From Picard and Simart's books to periods of algebraic cycles.

Hossein Movasati

IMPA, MPIM,
www.impa.br/$\sim$hossein/

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Abstract:

The origin of Hodge theory goes back to many works on elliptic, abelian and multiple integrals (periods). In particular, Picard and Simart's book "Théorie des fonctions algébriques de deux variables indépendantes. Vol. I, II." published in 1897, 1906, paved the road for modern Hodge theory. The first half of the talk is mainly about these books, for instance, I am going to explain how Lefschetz was puzzled with the computation of Picard rank (by Picard and using periods) and this led him to consider the homology classes of curves inside surfaces. This was ultimately formulated in Lefschetz (1,1) theorem and then the Hodge conjecture. In the second half of the talk I will discuss periods of algebraic cycles and will give some applications in identifying some components of the Noether-Lefschetz and Hodge locus. The talk is based on my book: A course in Hodge Theory: With Emphasis on Multiple Integrals,

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É. Picard and G. Simart. Théorie des fonctions algébriques de deux variables indépendantes. Vol. I, II. Gauthier-Villars. 1897, 1906.

From these books we have:Picard-Lefschetz theory,
Picard-Fuchs equations, Picard rank, Picard group.

Georges Simart (1846-1921)

In Picard-Simart book he is introduced as ‘capitaine de frégate, répétiteur a l’École Polytechnique’. In the preface of the same book, which is only signed by Picard, one reads “Mon ami, M. Simart, qui m’a déjà rendu de grands services dans la publication de mon Traité d’Analyse, ayant bien voulu me promettre son concours, a levé mes hésitations. J’ai traité cet hiver dans mon cours de la Théorie des surfaces algébriques, et nous avons, M. Simart et moi, rassemblé ces Leçons dans le Tome premier, que nous publions aujourd’hui".

After many works on elliptic and abelian integrals by Augustin Louis Cauchy (1789-1857), Niels Henrik Abel (1802-1829), Carl Gustav Jacob Jacobi (1804-1851) and Georg Friedrich Bernhard Riemann (1826-1866), Jules Henri Poincaré (1854-1912), among many others, it was time to go to higher dimensions.

The main aim of these books is to study the integrals:

$$\iint \frac{P(x,y,z)dxdy}{f'_z}, \quad P, f \in \mathbb{C}[x,y,z] \tag{1}$$

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and in general the integration of any meromorphic differential form over ??.In particular

$$\int \frac{P dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1}}{f^k}, \quad P, f \in \mathbb{C}[x_1, x_2, \ldots, x_{n+1}]. \tag{2}$$

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“M. Picard a donné à ces integrales le nom de périodes; je ne saurais l’en blâmer puisque cette dénomination lui a permis d’exprimer dans un langage plus concis les intéressants résultats auxquels il est parvenu. Mais je crois qu’il serait fàcheux qu’elle s’introduisit définitivement dans la science et qu’elle serait propre à engendrer de nombreuses confusions", (H. Poincaré 1887).

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For any 'smooth' projective variety X, and in particular a hypersurface in $\mathbb{P}^{n+1}$ given by $f = 0$:

1. The study of integration domain was developed into, Poincaré's Analysis Situs and later into singular (co)homology theories ($H_q(X, \mathbb{Z}), H^q(X, \mathbb{Z})$) and homological algebra.
2. The study of integrands were developed into de Rham cohomology ($H^q_{\mathrm{dR}}(X)$), and later to many cohomology theories in Algebraic Geometry.

Origin of the Hodge conjecture

Origin of the Hodge conjecture

Généralités.

1. Nous appellerons *surface hyperelliptique* toute surface telle que les coordonnées x, y, z d'un de ses points puissent s'exprimer par des fonctions uniformes de deux paramètres ayant quatre paires de périodes.

M. Appell a établi [1], ce qu'on déduit d'ailleurs d'un théorème célèbre de MM. Poincaré et Picard, qu'on peut mettre x, y, z sous la forme

$$x = \frac{\theta_1(u,v)}{\theta_0(u,v)}, \qquad y = \frac{\theta_2(u,v)}{\theta_0(u,v)}, \qquad z = \frac{\theta_3(u,v)}{\theta_0(u,v)},$$

les fonctions θ_h étant uniformes, et vérifiant les relations

$$(1)\quad \left\{\begin{aligned} &\theta_h(u+2\pi i, v) = \theta_h(u, v+2\pi i) = \theta_h(u,v) \\ &\theta_h(u+a, v+b) = e^{-mu+\alpha}\theta_h(u,v) \\ &\theta_h(u+b, v+c) = e^{-mv+\beta}\theta_h(u,v) \end{aligned}\right\} (h = 0, 1, 2, 3).$$

Figure: Pages 32 of G. Humbert (1859-1921), Théorie générale des surfaces hyperelliptiques, Journal de mathématiques pures et appliquées 4e série, tome 9 (1893), p. 29-170.

