"This is a theory which in the last ten years I worked only its particular examples".

Modular and automorphic forms & beyond.

1. Basics of alg. de Rham cohomology
2. A moduli space & conjecture & the algebra of differential modular forms
3. Gauss-Manin connection in disguise.
4. Special cases:

$X_0 \subseteq \mathbb{P}^N$, X_0/k, $\mathrm{char}(k)=0$, X_0 smooth, For simplicity take $k=\mathbb{Q}$.

$$\begin{array}{l} H^m_{dR}(X_0) := \mathbb{H}^m\left(\cdots \to \Omega^i_{X_0} \xrightarrow{d} \Omega^{i+1}_{X_0} \xrightarrow{d} \cdots\right) \\ \quad \| \qquad\qquad\qquad\qquad \uparrow \\ F_0 \qquad\qquad\qquad\qquad\quad i \\ \cup \\ F^i = \mathrm{Im}\left(\mathbb{H}^m(\cdots 0 \to \Omega^i_{X_0} \to \Omega^i_{X_0} \to \cdots)\right) \end{array}$$

explaining classical de Rham cohomology & Hodge filtration.

cup product

$$H^{m_1}_{dR}(X_0) \times H^{m_2}_{dR}(X_0) \to H^{m_1+m_2}(X_0) \qquad \omega_1, \omega_2 \mapsto \omega_1 \cup \omega_2$$

polarization: $\theta_0 \in H^2_{dR}(X_0)$

Trace map: $\mathrm{Tr}: H^{2n}_{dR}(X_0) \xrightarrow{\sim} k$, $\mathrm{Tr}(\omega) := \frac{1}{(2\pi i)^n} \int_{X_0} \omega$ $\quad k=\mathbb{C}$.

$\mathbb{H}$ $\quad$ H

Proposition: For another variety X in an irreducible component of the Hilbert scheme of X_0 we have

↳ the case of hypersurfaces.

$$i : \left(H^*_{dR}(X), F^*, \cup, \theta\right) \xrightarrow{\sim} \left(H^*_{dR}(X_0), F_0^*, \cup, \theta_0\right)$$

$$\begin{array}{ll} H^2_{dR}(X) \to H^2_{dR}(X_0) & \\ \cup\!\! | & \cup\!\! | \\ H^2(X,\mathbb{Z}) & H^2_{dR}(X_0) \quad \text{no map at this level.} \end{array}$$

θ, θ_0 are defined$/\mathbb{Z}$

Definition: An enhanced variety is the pair

$$(X, i), \quad i \text{ as above}$$

Remark: One may add more structure. In the case of elliptic curve, torsion points lattice polarization etc.

$T :=$ the moduli space of (X, i)
$X \in H$, i as before.

The algebraic group
$$G_0 := Aut(H^*_{dR}(X_0), F_0^*, U, \theta_0)$$

acts on from the left:
$$g \cdot (X, \alpha) = (X, g \circ i)$$

Conjecture: T has a natural structure of a (quasi-affine) variety over k such that its k-rational points corresponds to pairs (X, α) defined over k and the action of G_0 is algebraic.

Definition: $O_T :=$ the k-Algebra of (global) regular functions on T ($T \subseteq Spec(O_T)$) is a good candidate for a vast generalization of modular/autom forms and their derivations.

Remark: $G \backslash T$ is the classical moduli space of X. This is usually a stack, However, T is expected to be a usual variety

Gauss-Manin Connection: we expect that the universal family over T exists.

$$\mathcal{X} \longrightarrow T$$

$H^*_{dR}(\mathcal{X}/T)$ = global sections of the cohomology bundle.
this is an $\mathcal{O}_T$-module.

$$\left(H^*_{dR}(\mathcal{X}/T), F^*, U, \theta\right) \overset{*}{\simeq} \left(H^*_{dR}(X_0), F^*_0, U, \theta_0\right) \otimes_k \mathcal{O}_T$$

