theta} \), then either the conclusion of (iii) holds or the equation (14) becomes the inequality “greater or equal” for \( \theta_1 = \overline{\theta} \).

**Proof.** (i) From the definition of \( \delta_s \), \( \delta'_s(\xi) = -f_x(\xi, \theta_1)p(\theta_1) + \mu(\xi)v_x(\xi, \theta_1) \), for all \( \xi \in [\xi_1, \xi_2] \). Let \( \xi^* \in [\xi_1, \xi_2] \) be such that \( \mu(\xi) = 0 \), for all \( \xi \in [\xi^*, \xi_2] \) (which coincides with the point where the vertical line \( \theta = \theta_1 \) crosses the graph of \( \overline{\psi} \)). By A1 and A2, \( \delta''_s(\xi) > 0 \) and, since \( \delta'_s(\xi_2) = -f_x(\xi_2, \theta_1)p(\theta_1) \leq 0 \), we conclude that \( \delta'_s(\xi) < 0 \), for all \( \xi \in (\xi^*, \xi_2] \). Using the uniqueness of \( x_u \) (condition C2) and condition (18) below for \( \theta = \psi_b^*(\xi, \theta_1) > \psi_b^*(\xi) \), we have that \( \theta_1 = \varphi(\xi, \theta) < \psi_b^*(\xi) \) and

\[
\mu(\xi) = \frac{f_x(\xi, \psi_b^*(\xi, \theta_1))p(\psi_b^*(\xi, \theta_1))}{v_x(\xi, \psi_b^*(\xi, \theta_1))} > \frac{f_x(\xi, \theta_1)p(\theta_1)}{v_x(\xi, \theta_1)},
\]

for all \( \xi \in (\xi_1, \xi^*] \). Then, \( \delta'_s(\xi) > 0 \) and, since \( \delta_s(\xi_1) = 0 \), \( \delta_s(\xi) > 0 \) for all \( \xi \in (\xi_1, \xi^*] \).

(ii) In particular, if \( \theta_1 = \overline{\theta}_1 \) (resp. \( \theta_2 = \theta, \theta < \overline{\theta} \)), then \( \xi^* = \xi_2 \) and \( \delta_s(\xi) > 0 \) (resp. \( \xi^* = \xi_1 \) and \( \delta_s(\xi) < 0 \)) for all \( \xi \in (\xi_1, \xi_2] \).

(iii) From item (ii) and the intermediate value theorem, there exists \( \theta_1 \in (\overline{\theta}_1, \theta_u) \) such that the corresponding function \( \delta_s \) satisfies \( \delta_s(\xi_2) = 0 \), which is exactly the equation (14).

(iv) If for \( \theta_1 = \overline{\theta} \) the function \( \delta_s \) is such that \( \delta_s(\xi_2) \geq 0 \), then, by item (i), \( \delta_s(\xi) \geq 0 \), for all \( \xi \in [\xi_1, \xi_2] \). Otherwise, we can apply the same argument made in (iii) to prove the existence of \( \theta_1 \in (\overline{\theta}_1, \theta_u) \) for which equation (14) holds. ■

**Optimal bunching (optimal \( \xi \)).** Let \([\theta_2, \theta_3]\) be a jump interval for \( \psi_s^* \) at \( \xi \), i.e., \( \theta_2 \) and \( \theta_3 \) are the right- and left-hand side limits of \( \psi_s^* \) at \( \xi \), respectively. If \( \xi \) is optimal, then

\[
\{\theta_2, \theta_3\} \subset \psi_s^*(\xi) = \arg \max_{\varphi(\xi, \theta)} L(\xi, \varphi, \psi_s^*, \mu)
\]

and, in particular, we must have \( L(\xi, \theta_2, \psi_s^*, 0) = L(\xi, \theta_3, \psi_s^*, \mu(\xi)) \) or, using the Lagrangian (L),