Riemann's theta series:

Let $a, b \in \mathbb{R}^g$ and $\mathbb{H}_g$ be the genus g Siegel domain. It is the set of $g \times g$ symmetric matrices τ over the complex numbers whose imaginary part is positive definite. The Riemann theta function $\theta \begin{bmatrix} a \\ b \end{bmatrix}$ with characteristics (a, b) is the following holomorphic function:

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} \quad : \quad \mathbb{C}^g \times \mathbb{H}_g \to \mathbb{C}$$

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) \quad := \quad \sum_{n \in \mathbb{Z}^g} \exp\left(\frac{1}{2}(n+a)^{\mathrm{t}} \cdot \tau \cdot (n+a) + (n+a)^{\mathrm{t}} \cdot (z+b) \right)$$

Let $\Lambda := \{\tau m + n \mid m, n \in \mathbb{Z}^g\} \cong \mathbb{Z}^{2g}$ and $\mathbb{C}^g/\Lambda$ be the corresponding complex compact torus.

Hyperelliptic surfaces:

Let us now consider $g = 2$. Let also $\theta_i, \quad i = 1, 2, 3, 4$ be a collection of linearly independent Riemann's theta functions with the same characteristics a, b. G. Humbert is talking about the image X of the map:

$$\mathbb{C}^2/\Lambda \to \mathbb{P}^3, \; z \mapsto [\theta_1; \theta_2; \theta_3; \theta_4]. \tag{3}$$

This is an algebraic surface with singularities.

Algebraic Geometry of Humbert and many others around 1900 was limited to affine varieties and so instead of the projective geometry notation, one was mainly interested on the surface

$$U := \left\{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 0\right\} \subset X, \tag{4}$$

where f is the polynomial relation between the following quotient of theta series:

$$(x, y, z) = \left(\frac{\theta_2(z, \tau)}{\theta_1(z, \tau)}, \frac{\theta_3(z, \tau)}{\theta_1(z, \tau)}, \frac{\theta_4(z, \tau)}{\theta_1(z, \tau)}\right) \tag{5}$$

Why such a complicated example? Why not simply the Fermat surface:

$$X : x^d + y^d + z^d + w^d = 0.$$

Answer: At the time of Humbert and Picard we did know the topology of $\mathbb{C}^2/\Lambda$: It is the product of four circles. But we did not know the topology of a general surface and in particular Fermat surface!

No more hyperelliptic surface:

In modern language Humbert and Picard studied singular models for abelian surfaces: A purely algebraic approach to the topic of equations for abelian varieties was introduced by Mumford in a series of articles 1966, which in turn, are based on many works of Baily, Cartier, Igusa, Siegel, Weil. It does not seem to me that the contributions of Humbert and Picard as founders of the subject is acknowledged in the modern treatment of equations defining abelian varieties.

The modern meaning of hyperelliptic surface seems to appear first in the work of Bombieri and Mumford in 1977. In a private letter Mumford explains this: “I believe Enrico and I knew that the word “hyperelliptic" had been used classically as a name for abelian surfaces but we felt that this usage was no longer followed, i.e. after Weil’s books, the term “abelian varieties" had taken precedence. So the word “hyperelliptic" seemed to be a reasonable term for this other class of surfaces."

To close personal recollections, let me tell you what made me turn with all possible vigor to topology. From the ρ_0 formula of Picard, applied to a hyperelliptic surface Φ (topologically the product of 4 circles) I had come to believe that the second Betti number $R_2(\Phi)=5$, whereas clearly $R_2(\Phi)=6$. What was wrong? After considerable time it dawned upon me that Picard only dealt with *finite* 2-cycles, the only useful cycles for calculating periods of certain double integrals. Missing link? The cycle at infinity, that is the plane section of the surface at infinity. This drew my attention to cycles carried by an algebraic curve, that is to *algebraic* cycles, and $\cdots$ the harpoon was in!

Figure: S. Lefschetz, A page of mathematical autobiography. Bull. Amer. Math. Soc., 74:854–879, 1968.