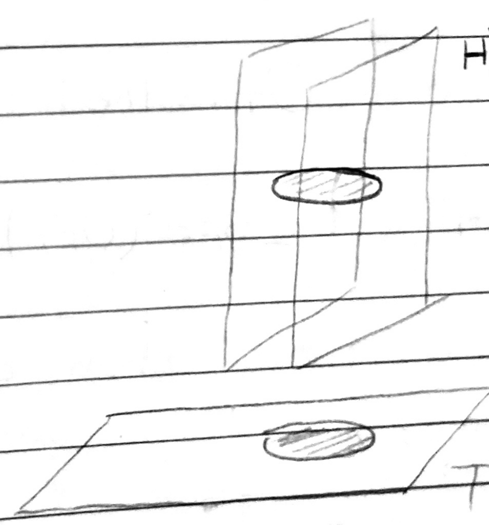
Gauss-Manin

$$H^m_{dR}(\mathcal{X}/T) \longrightarrow \Omega^1_T \otimes_{\mathcal{O}_T} H^m_{dR}(\mathcal{X}/T)$$

analytic definition

$H^m_{dR}(X_t) \cong H^m_{dR}(X_{t_0}, \mathbb{Z})$

sections with values in are flat, that is $\nabla\alpha = 0$.



A basis of $H^*_{dR}(X_0)$ 1. compatible with the Hodge filtration.

2. " with the Lefschetz decomposition

in particular a basis of $H^{2n}_{dR}(X_0)$ is given by θ^n.

we use the isomorphism $*$ and obtain a basis of $H^*_{dR}(\mathcal{X}/T)$ and with $\cup$ products in k (and not $\mathcal{O}_T$).

$\alpha_{m_1,i}$ $i=1,2,\dots$ a basis of $H^{m_1}_{dR}(\mathcal{X}/T)$

$\alpha_{m_2,i}$ " " $H^{m_2}_{dR}(\mathcal{X}/T)$

$$[\alpha_{m_1} \cup \alpha_{m_2}^{tr}] = \sum_i \underbrace{\Phi_{m_1,m_2,i}}_{\text{constant.}} \alpha_{m_1+m_2,i}$$

$$\nabla \begin{bmatrix} \alpha_{m,1} \\ \alpha_{m,2} \\ \vdots \\ \alpha_{m,b_m} \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} h^{m0} & h^{m-1,1} & h^{m-2,2} & \cdots \end{array} \\ \begin{bmatrix} & & 0 & 0 \\ & & & 0 \\ & A_{ij} & & \\ & & & \end{bmatrix} \end{array} \begin{bmatrix} \alpha_{m,1} \\ \alpha_{m,2} \\ \vdots \\ \alpha_{m,b_m} \end{bmatrix}$$

Griffiths transversality.

The entries of A are differential 1-forms in T; $A_{ij} \in \Omega^1_T$.

The $\mathcal{O}_T$-module of vector fields in $T = \mathcal{O}_T = (\Omega^1_T)^\vee :=$

$\{\Omega^1_T \to \mathcal{O}_T,\ \mathcal{O}_T\text{-linear}\}$.

Moduler vector fields: $R \in \mathcal{O}_T$ is called moduler if

$$\nabla_R \begin{bmatrix} \alpha_{m,1} \\ \alpha_{m,2} \\ \vdots \\ \alpha_{m,b_m} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & & 0 & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_R} \begin{bmatrix} \alpha_{m,1} \\ \alpha_{m,2} \\ \vdots \\ \alpha_{m,b_m} \end{bmatrix}$$

Conjecture: The $\mathcal{O}_T$-module of moduler vector fields is free of rank $\dim(G_0 \backslash T)$

Proposition: There is an embedding of k-vector spaces

$$\mathrm{Lie}(G) \hookrightarrow \mathcal{O}_T$$

such that for $v \in \mathrm{Lie}(G)$

$$A_v = \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix} \quad \text{is constant.}$$

Hypersurfaces: f homogenous polynomial of degree d in $x_0, x_1, \dots x_{n+1}$ ver.

$$X \subseteq \mathbb{P}^{n+1} \quad \text{given by} \quad f(x_0, x_1, \dots x_{n+1})$$

Hilbert scheme of X_0 = affine subset of $\mathbb{A}^N_k \quad N := \binom{n+1+d}{d}$

$$t = (t_\alpha) \rightsquigarrow X_t : f_t = 0, \quad \text{where } f = \sum_\alpha t_\alpha x^\alpha$$

where x^α runs through all monomials of degree d in $x_0, x_1, \dots x_{n+1}$.

Kodair-Spencer: $n \geqslant 2$, $d \geqslant 3$ and $(n,d) \neq (2,4)$.

1. Any deformation of X_0 as a complex manifold is still a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$
2. Two hypersurfaces X_1, X_2 are isomorphic if and only if there is $f \in \mathrm{Aut}(\mathbb{P}^{n+1}) \simeq PGL(n+2, \mathbb{C})$ which sends X_1 to X_2.