\[
0 = L(\xi, \theta_3, \psi_s^*, \mu(\xi)) - L(\xi, \theta_3, \psi_s^*, 0) + L(\xi, \theta_3, \psi_s^*, 0) - L(\xi, \theta_2, \psi_s^*, 0) = \int_{\theta_2}^{\theta_3} \partial_\mu L(\xi, \theta, \psi_s^*, \mu)d\theta + \int_{\theta_2}^{\theta_3} \partial_\psi L(\xi, \theta, \psi_s^*, 0)d\theta = \mu(\xi)v_x(\xi, \theta_3) + \int_{\theta_2}^{\theta_3} f_x(\xi, \theta)p(\theta)d\theta.
\]
Analogously, if \([\theta_3, \overline{\theta}]\) is an optimal jump for \(\psi^*_s\) at \(\overline{\xi}\), then \(L(\overline{\xi}, \psi^*_b, \theta_3, \mu(\overline{\xi})) = L(\overline{\xi}, \psi^*_b, \overline{\theta}, 0)\) or

\[0 = -\mu(\overline{\xi})v_x(\overline{\xi}, \theta_3) + \int_{\theta_3}^{\overline{\theta}} f_x(\overline{\xi}, \theta)p(\theta)d\theta.\]

Adding up these two last equations and using that \(v_x(\overline{\xi}, \hat{\theta}_3) = v_x(\overline{\xi}, \theta_3)\), we have the horizontal “ironing principle”

\[\int_{\theta_2}^{\hat{\theta}_3} f_x(\overline{\xi}, \theta)p(\theta)d\theta + \int_{\theta_3}^{\hat{\theta}_3} f_x(\overline{\xi}, \theta)p(\theta)d\theta = 0,\]

where \(\theta_2, \hat{\theta}_3\) and \(\theta_3\) are shown in the right graph of Figure 4. In particular, if \(\overline{\xi}\) is below the crossing point of \(x_0\) and \(x_1\), then we have a degenerate case of unique bunching interval (i.e., \(\hat{\theta}_3 = \theta_3\)). Other degenerate case is when there is no interval with the relaxed solution (i.e., \(\theta_2 = \overline{\theta}\) and \(x_1(\theta) < \overline{\xi}\)).

Finally, we claim that both ironing principles cannot hold at the same time, i.e., the solution of Problem (UP) can present only one ironing procedure. Otherwise, from Lemma 4 (i) (in particular, its proof), for the optimal \(\overline{\xi}\) the optimal vertical ironing line must cross the curve \(\psi\), which implies that \(\overline{\xi}\) must be above this crossing point. Thus, the bunching interval would be completely below the curve \(x_1\) (i.e. \(\hat{\theta}_3 < \overline{\theta}\) in the right graph of Figure 4). In this case, the integral in (HI) would be strictly positive due to A1. This implies that \(\overline{\xi}\) cannot be optimal unless \(\overline{\xi} = \overline{x}\).

Therefore, the solution of Problem (UP) must have one of the shapes described in the text.

**The second-best solution.** Next we determine whether the solution of Problem (UP) is implementable. For this we need the following lemma which shows that to get implementability, besides conditions (M') and (UC'), we only have to check the following condition that is complementary to (UC'): Let \(\psi_b\) and \(\psi_s\) be a pair of type assignment functions of \(x \in X\). Then, we that they satisfy the complementary relaxed U-shaped condition if \(\xi \in X\) is a point of continuity for both \(\psi_b\) and \(\psi_s\), then

\[\psi_b(\xi) > \theta \implies v_x(\xi, \psi_b(\xi)) \geq v_x(\xi, \psi_s(\xi)).\]  

\[(UC'')\]

Notice that conditions (UC') and (UC'') are equivalent to condition (UC) in intervals where the decision functions have interior type assignment functions (i.e., \(\psi_b(\xi) > \overline{\theta}\) and \(\psi_s(\xi) > \overline{\theta}\)). Otherwise, condition (UC'') avoids the problem raised in Figure 1 for other global IC constraints besides (UC) that may be binding.
Lemma 5 (Sufficient conditions for implementability) Assume A2. Suppose that \( \psi_b \) and \( \psi_s \) satisfy conditions \((M')\), \((UC')\) and \((UC'')\). Then, \( x \) is implementable.