Lefschetz puzzle:

Let us redefine X to be the abelian surface $\mathbb{C}^2/\Lambda$, let Y be the subvariety of X which is the zero locus of θ_1 (the curve at infinity) and $U := X \setminus Y$. From the long exact sequence of $U \subset X$ we get

$$\begin{array}{ccccc} H_2(U) & \hookrightarrow & H_2(X) & \to & H_0(Y) \cong \mathbb{Z} \\ 5 & & 6 & & 1 \end{array}$$

The second map is the intersection with Y. The homology class of Y in X is a one dimensional subspace of $H_2(X)$ and $1 = 6 - 5$.

Algebraic de Rham cohomology:

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$$\dim H^2_{\mathrm{dR}}(U) = 5.$$

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The algebraic de Rham cohomology is mainly attributed to A. Grothendieck (1966). Grothendieck's contribution is just a final remark on a creation of a mathematical object! At the time Picard computed this, neither de Rham nor Grothendicek was born!

Let us consider the projective space $\mathbb{P}^{n+1}$ with the coordinates $[x_0 : x_1 : \cdots : x_{n+1}]$, a smooth hypersurface in $\mathbb{P}^{n+1}$ given by $f = 0$, where f is a homogeneous polynomial of degree d in $x_0, x_1, \cdots, x_{n+1}$, and

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$$\omega_i := \mathrm{Resi}\left(\frac{x^i \cdot \sum_{j=0}^{n+1}(-1)^j x_j \; dx_0 \wedge \cdots \wedge \widehat{d}x_j \wedge \cdots \wedge dx_{n+1}}{f^k}\right)$$

with $x^i = x_0^{i_0} \cdots x_{n+1}^{i_{n+1}},\; i_0, i_1, \ldots, i_{n+1} \in \mathbb{N}_0$, $k := \frac{n+2+\sum_{e=0}^{n+1} i_e}{d} \in \mathbb{N}$, and

$$\mathrm{Resi} : H^{n+1}_{\mathrm{dR}}(\mathbb{P}^{n+1} - X) \xrightarrow{\sim} H^n_{\mathrm{dR}}(X)_0$$

is the residue map.

After a theorem of Griffiths in 1970, we can introduce (*n* an even number):

Definition

A topological cycle $\delta \in H_n(X, \mathbb{Z})$ *is called a Hodge cycle if*

$$\int_\delta \omega_i = 0, \ \ \forall i \quad \textit{with} \quad \frac{n+2+\sum_{e=0}^{n+1} i_e}{d} \leq \frac{n}{2}.$$

Hodge Conjecture for hypersurfaces:

A cycle $\delta \in H_n(X, \mathbb{Z})$ is Hodge if and only if there is an algebraic cycle

$$Z := \sum_{i=1}^{s} a_i Z_i, \; \dim(Z_i) = \frac{n}{2}, \;\; a_i \in \mathbb{Z}$$

and $a \in \mathbb{N}$ such that

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The first two chapters of my book search for the origin of the Hodge conjecture in the Picard-Simart Book.

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For a smooth surface $X \subset \mathbb{P}^3$ the $\mathbb{Z}$-module of Hodge cycles, the Neron-Severi group and the Picard group are all the same.

Why is the Hodge conjecture difficult?

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“But the whole program [Grothendieck’s program on how to prove the Weil conjectures] relied on finding enough algebraic cycles on algebraic varieties, and on this question one has made essentially no progress since the 1970s.... For the proposed definition [of Grothendieck on a category of pure motives] to be viable, one needs the existence of “enough” algebraic cycles. On this question almost no progress has been made, (P. Deligne 2014)....la construction de cycles algébriques intéressants, les progrès ont été maigres, (P. Deligne 1994). "

Fermat varieties:

Fermat variety of dimension n and degree d:

$$X_n^d : \quad x_0^d + x_1^d + \cdots + x_{n+1}^d = 0. \tag{7}$$

Fermat surface of degree d:

$$X_2^d : \quad x^d + y^d + z^d + w^d = 0.$$

Well-known curves of the Fermat surface:

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$$\begin{cases} x^{\frac{d}{3}} + y^{\frac{d}{3}} + z^{\frac{d}{3}} = 0, \\ w^3 - 3^{\frac{3}{d}} \zeta_{\frac{d}{3}} xyz = 0 \end{cases} \tag{8}$$

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Other curves can be produced using the automorphism group of the Fermat surface.

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If $d \leq 100$ and $(d, 6) = 1$, then the Neron-Severi group of X is generated by lines on X.

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For the above affirmation we only need the hypothesis $d \leq 4$ or $(d, 6) = 1$.

Theorem (Aoki 1988)

For $d \leq 11$ lines together with Aoki-Shioda curves generate the Neron-Severi group of X over $\mathbb{Q}$.