The second item says that the moduli of X_0 is

$$H / PGL(n+2, \mathbb{C})$$

counting parameters: $\quad \dim(\;''\;) = \binom{n+1+d}{d} - (n+2)^2$

for $n=1$ item 2 fails.
for $(n,d) = (2,4)$ item 1 fails.

Remark: If $\dim H^2_{dR}(X) = 1$, we do not need to mention the polarization in our definition.

$X_1, X_2 \in H$, $\quad H^2_{dR}(X_1, \mathbb{Z}) \xrightarrow{\sim} H^2_{dR}(X_2) \simeq \mathbb{Z}$

$\to$ generated by δ_1 such that $a\delta_1$ for some $a \in \mathbb{N}$ algebraic

$\theta_1 = n\delta_1, \; \theta_2 = n\delta_2$

Weierstrass form: Any elliptic curve $/k$ can be written in the

$$y^2 = 4x^3 - t_2 x - t_3, \quad t_2, t_3 \in k, \quad \Delta := 27t_3^2 - t_2^3 \neq 0.$$

$T :=$ the moduli space of (E, α, ω), E an elliptic curve $/k$

$\alpha, \omega \in H^1_{dR}(E)$, a basis

α is holomorphic 1-form on E

$$\langle \alpha, \omega \rangle := \frac{1}{2\pi i} \int_E \alpha \cup \omega = 1 \quad (k = \mathbb{C})$$

In fact, we have a universal family

$$T := \operatorname{Spec}\left(\mathbb{Q}\left[t_1, t_2, t_3, \frac{1}{27t_3^2 - t_2^3}\right]\right)$$

$t \in T(k)$, E_t: $y^2 = 4(x - t_1)^3 - t_2(x - t_1) - t_3$ ①

$\shortparallel$

$\{(t_1, t_2, t_3) \in k^3 \mid 27t_3^2 - t_2^3 \neq 0\}$

$$\alpha = \left[\frac{dx}{y}\right], \quad \omega = \left[\frac{x\,dx}{y}\right].$$

$$\mathcal{X} \longrightarrow T$$

$\mathcal{X} \subseteq \mathbb{P}^2_k \times T$ given by ① and this map is projection

Gauss-Manin connection

$$\nabla \begin{pmatrix} \alpha \\ \omega \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{3}{2} t_1 \alpha - \frac{1}{12} d\Delta, & \frac{3}{2} \alpha \\ * & \frac{3}{2} t_1 \alpha + \frac{1}{12} d\Delta \end{pmatrix}}_{A} \otimes_{\mathcal{O}_T} \begin{pmatrix} \alpha \\ \omega \end{pmatrix}$$

$$* = \Delta\, dt_1 - \frac{1}{6} t_1 d\Delta - \left(\frac{3}{2} t_1^2 + \frac{1}{8} t_2\right)\alpha$$

What is this in a languge that Gauss could understand?

The algebraic group: $\mathbb{G}_a = (K, +)$, $\mathbb{G}_m = (K, \cdot)$

$$G_0 := \left\{ \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \;\middle|\; k' \in \mathbb{G}_a,\ k \in \mathbb{G}_m \right\}.$$

G_0 acts on T

$$(E, \alpha, \omega) \cdot g = (E, k\alpha, k'\alpha + k^{-1}\omega)$$

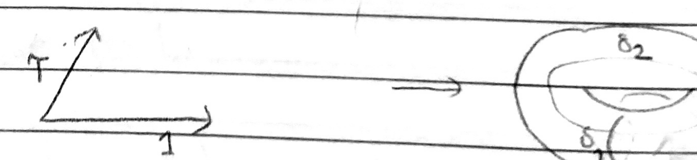
G_0 acts on T

$$(t_1, t_2, t_3) \cdot g = (t_1 \cdot k^{-2} + k' k^{-1}, t_2 k^{-4}, t_3 k^{-6})$$

How we get the holomorphic (quasi) modular forms:

$$\mathbb{H} \longrightarrow T$$

$$\tau \longrightarrow \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z},\ \alpha,\ \omega$$



$\alpha = dz$,

$$\frac{1}{\sqrt{2\pi i}} \begin{pmatrix} \int_{\delta_1} \alpha & \int_{\delta_1} \omega \\ \int_{\delta_2} \alpha & \int_{\delta_2} \omega \end{pmatrix} = \begin{pmatrix} \tau & -1 \\ 1 & 0 \end{pmatrix}.$$