Proof. We have to show that there exists a non-linear tariff \( T(\xi) \) such that for each \( \theta \in \Theta \) and \( \xi \in x(\theta) \)

\[
v(\xi, \theta) + T(\xi) \geq v(\hat{\xi}, \theta) + T(\hat{\xi}),
\]

for all \( \hat{\xi} \in x(\Theta) \). By the fundamental theorem of calculus, this is equivalent to

\[
\int_{\xi}^{\hat{\xi}} [v_x(\tilde{\xi}, \theta) + T'(\tilde{\xi})]d\tilde{\xi} \geq 0.
\]

For such \( x \), take the corresponding type assignments \( \psi_b \) and \( \psi_s \). Let us consider the following cases:

(a) \( x(\bar{\theta}) \geq x(\check{\theta}) \). By Remark 1, we can define \( T'(\tilde{\xi}) = -v_x(\tilde{\xi}, \psi_s(\tilde{\xi})) \) for a.e. \( \xi \in x(\Theta) \). There are two possibilities:

- \( \theta = \psi_s(\xi) \). Then, implementability of \( x \) is equivalent to

\[
\int_{\xi}^{\hat{\xi}} [v_x(\tilde{\xi}, \theta) + T'(\tilde{\xi})]d\tilde{\xi} = \int_{\xi}^{\hat{\xi}} \int_{\psi_s(\xi)}^{\psi_s(\tilde{\xi})} v_x(\tilde{\xi}, \theta)d\tilde{\theta}d\tilde{\xi} \geq 0,
\]

for a.e. \( \hat{\xi} \in x(\Theta) \). Since \( x_0 \) is a decreasing function, the area delimited by the previous double integral is contained in \( \text{CS}_+ \), for a.e. \( \hat{\xi} \in x(\Theta) \). Therefore, it is non-negative if and only if \( \psi_s \) is non-decreasing which is ensured by \((M')\).

- \( \theta = \psi_b(\xi) \). We claim that

\[
\int_{\xi}^{\hat{\xi}} v_x(\tilde{\xi}, \theta)d\tilde{\xi} \geq \int_{\xi}^{\hat{\xi}} v_x(\tilde{\xi}, \psi_s(\xi))d\tilde{\xi},
\]

for a.e. \( \hat{\xi} \in x(\Theta) \). Indeed, defining the difference function as

\[
a(\xi, \hat{\xi}) = \int_{\xi}^{\hat{\xi}} [v_x(\tilde{\xi}, \theta) - v_x(\tilde{\xi}, \psi_s(\xi))]d\tilde{\xi},
\]

we have that \( a(\xi, \xi) = \partial_\xi a(\xi, \xi) = 0 \) by \((UC')\) and \((UC'')\). Moreover,

\[
\partial^2_\xi a(\xi, \hat{\xi}) = v_{xx}(\hat{\xi}, \psi_s(\xi)) - v_{xx}(\hat{\xi}, \theta) > 0
\]

since \( v_{xx} > 0 \) and \( \psi_s(\xi) > \theta \). Hence, \( a(\xi, \hat{\xi}) \geq 0 \) for all \( \hat{\xi} \in x(\Theta) \).