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In the literature we do not have a complete list of algebraic curves which generate the Neron-Severi group of X_2^{12}, (N. Aoki, in a private letter has told me that he will soon publish an article on this!)

Periods of Hodge/algebraic cycles:

For a Hodge cycle $\delta \in H_n(X, \mathbb{Z})_0$, we define its periods

$$\mathsf{p}_i := \frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}} \int_{\delta} \omega_i, \tag{9}$$

$$\sum_{e=0}^{n+1} i_e = (\frac{n}{2} + 1)d - (n+2).$$

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Theorem (P. Deligne 1982)

If X and the algebraic cycle Z are defined over a field $\mathrm{k} \subset \mathbb{C}$ *and* $\delta = [Z]$ *then*

$$\mathrm{p}_i \in \mathrm{k}.$$

If X is the Fermat variety then p_i*'s are in an abelian extension of* $\mathbb{Q}(\zeta_d)$.

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This observation naturally leads us to the notion of absolute Hodge cycles due to P. Deligne.

Let us consider

$$X_t : x_0^d + x_1^d + \cdots + x_{n+1}^d - \sum_{j \in I_d} t_j x^j = 0, \qquad (10)$$

where

$$I_d := \{j \in \mathbb{N}_0^{n+1},\ \sum_{e=0}^{n+1} j_e = d,\ 0 \le j_e \le d-2\}.$$

Here $t = (t_j,\ j \in I_d) \in \mathsf{T} := \mathbb{C}^{\# I_d}$ and

$$X_0 = X_n^d.$$

Hodge and Noether-Lefschetz locus:

Let $\delta_t \in H_n(X_t, \mathbb{Z}), \quad t \in (\mathrm{T}, 0)$ be a continous family of cycles and δ_0 be a Hodge cycle. Let also $\omega_1, \omega_2, \ldots, \omega_a$ be differential forms as in the definition of a Hodge cycle.

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Definition

The (analytic) Hodge locus passing through 0 and corresponding to δ is the following local analytic scheme:

$$\mathcal{O}_{V_{\delta_0}} := \mathcal{O}_{\mathsf{T},0} \Big/ \left\langle \int_{\delta_t} \omega_1, \int_{\delta_t} \omega_2, \cdots, \int_{\delta_t} \omega_a \right\rangle . \tag{11}$$

The underlying analytic variety is:

$$\left\{ t \in (\mathsf{T}, 0) \;\middle|\; \int_{\delta_t} \omega_1 = \int_{\delta_t} \omega_2 = \cdots = \int_{\delta_t} \omega_a = 0 \right\}. \tag{12}$$

In the two dimensional case, that is $\dim(X_t) = 2$, Hodge locus is usually called Noether-Lefschetz locus.

Theorem

The Zariski tangent space of the Hodge locus V_{δ_0} passing through the Fermat point $0 \in \mathsf{T}$ and corresponding to the Hodge cycle δ_0 is given by $\ker([\mathsf{p}_{i+j}]^{\mathsf{t}})$.

Theorem

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The origin of the above theorem lies in the so called infinitesimal variation of Hodge structures (IVHS) of families of projective varieties developed by P. Griffiths and his collaborators in 1983. C. Voisin in her book in 2003 relates IVHS with the Zariski tangent space of the Hodge locus. For Fermat varieties one has to do further computations (done by the author in 2015) to get this Theorem.

Theorem (M. 2015)

For any smooth hypersurface of degree d and dimension n in a Zariski neighborhood of the Fermat variety with $d \geq 2 + \frac{4}{n}$ and a linear projective space $\mathbb{P}^{\frac{n}{2}} \subset X$, deformations of $\mathbb{P}^{\frac{n}{2}}$ as an algebraic cycle and Hodge cycle are the same.

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The proof of the above theorem reduces to the following: for $\mathbb{P}^{\frac{n}{2}} \subset X^d_n$ and

$$\mathsf{p}_i := \frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}} \int_{\mathbb{P}^{\frac{n}{2}}} \omega_i, \tag{13}$$

with

$$\sum_{e=0}^{n+1} i_e = (\frac{n}{2}+1)d - (n+2).$$

show that

$$\operatorname{rank}([\mathsf{p}_{i+j}]) = \binom{\frac{n}{2}+d}{d} - (\frac{n}{2}+1)^2. \tag{14}$$

For further reading see my book:

A course in Hodge theory: with emphasis on multiple integrals

http://w3.impa.br/ hossein/myarticles/hodgetheory.pdf