The pull-back of t_1, t_2, t_3 are Eisenstein series $a_1 E_2, a_2 E_4, a_3 E_6$, resp

$$E_{2i}(q) = 1 + b_i \sum_{n=1}^{\infty} \left(\sum_{d \mid n} d^{2i-1} \right) q^n \qquad i = 1, 2, 3$$

$(a_1, a_2, a_3) = \left(\frac{2\pi i}{12}, 12\left(\frac{2\pi i}{12}\right)^2, 8\left(\frac{2\pi i}{12}\right)^3\right)$

$(b_1, b_2, b_3) = (-24, 240, -504)$

The functional equations ① are translated into

$$(c\tau+d)^{-2i} E_{2i}\left(\frac{a\tau+b}{c\tau+d}\right) = E_{2i}(\tau),$$

$$~~(c\tau+d)^{-2} E_2\left(\frac{a\tau+b}{c\tau+d}\right) = c(c\tau+d)^{-1}\left(\frac{-12}{2\pi i}\right)~~$$

Gauss-Manin Connection in general + the case of elliptic curves.

There are unique vector fields e, f, h in T such that

$$A(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A(e) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$e = -\left(t_1^2 - \frac{1}{12} t_2\right)\frac{\partial}{\partial t_1} - (4t_1t_2 - 6t_3)\frac{\partial}{\partial t_2} - \left(6t_1t_3 - \frac{1}{2}t_2^2\right)\frac{\partial}{\partial t_3}$$

$$h = -6t_3\frac{\partial}{\partial t_3} - 4t_2\frac{\partial}{\partial t_2} - 2t_1\frac{\partial}{\partial t}, \quad f = \frac{\partial}{\partial t_1}$$

This is the Lie algebra sl_2:

$$[h,e] = 2e \qquad [h,f] = -2f, \qquad [e,f] = h$$

$\mu_d := \{1, \xi_d, \dots, \xi_d^{d-1}\}$, $\mu = \{(\xi_0, \dots \xi_{n+1}) \in \mu_d^{n+2} \mid \xi_0 \xi_1 \dots \xi_{n+1} = 1\}$

μ acts on $\mathbb{P}^{n+1}$

$$(\xi_0, \xi_1, \dots), [x_0, x_1, \dots] \mapsto [\xi_0 x_0, \xi_1 x_1, \dots, \xi_{n+1} x_{n+1}]$$

The space of Hypersurfaces invariant under this action:

$$x_0^{i_0} x_1^{i_1} \cdots x_{n+1}^{i_{n+1}} \Rightarrow \xi_0^{i_0} \cdots \xi_{n+1}^{i_{n+1}} \text{ "}$$

$\Rightarrow \xi_0^{i_0} \cdots \xi_{n+1}^{i_{n+1}} = 1 \Rightarrow$ if $d = n+2$ we have only the hyp.

$$a_0 x_0^{n+2} + a_1 x_1^{n+2} + \dots + a_{n+1} x_{n+1}^{n+2} + a_{n+2} x_0 x_1 x_2 \dots x_{n+1} = 0$$

After making linear transformation $x_i \to k_i x_i$ we get the Dwork family

$$x_0^{n+2} + x_1^{n+2} + \dots + x_{n+1}^{n+2} + (n+2)\psi x_0 x_1 x_2 \dots x_{n+1} = 0, \quad \psi^{n+2} \neq 1$$

AMSY $\to$ arbitrary CY.

$n = 1, 2, 3, \dots$ Y. Nikdelan + M., $\Rightarrow$ $n = 3$.

$$H_{dR}^{n}(X)_{eq} := \{\omega \in H_{dR}^n(X) \mid \mu^* \omega = \omega\}$$

$$\dim H_{dR}^n(X)_{eq} = n+1.$$

$$H_{dR}^n(X)_{eq} \times H_{dR}^n(X)_{eq} \to K$$

$$(\alpha, \omega) \xmapsto{K = \mathbb{C}} \frac{1}{2\pi i} \int \alpha \cup \omega$$

$$\{0\} = F^{n+1} \subseteq F^n \subseteq \cdots \subseteq F^1 \subseteq F^0 = H_{dR}^n(X)_{eq}$$

$$\dim F^i = n+1-i$$

$n = 3$.