Subtracting \( \int_{\xi}^{\hat{\xi}} v_x(\tilde{\xi}, \psi_s(\tilde{\xi}))d\tilde{\xi} \) from each side of (15), the implementability for \( \theta = \psi_b(\xi) \) follows from the implementability for the case \( \theta = \psi_s(\xi) \).
(b) \(x(\overline{\theta}) < x(\overline{\theta})\). Again by Remark 1, we can define \(T'(\tilde{\xi}) = -v_x(\tilde{\xi}, \psi_b(\tilde{\xi}))\) for a.e. \(\tilde{\xi} \in x(\overline{\theta})\). Define \(\tilde{\psi}_b\) such that
\[
v_x(\xi, \tilde{\psi}_b(\xi)) = v_x(\xi, \overline{\theta}),
\]
for all \(\xi > x(\overline{\theta})\), and identical to \(\psi_b\), for all \(\xi \leq x(\overline{\theta})\). By \((UC')\) and \((UC'')\),
\[
v_x(\xi, \tilde{\psi}_b(\xi)) = v_x(\xi, \overline{\theta}) \leq v_x(\xi, \psi_b(\xi)),
\]
for all \(\xi > x(\overline{\theta})\). This and A2 imply that \(\tilde{\psi}_b \geq \psi_b\). From Remark 2, \(\tilde{\psi}_b\) is non-increasing (it is \(\psi_b\) for \(\xi < x(\overline{\theta})\) and jumps downward at \(x(\overline{\theta})\)). From the case (a), the decision associated to the type assignment functions \(\psi_b\) and \(\psi_s\), \(\tilde{x}\), is implementable because \(\tilde{x}(\overline{\theta}) = \tilde{x}(\overline{\theta})\). From this and the fact that \(x\) satisfies (M) and (UC), the only IC constraints that remain to be checked for \(x\) are the ones illustrated in Figure 1 (the other IC constraints are similar to the case (a)). This is equivalent to check that
\[
\int_{\xi}^{\tilde{\xi}} [v_x(\tilde{\xi}, \theta) + T'(\tilde{\xi})] d\tilde{\xi} = \int_{\xi}^{\tilde{\xi}} \int_{\psi_b(\tilde{\xi})}^{\psi_b(\xi)} v_x(\tilde{\xi}, \theta) d\overline{\theta} d\tilde{\xi} \geq 0,
\]
for \(\theta = \psi_b(\xi)\) and \(\tilde{\xi} \geq \xi\). However, since \(\tilde{\psi}_b \geq \psi_b\), A2 and the implementability of \(\tilde{x}\) imply
\[
\int_{\xi}^{\tilde{\xi}} \int_{\psi_b(\tilde{\xi})}^{\psi_b(\xi)} v_x(\tilde{\xi}, \theta) d\overline{\theta} d\tilde{\xi} \geq \int_{\xi}^{\tilde{\xi}} \int_{\tilde{\psi}_b(\tilde{\xi})}^{\tilde{\psi}_b(\xi)} v_x(\tilde{\xi}, \theta) d\overline{\theta} d\tilde{\xi} \geq 0
\]
since \(\tilde{\psi}_b(\tilde{x}(\theta)) = \theta\). This completes the proof of implementability of \(x\).

The following condition avoids that other global IC constraints than \((UC)\) bind for \(x^*\). For instance, if the decreasing part of \(x_1\) is close enough to the limit curve \(x_0\), condition C3 may fail because of the problem raised in Figure 1.

C3. The type assignment functions \((\psi_b^*, \psi_s^*)\) satisfy condition \((UC'')\).

Condition C3 holds under \(x_1(\overline{\theta}) < x_1(\overline{\theta})\). Otherwise, there exists \(\xi \in X\) such that \(\psi_b^* > \overline{\theta}\) and condition \((UC')\) is slack for Problem (RUP). Hence, \(\psi_b^* = \psi_b^1\) and \(\mu(\xi) = 0\). A1 and \(x_1(\overline{\theta}) < x_1(\overline{\theta})\) imply
\[
\partial_{\psi_b} L(\xi, \psi_b^*, \overline{\theta}, 0) = -f_x(\xi, \overline{\theta}) p(\overline{\theta}) < 0
\]
and then \(\psi_s^* = \psi_s^1 < \overline{\theta}\). Lemma 3 (ii) (specially its proof) shows that this cannot be the case.

\(^{25}\) The intuition of this part of the proof can be explained using Figure 1. Suppose now that \(\tilde{x} = x_1\) is an implementable decision and \(x\) is non-increasing, and completely below \(\tilde{x}\). Then, \(x\) is also implementable. Indeed, the typical weighted hatched area of Figure 1 will be always non-negative for \(x\) once it is for \(\tilde{x}\).
Proof of Theorem 3. We only have to check that the solutions or Problem (UP) satisfy the conditions of Lemma 5. All conditions but (UC") are trivially satisfied. Condition C3 implied by C2 (i.e., by $x_1(\theta) < x_1(\overline{\theta})$), however, ensures condition (UC").

Sufficient condition for the uniqueness of $x_u$. Let $\xi_0$ be the parameter where $x_0$, $x_1$ and $x_u$ cross. Suppose that for each $\xi \geq \xi_0$ and $\theta \neq x_0^{-1}(\xi)$:  

$$\frac{\partial}{\partial \theta} \left( \frac{f_x(\xi, \theta)}{v_{x\theta}(\xi, \theta)} p(\theta) \right) > 0. \quad (16)$$

For each such $\xi$ let us make the following change of variables:

$$\Psi_j = v_x(\xi, \psi_j), \text{ for } j = b, s. \quad (17)$$

In the new variables $\Psi_b$ and $\Psi_s$, the constraint (UC") becomes the linear constraint: $\Psi_b \leq \Psi_s$. By the implicit function theorem, the derivative of the objective function of Problem (RUP) with respect to $\Psi_b$ and $\Psi_s$ are

$$\frac{f_x(\xi, \psi_b)}{v_{x\theta}(\xi, \psi_b)} p(\psi_b) \text{ and } -\frac{f_x(\xi, \psi_s)}{v_{x\theta}(\xi, \psi_s)} p(\psi_s),$$

respectively. Since $v_x(\xi, \cdot)$ is decreasing on $[\theta, x_0^{-1}(\xi)]$ and increasing on $[x_0^{-1}(\xi), \overline{\theta}]$, condition (16) ensures that the objective function of Problem (RUP) is strictly concave in $\Psi_b$ and $\Psi_s$. Moreover, the cross derivative with respect to $\Psi_b$ and $\Psi_s$ is zero, which implies that it is a strictly concave function. Therefore, in these new variables Problem (RUP) is a strictly concave program which implies the uniqueness of solution. Since (17) defines a monotone transformation, the solution is unique in the original variables as well.

Jullien [11] assumes the “potential separation property” that has a similar role of our condition C2. Although we prove below that $\psi^*_b$ is non-increasing, we cannot ensure that $\psi^*_s$ is always non-decreasing (as a function of $\xi$). Indeed, condition (M") binds for Problem (RUP) when the solutions are at the boundary.

Monotonicity of the type assignment function $\psi^*_b$. Let us consider the

26 For private value model (i.e., $u_{x\theta} = 0$), a simple calculation of the right and left limits of condition (16) at $x_0^{-1}(\xi)$ shows that

$$\lim_{\theta \to (x_0^{-1}(\xi))^+} \frac{\partial}{\partial \theta} \left( \frac{f_x(\xi, \theta)p(\theta)}{v_{x\theta}(\xi, \theta)} \right) = +\infty$$

for all $\xi \geq \xi_0$, and the reverse asymptotic limits occur for $\xi < \xi_0$. In particular, condition (16) can only hold for $\xi \geq \xi_0$. Jullien [11] uses a similar condition to ensure the potential separation assumption in his paper.
interior solution case. (The boundary solution case is an immediate consequence of Lemma 3.) Defining, for each \( \xi \), \( D(\xi, \theta) = F(\xi, \theta) - F(\xi, \varphi(\xi, \theta)) \)
and \( \varphi(\xi, \theta) \neq 0 \) implicitly by \( v_x(\xi, \theta) = v_x(\xi, \varphi(\xi, \theta)) \), we have that
\[
\psi_\theta^*(\xi) \in \arg \max_{\theta \in [\hat{\xi} , \xi]} D(\xi, \theta)
\]
is a critical point of \( D(\xi, \theta) \), i.e., \( D_\theta(\xi, \psi_\theta^*(\xi)) = 0 \). Taking the total derivative of with respect to \( \xi \) we get
\[
D_{\theta \theta}(\xi, \psi_\theta^*(\xi)) \frac{d\psi_\theta^*(\xi)}{d\xi}(\xi) = -D_{\theta x}(\xi, \psi_\theta^*(\xi)).
\]
Using the definition of \( F \) in (10),
\[
D_\theta(\xi, \theta) = f_x(\xi, \theta)p(\theta) - f_x(\xi, \varphi)p(\varphi)\varphi(\xi, \theta)
\]
and
\[
D_{\theta x}(\xi, \theta) = f_{xx}(\xi, \theta)p(\theta) - f_{xx}(\xi, \varphi)p(\varphi)\varphi(\xi, \theta) - [f_{x \theta}(\xi, \varphi)p(\varphi) + f_x(\xi, \varphi)p'(\varphi)]\varphi(\xi, \theta)^2 - f_x(\xi, \varphi)p(\varphi)\varphi_{\theta x}(\xi, \theta),
\]
where \( \varphi = \varphi(\xi, \theta) \) and \( \varphi_{\theta}(\xi, \theta) = v_x(\xi, \theta)/v_{x \theta}(\xi, \varphi) < 0 \).
Since \( f_x(\xi, \varphi)p(\varphi) = \mu(\xi)v_{x \theta}(\xi, \varphi) \) from (12) and \( \varphi_{\theta x}(\xi, \theta)v_{x \theta}(\xi, \varphi) = v_{x \theta}(\xi, \theta) - \varphi_{\theta}(\xi, \theta)v_{x \theta}(\xi, \varphi) - \varphi_{\theta}(\xi, \theta)^2 v_{x \theta}(\xi, \varphi) \), we have that
\[
D_{\theta x}(\xi, \theta) = f_{xx}(\xi, \theta)p(\theta) - \mu(\xi)v_{x \theta}(\xi, \varphi) - [f_{x \theta}(\xi, \varphi)p(\varphi) - \mu(\xi)v_{x \theta}(\xi, \varphi)]\varphi(\xi, \theta)
- [f_{x \theta}(\xi, \varphi)p(\varphi) + f_x(\xi, \varphi)p'(\varphi) - \mu(\xi)v_{x \theta}(\xi, \varphi)]\varphi(\xi, \theta)^2.
\]
The second-order condition of the Lagrangian \( L \) implies that
\[
f_{x \theta}(\xi, \varphi)p(\varphi) + f_x(\xi, \varphi)p'(\varphi) - \mu(\xi)v_{x \theta}(\xi, \varphi) \geq 0,
\]
where \( \varphi = \varphi(\xi, \psi_\theta^*(\xi)) \). Therefore, A1 and A2 imply that \( D_{\theta x}(\xi, \psi_\theta^*(\xi)) < 0 \).
By the second-order condition, \( D_{\theta \theta}(\xi, \psi_\theta(\xi)) < 0 \). Then, \( \psi_\theta^*(\xi) \) is decreasing.\(^{27}\)