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$T :=$ moduli space of $(X, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ X invariant under μ.
$\alpha_1, \dots \alpha_4$ a basis of $H^3_{dR}(X)_{eq}$ compatible with the Hodge filtration
$\alpha_1 \in F^3$, $\alpha_1, \alpha_2 \in F^2$, $\alpha_1, \alpha_2, \alpha_3 \in F^1$, ...

$$[\langle \alpha_i, \alpha_j \rangle] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$\dim T = 7 = 6+1.$

Take affine chart $x_0 = 1$, $\omega = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{df} = \frac{dx_1 \wedge dx_2 \wedge dx_3}{(f)_{x_4}}$.

$$f = t_4 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + 5 t_0 x_1 x_2 x_3 x_4$$

(X_{t_0, t_4}, ω) is the universal family of (X, α), X as befor, α a $(3,0)$-form.

$$\omega_i = \frac{\partial^{i-1}}{\partial t_0^{i-1}} \omega \qquad i = 1, 2, 3, 4.$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_9 & t_8 & 0 & 0 \\ t_7 & t_6 & t_5 & 0 \\ t_1 & t_2 & t_3 & t_{10} \end{pmatrix}$$

$$t_9 = \frac{-5^5 t_0^4 - t_3}{t_5}$$

$$t_8 = \frac{5^4 (t_0^5 - t_4)}{t_5}$$

$$t_7 = \frac{(5^5 t_0^4 + t_3) t_6 - (5^5 t_0^3 + t_2) t_5}{5^4 (t_4 - t_0^5)}$$

$$t_{10} = (t_4 - t_0^5) \cdot 5^4$$

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$$T := \operatorname{Spec}\left[\mathbb{Q}\left[t_0, t_1, \dots t_6, \frac{1}{t_5 (t_4 - t_0^5)}\right]\right)$$

$\downarrow$

algebra of differential CY-moduler forms attached to mirror quintic.

Theorem: There are unique vector fields $R_0, R_1, \dots, R_6$ on T such that

$$A_{R_0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ & & Y & 0 \\ & & & -1 \\ & & & 0 \end{pmatrix} \quad A_{R_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A_{R_2} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_{R_3} = \begin{pmatrix} 0 & & & \\ -1 & & & \\ 0 & & & \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad A_{R_4} = \begin{pmatrix} & & & \\ & & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix} \quad A_{R_5} = \begin{pmatrix} 0 & & & \\ 0 & & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}$$

$$A_{R_6} = \begin{pmatrix} 0 & & & \\ 0 & & & \\ 0 & & & \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

where Y is a reguler function on T; We can compute all R_i's and Y.

$$Y = \frac{5^8(t_4 - t_0^5)^2}{t_5^3} \qquad R_0 = \begin{cases} \dot{t}_0 = \frac{1}{t_5}\left(6 \cdot 5^4 t_0^5 + t_0 t_3 - 5^4 t_4\right) \\ \dot{t}_1 = \frac{1}{t_5}(\dots \\ \vdots \\ \dot{t}_4 = \frac{1}{t_5}\left(5^6 t_0^4 t_4 + 5 t_3 t_4\right) \\ \dot{t}_5 \\ \dot{t}_6 = \frac{1}{t_5}\left(3 \cdot 5^4 t_0^4 t_6 - \dots\right) \end{cases}$$

take t_i's as formal power series $t_i = \sum_{n=0}^{\infty} t_{i,n} q^n$, $t_{0,0} = \frac{1}{5}$, $t_{0,1} = 24$, $t_{4,0} = 0$

substitute in R_0 $\dot{t}_i = 5q\frac{\partial}{\partial q}$ then you can compute all $t_{i,j}$'s.

$$-5^3 \cdot Y = 5 + 2875 \frac{q}{1-q} + 609250 \cdot 2^3 \frac{q^2}{1-q^2} + \dots + n_d d^3 \frac{q^d}{1-q^d} + \dots$$

Take $t \in T$, $(X_t, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$. take a symplectic basis $\delta_1, \delta_2, \delta_3, \delta_4 \in H_3(X_t, \mathbb{Z})$

The loci of t such that

$$\left[\int_{\delta_j} \alpha_i\right]_{4\times 4} = \begin{pmatrix} \tau_0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \tau_1 & \tau_3 & 1 & 0 \\ \tau_2 & -\tau_0\tau_3+\tau_1 & -\tau_0 & 1 \end{pmatrix}$$