Using \( D_\theta(\xi, \theta) \) and \( \varphi_{\theta}(\xi, \theta) \), the uniqueness of \( x_u \) implies that
\[
\frac{f_x(\xi, \theta)}{v_{x \theta}(\xi, \theta)}p(\theta) - \frac{f_x(\xi, \varphi(\xi, \theta))}{v_{x \theta}(\xi, \varphi(\xi, \theta))}p(\varphi(\xi, \theta)) \leq 0 \iff \theta \leq \psi_\theta^*(\xi). \tag{18}
\]
\(^{27}\) We cannot have the same conclusion for \( \psi_\theta^* \). The reason is that, in this case, the first and second lines of the last expression of \( D_{\theta x} \) have opposite signs.
Appendix C. Example (labor contract)

Suppose that a firm, monopolistic in the labor market, can hire a manager with a vector of unknown abilities \((\theta, \eta)\) to deliver outcome \(x\) through the technology:

\[
x = \theta e + \alpha \eta,
\]

where \(e\) is the manager’s unknown effort, \(\theta\) is her marginal productivity of effort and \(\eta\) is a fixed factor with common knowledge shadow price \(\alpha\).

The firm cannot contract directly on the abilities, but it contracts on an exogenous verifiable fixed signal \(s\) (schooling, interview, etc.) that mixes these abilities:

\[
s = \theta + \eta.
\]

Given \(s\), a higher marginal return to effort (the slope \(\theta\) in the production function) means a lower \(\eta\) (and hence a lower intercept \(\alpha \eta\) in the production function). Therefore, the effort required by an agent with abilities \((\theta, s - \theta)\) to achieve output level \(x\),

\[
e(x, \theta) = \theta^{-1}(x - \alpha \eta) = \theta^{-1}(x - \alpha s) + \alpha,
\]

is not necessarily decreasing in \(\theta\).

Suppose that the manager’s cost of effort is quadratic. If \(t\) is the manager’s wage, her utility function has the quasi-linear form of Section 2, \(V = v(x, \theta) + t\), with

\[
v(x, \theta) = -e(x, \theta)^2.
\]

The firm is on the other hand a profit maximizer with utility function \(U = x - t\).

Let us check the validity of the SMC:

\[
v_x \theta(x, \theta) = 2\theta^{-2}(2e(x, \theta) - \alpha) = 2\theta^{-2}(2\theta^{-1}(x - \alpha s) + \alpha).
\]

When outputs and efforts are low (high), the intercept can be more (less) important than the slope. In this case, increasing (decreasing) the slope \(\theta\) - but reducing (improving) the intercept - can make more (less) effort needed to achieve the same level of output, and hence increases (decreases) the marginal cost of output: \(v_x \theta < (>) 0\).