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t_i restricted to this locus and written as a function of $q = e^{2\pi i \tau_0}$ are as before

$$F_g = \frac{Q_g}{(t_4 - t_0^5)^{2g-2}\, t_5^{3g-3}}$$, Q_g a homogenous polynomial of degree $69(g-1)$ in the ring

$\mathbb{Q}[t_0, t_1, \ldots t_6]$ $\quad$ $\deg t_i = 3(i+1)$ $\quad i = 0, 1, \ldots, 4$

$\deg t_5 = 11$, $\deg t_6 = 8$

$$F_1 = \ln\left(t_4^{\frac{25}{12}} (t_4 - t_0^5)^{\frac{-5}{12}} t_5^{\frac{1}{2}} \right)$$

$$\boxed{F_g = \sum_{d=0}^{\infty} N_{g,d}\, q^d \qquad F_1 = \frac{25}{12} \ln q + \sum_{d=1}^{\infty} N_{1,d}\, q^d}$$

If you know

$$\#\left\{ (i_0, i_1, i_2, i_3, i_4, i_6) \in \mathbb{N}_0^7 \;\middle|\; \begin{matrix} 3i_0 + 6i_1 + 9i_2 + 12i_3 + 15i_4 + \\ 11i_5 + 8i_6 = 69(g-1) \end{matrix} \right\}$$

GW invariant at each genus you can compute all of them.

BCOV anomaly equation:

$$R_i F_g = 0 \quad i = 1, 3$$

$$R_2 F_g = (2g-2) F_g$$

$$R_4 F_g = \frac{1}{2}\left(R_0^2 F_{g-1} + \sum_{r=1}^{g-1} R_0 F_r\, R_0 F_{g-r} \right).$$

We cannot compute $p(t_0,t_4)/(t_4-t_0^5)^{2g-2}$ $\deg P_g = 36(g-1)$
part of F_g.

$\| \quad [\frac{12}{5}(g-1)]+1$ coef.

If we know $\#\{(i_0,i_4)\in \mathbb{N}_0^2 \mid i_0+5i_4 = 12(g-1)\}$ GW invariant at each genus we can compute all of them.

$$\frac{P(t_0, t_4-t_0^5)}{(t_4-t_0^5)^{2g-2}} = \underbrace{\text{poles in } t_4-t_0^5}_{} + \sum_{a+5b=2g-2,\ a,b\in\mathbb{N}_0} t_0^a (t_4-t_0^5)^b$$

Using Gap condition we can even fix all the coefs here
Klemm+ ....

In total, we have $\left[\frac{2g-2}{5}\right]$ ambiguities that apply must be calculated for the enumerative geometry of GW.

Klemm+ ... Castelnuovo bound $\rightarrow$ $g \leq 51$

Fourth lecture: Abelian varieties. A/k

$k=\mathbb{C}$ $\quad A \xrightarrow{\sim} \mathbb{C}^n/\Lambda$, $\quad \Lambda \subseteq \mathbb{C}^n$ of rang $2n$.

Not every torus can be embedded in $\mathbb{P}^N$.

$$H^m_{dR}(X) = \Lambda^m_{i=1} H^1_{dR}(X), \quad m=1,2,\ldots,2n$$

its betti numbers $\quad 1, \binom{2n}{1}, \binom{2n}{2}, \ldots \binom{2n}{2n-1}, 1.$

$$\{0\} = F^2 \subseteq F^1 \subseteq F^0 = H^1_{dR}(X)$$

$\dim F^1 = n$, take a basis $\alpha_1, \alpha_2 \ldots \alpha_n$ of F^1

$\alpha_1 \ldots \alpha_{n+1}, \alpha_{2n}$ $H^1_{dR}(X)$

$H^m_{dR}(X)$ is gen. by

$$\alpha_i = \alpha_{i_1} \wedge \alpha_{i_2} \wedge \ldots \alpha_{i_m} \quad 1 \leq i_1 \leq i_2 \leq \ldots \leq i_m \leq 2n$$

F^p/F^{p+1} is gen. by α_i's with

$$1 \leq i_1 < i_2 \ldots i_p \leq n < i_{p+1} < \cdots < i_m \leq 2n.$$

Polarization: $\Theta \in H^2_{dR}(A)$, which is in $H^2(X, \mathbb{Z})$

$$\langle \cdot, \cdot \rangle : H^1_{dR}(X) \times H^1_{dR}(X) \longrightarrow k$$
$$(\alpha, \beta) \longmapsto \frac{1}{(2\pi i)^n} \int_X \alpha \cup \beta \cup \Theta^{n-1}$$

This satisfies Riemann relation

$$-\langle \omega, \bar{\omega} \rangle > 0, \quad \omega \in F^1.$$

Propostion: We can take $\alpha_1, \alpha_2, \ldots \alpha_{2n}$ in such a way that

$$\theta = \alpha_1 \wedge \alpha_{n+1} + \cdots + \alpha_n \wedge \alpha_{2n} \overset{\shortparallel}{} \Phi.$$

and in particular

$$[\langle \alpha_i, \alpha_j \rangle] = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

An enhared abelian variety is $(X, \theta, \alpha_1, \alpha_2, \ldots \alpha_{2n})$, ... as before

We will consider moduli of T enhanced abelian varieties.

$\dim T = 2n^2+n$.

$$G_0 = \left\{ \begin{pmatrix} k & k' \\ 0 & k^{-tr} \end{pmatrix} \in GL(2n, k) \;\middle|\; k k'^{tr} = k' k^{tr} \right\} . \curvearrowright T$$

$$\dim G_0 = 2n^2 - \frac{n(n-1)}{2} \qquad \dim T/G_0 = \frac{n^2+n}{2}$$

$n=2$ $=3$.

O_T := the algebra of differential Siegel moduler forms

S = moduli of (X, θ, ω), θ is a principal polarization
ω is a holom. n-form on X.

in the Lang. of prev. things $\omega = \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n$, $h^{n0}(X) = 1$.

$$T \longrightarrow S, \qquad O_S \hookrightarrow O_T.$$

$$(X, \theta, \alpha_1 \ldots \alpha_{2n}) \longmapsto (X, \theta, \alpha_1 \wedge \ldots \wedge \alpha_n)$$

$\downarrow n=2$.

$$O_S \cong \mathbb{C}[E_4, E_6, \mathscr{C}_{10}, \mathscr{C}_{12}, \mathscr{C}_{35}] \Big/ \mathscr{C}_{35}^2 = P(E_4, E_6, \mathscr{C}_{10}, \mathscr{C}_{12}, \mathscr{C}_{35})$$

P explicit poly.

Siegel upper hal plane

$$\mathbb{H}_g := \{ T \in Mat(n \times n, \mathbb{C}) \mid T^{tr} = T \quad Im(T) \text{ is a postive matrix.}$$

$$\mathring{\alpha} : \mathbb{H}_g \longrightarrow T \qquad \iota(T) = (X, \overset{\theta}{\alpha_1}, \alpha_2, \ldots \alpha_{2n})$$

$$\left[\int_{\delta_j} \alpha_i\right] = \begin{bmatrix} T & -I \\ I & 0 \end{bmatrix}$$

if you change the type of polarization diff. constant will appear here.

Moduler vector fields:

Theorem: There are unique vector fields $R_{i,j}$ on T, $i,j = 1, \ldots n$, $i \leq j$

$$A_{R_{ij}} = \begin{pmatrix} 0 & \begin{smallmatrix} & -1 \\ -1 & \end{smallmatrix} \\ 0 & 0 \end{pmatrix} \qquad \begin{matrix} (i, n+j) \text{ entry} \\ (j, n+i) \quad '' \end{matrix}$$

$$\nabla_{R_{ij}} \alpha_i = -\alpha_{n+j} \qquad \nabla \alpha_j = -\alpha_{n+i}$$

Ref: Gauss-Manin connection in disguise: Calabi-Yau modular forms, 2017.

Eisenstein series $E_{2i}(\tau) := 1 - \frac{4i}{B_{2i}} \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2i-1} \right) e^{2\pi i \tau \cdot n}$ $\quad 2k \geq 4$

f a modular form of weight $2k$ for $SL(2,\mathbb{Z})$

$$(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \ \tau \in \mathbb{H} \qquad + \text{finit growth.}$$

Geometric version: k, $\mathrm{char}(k)=0$

$$f(E/k \text{ elliptic curve}, \alpha \text{ regular dif. 1-form}) \in k$$

$$f(E, a\alpha) = a^{-i} f(E,\alpha), \qquad a \in G_m = (k,\cdot)$$

Eisenstein series E_2?