If the shadow price of \(\eta\) is zero \((\alpha = 0)\), then the marginal cost of production decreases with \(\theta\) \((v_x \theta > 0)\), which implies the SMC and that implementable outputs are non-decreasing in \(\theta\). This is the standard case since \(\theta\) is productivity, \(\eta\) has no impact on the firm’s technology and \(s\) is an uninformative signal of the manager’s productivity.

On the other hand, if \(\alpha\) is high enough, then the marginal cost of production increases with \(\theta\) \((v_x \theta < 0)\) because \(\eta\) is now relatively more valuable than \(\theta\).
for the firm’s technology. A given signal $s$ may convey bad news if $\theta$ is high since it means low $\eta$. Then, the SMC holds again but outputs decrease with $\theta$ for every implementable contract.

For intermediate values of $\alpha$ we may have both signs: low (high) $\theta$ implies that the marginal cost of production increases (decreases) with $\theta$. Indeed, we can compute the limit curve $x_0$ between $CS_-$ (below $x_0$) and $CS_+$ (above $x_0$), i.e., the curve implicitly defined by $v_{x\theta}(x, \theta) = 0$ in the positive quadrant:

$$x_0(\theta) = \alpha[s - 0.5\theta]^+, \tag{37}$$

where $[a]^+ = \max\{a, 0\}$.

Given $s$, suppose that $\theta$ is uniformly distributed (the conditional distribution of $\eta$ is then uniform as well). Consider a particular example: $\Theta = [1, 1.5]$. The relaxed solution is given by:

$$x_1(\theta) = \min\\{\alpha s, \alpha s - \frac{2\alpha - \theta^2}{2(2 - \theta)}\}.$$

Observe that $x_1$ satisfies condition C1 and crosses $x_0$ exactly at $\theta = \alpha$. Hence, $\alpha \in (1, 1.5)$ if and only if the SMC does not hold.

The left and right graphs below give, besides $x_0$ and $x_1$, the optimal decisions characterized by Theorem 3 for $s = 1.5$ and $\alpha = 1.15$ and 1.3, respectively.

---

**The sign of the derivatives and the critical U-shaped decision.** An easy inspection gives

$$f_{xx}(\xi, \theta) = 2(\theta - 2) < 0,$$

for $\theta \in [1, 1.5]$ which implies A1. The third derivatives of $v$, $v_{x^2}(\xi, \theta) = 4\theta^{-3} > 0$ and $v_{x^2}(\xi, \theta) = 2\theta^{-3}[6\theta^{-1}(\alpha s - \xi) - 2\alpha]$ (positive along $x_0$), imply A2. Since
\[ v_x(\xi, \theta) = -2\theta^{-1}e(\xi, \theta), \] condition (UC) gives
\[ \hat{\theta} = \varphi(\theta, \xi) = (\alpha(\alpha s - \xi)^{-1} - \theta^{-1})^{-1}. \]

Thus, the U-shaped part characterized by Theorem 2 is a root of the following third degree polynomial (the monotonicity condition eliminates the other two roots):
\[-\frac{\alpha^2}{4} + \alpha\theta^{-1} \left( \frac{1}{2} + \alpha\theta^{-1} \right) y - \theta^{-2} \left( \frac{1}{2} + 2\alpha\theta^{-1} \right) y^2 + \theta^{-4} y^3 = 0,\]
where \( y = \alpha s - \xi \) and condition C2 is easily satisfied.

**Appendix D. Suplements (demand profile approach and example)**

**Demand profile approach.** For simplicity, let us assume that \( u \) does not depend directly on \( \theta \) (private value case). Define the cumulative aggregate demand functional:
\[ D(\tau, \xi) = \operatorname{Prob} [-v_x(\xi, \theta) \leq \tau], \]
where \(-v_x\) is the type-\( \theta \) marginal cost of \( \xi \). For each \( \xi \), the principal’s program is equivalent to determine the marginal tariff that solves
\[ \max_{\tau} D(\tau, \xi)[u(\xi) - \tau]. \]

The first-order condition gives the well-known equalization of the markup to the inverse of the demand elasticity with respect to \( \tau \):
\[ \frac{\tau - u_x(\xi)}{\tau} = \frac{1}{\epsilon(\tau, \xi)}, \tag{19} \]
where \( \epsilon(\tau, \xi) = -\tau D_\tau(\tau, \xi)/D(\tau, \xi) \).

If there is only one marginal type \( \theta \) choosing \( \xi \), then \( D(\tau, \xi) = P(\theta) \). Using the implicit function theorem on \( v_x(\xi, \theta) + \tau = 0 \), we obtain a formula for the elasticity:
\[ \epsilon(\tau, \xi) = \frac{p(\theta)}{P(\theta)} \frac{\tau}{v_x(\xi, \theta)}, \]
which is proportional to the hazard rate \( p(\theta)/P(\theta) \). Plugging this into the expression (19) gives exactly the first-order condition of the relaxed problem (6).

On the other hand, suppose that \( \xi \) belongs to a U-shaped part. That is, there are two marginal pooling types, \( \hat{\theta} < \theta \), choosing \( \xi \). Thus, \( D(\tau, \xi) = P(\theta) - P(\hat{\theta}) \) and arguing in same way as before we get
\[ \epsilon(\tau, \xi) = \frac{p(\theta) + \lambda(\xi, \theta)p(\hat{\theta})}{P(\theta) - P(\hat{\theta})} \frac{\tau}{v_x(\xi, \theta)}, \]

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where \( \lambda(\xi, \theta) = v_{x\theta}(\xi, \theta)/v_{x\theta}(\xi, \hat{\theta}) \) and \( v_x(\xi, \theta) = v_x(\xi, \hat{\theta}) = -\tau \). This is exactly the content of Theorem 2. Since \( \lambda(\xi, \theta) < 0 \) and \( P(\hat{\theta}) \geq 0 \), the hazard rate here is lower than the corresponding in the separation case. This explains the upward jump at the transition point.

**Example showing how restrictive is \( \mathcal{X} \).** Let \( \Theta = [0, 1] \) with uniform distribution, \( v(x, \theta) = \frac{1}{2} \theta x(x - 1) \) and \( u(x, \theta) \) be such that the relaxed function is \( f(x, \theta) = x \left( \theta - \frac{2}{3} \right) \). Thus, the limit curve \( x_0 \) is the horizontal line \( \xi = 1/2 \) such that \( \text{CS}_{- (+)} \) is below (above) it and the relaxed solution is \( x_1(\theta) = \theta \). Therefore, the solution of Problem (UP) in \( \mathcal{X} \) is \( x^*(\theta) = \max\{x_0(\theta), x_1(\theta)\} \).

However, if we allow for discontinuous crossings (i.e., decisions not in \( \mathcal{X} \)), then one may find an incentive compatible decision that dominates \( x^* \):\(^{28}\)

\[
x^{**}(\theta) = \begin{cases} 
1/4, & \text{for } \theta \in [0, 1/2) \\
\max\{3/4, x_1(\theta)\}, & \text{for } \theta \in [1/2, 1].
\end{cases}
\]

Moreover, if we change the distribution of types, more crossing points may enhance welfare. Consider this other incentive compatible decision:

\[
x^{**}(\theta) = \begin{cases} 
5/8, & \text{for } \theta \in [0, 1/4) \cup [1/2, 5/8) \\
3/8, & \text{for } \theta \in [1/4, 1/2) \\
x_1(\theta), & \text{for } \theta \in [5/8, 1].
\end{cases}
\]

We see that, to keep incentive compatibility, the welfare gain to better approximate the relaxed solution on \([1/4, 1]\) is penalized by the loss on \([0, 1/4)\). However, if the density weight on \([0, 1/4)\) is much smaller than on \([1/4, 1]\), then \( x^{**} \) would increase the welfare gain to the principal compared with the previous decision. One might guess that we can keep doing this process. The questions are, for more general distributions and problems, how to determine the optimal decision and what would be the appropriate space of contracts.

**References**


\(^{28}\) The welfare loss of \( x^* \) w.r.t. \( x_1 \) is the triangle area delimited by the vertical line \( \theta = 0 \), \( x_0 \) and \( x_1 \), i.e., \( 1/8 \). On the other hand, the loss of \( x^{**} \) w.r.t. \( x_1 \) is three triangles of area of \( 1/32 \) each, i.e., \( 3/32 \) which is lower than \( 1/8 \). Moreover, one can easily check that (GIF) is always non-negative for \( x^{**} \), i.e., it is incentive compatible.