E/k, $\quad H^1_{dR}(E)$, $\quad \dim_k H^1_{dR}(E) = 2$,

$F^1 \subset H^1_{dR}(E)$, F^1 generated by regular 1-forms.

$\langle \cdot,\cdot \rangle \ H^1_{dR}(E) \times H^1_{dR}(E) \to k$ the cup product

$\alpha, \omega \in H^1_{dR}(E)$, $\alpha \in F^1$, $\langle \alpha, \omega \rangle = 1$, α, ω a basis.

$f(E,\alpha,\omega) \in k$ $\qquad f(E, a\alpha, a^{-1}\omega) = a^{-2} f(E,\alpha,\omega)$

$\qquad f(E,\alpha,\omega + a'\alpha) = f(E,\alpha,\omega) + a'$

this will correspond to $(c\tau+d)^{-2} E_2\left(\frac{a\tau+b}{c\tau+d}\right) = E_2(\tau) + c(c\tau+d)^{-1}$,

$\tau \in \mathbb{H}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

Moduli space of (E,α,ω): The universal family

$$E: \ y^2 = 4(x-t_1)^3 - t_2(x-t_1) - t_3, \quad \alpha = \left[\frac{dx}{y}\right], \ \omega = \left[\frac{x\,dx}{y}\right].$$

$$T := \mathrm{spec}\left(\mathbb{Q}\left[t_1, t_2, t_3, \frac{1}{27t_3^2 - t_2^3}\right]\right).$$

Ramanujan relations between Eisenstein series:

$$\nabla: H^1_{dR}(E/T) \longrightarrow \Omega^1_T \otimes_{O_T} H^1_{dR}(E/T)$$

R a vector field in T: $\nabla_R : H^1_{dR}(E/T) \circlearrowleft$

Proposition: There is a unique vector field R in T such that

$$\nabla_R \alpha = -\omega, \quad \nabla_R \omega = 0$$

In fact $\quad R = \left(t_1^2 - \frac{1}{12} t_2\right) \frac{\partial}{\partial t_1} + (4t_1t_2 - 6t_3) \frac{\partial}{\partial t_2} + \left(6t_1t_3 - \frac{1}{3} t_2^2\right) \frac{\partial}{\partial t_3}$

This can be done in general.

General picture: Quasi-modular forms attached to Hodge str. Fields Ins., 2013.

1. Principaly polarized A.V.
2. Lattice polarized K3 surfaces $\Bigg\} \longrightarrow$ ~~The algebra of certain automorphic forms and their derivations~~
3. Calabi-Yau varieties $\Longrightarrow$ q-expansions of physicists
4. others ??

Ask the audience to select one.

1. General context 2. P.P. A.V. 3. C.Y. varieties.

Abelian varieties: A/k $\dim_k H^1_{dR}(A) = 2n$, $n = \dim A$.

$H^m_{dR}(A) = \wedge H^1_{dR}(A)$ m-times

polarization $\theta \in H^2_{dR}(A)$.

$$0 \subseteq F^1 \subseteq F^0 = H^1_{dR}(A)$$

F^1 = reguler diff 1-forms in A $\dim_k F^1 = n$.

$$\langle \cdot, \cdot \rangle : H^1_{dR}(A) \times H^1_{dR}(A) \to k$$

when $k = \mathbb{C}$ $\langle \alpha, \beta \rangle = \frac{1}{(2\pi i)^n} \int_{A(\mathbb{C})} \alpha \cup \beta \cup \theta^{n-1}$

we can choose a basis

$\alpha_1, \alpha_2 \dots \alpha_n, \alpha_{n+1} \dots \alpha_{2n}$ of $H^1_{dR}(A)$

such that

1. $\alpha_1, \dots \alpha_n$ a basis of F^1
2. $\theta = \alpha_1 \wedge \alpha_{n+1} + \dots + \alpha_n \wedge \alpha_{2n}$

In perticular $[\langle \alpha_i, \alpha_j \rangle] = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

For fixed $\theta \in H^1(A, \mathbb{Z})$, moduli of $(A, \alpha_1, \dots \alpha_{2n})$. $\dim T = 2n^2 + n$.

(Fonseca) as an algebraic stack.

Theorem: global sections of $\mathcal{O}_T \simeq$ the algebra of defferential Siegel mod forms for $Sp(2n, \mathbb{Z})$ if θ comes from a principal polarization