Periods of Algebraic Cycles

Hossein Movasati

IMPA, MPIM,
www.impa.br/∼hossein/

Abstract:

The origin of Hodge theory goes back to many works on elliptic, abelian and multiple integrals (periods). The talk will start with some remarks on the books written by E. Picard and G. Simart in 1897 and 1906, where the origin of Hodge conjecture lies. Then I will discuss periods of algebraic cycles and will give some applications in identifying some components of the Noether-Lefschetz and Hodge locus. The talk is based on my book under preparation available in my webpage: A course in Hodge Theory: With Emphasis on Multiple Integrals

The origins of Hodge theory go back to the books:

The origins of Hodge theory go back to the books:

É. Picard and G. Simart. Théorie des fonctions algébriques de deux variables indépendantes. Vol. I, II. Gauthier-Villars. 1897, 1906.

From these books we have:

The origins of Hodge theory go back to the books:

É. Picard and G. Simart. Théorie des fonctions algébriques de deux variables indépendantes. Vol. I, II. Gauthier-Villars. 1897, 1906.

From these books we have:Picard-Lefschetz theory,
Picard-Fuchs equations, Picard rank, Picard group.

Georges Simart (1846-1921)

In Picard-Simart book he is introduced as ‘capitaine de frégate, répétiteur a l’École Polytechnique’. In the preface of the same book, which is only signed by Picard, one reads “Mon ami, M. Simart, qui m’a déjà rendu de grands services dans la publication de mon Traité d’Analyse, ayant bien voulu me promettre son concours, a levé mes hésitations. J’ai traité cet hiver dans mon cours de la Théorie des surfaces algébriques, et nous avons, M. Simart et moi, rassemblé ces Leçons dans le Tome premier, que nous publions aujourd’hui".

After many works on elliptic and abelian integrals by Augustin Louis Cauchy (1789-1857), Niels Henrik Abel (1802-1829), Carl Gustav Jacob Jacobi (1804-1851) and Georg Friedrich Bernhard Riemann (1826-1866), Jules Henri Poincaré (1854-1912), among many others, it was time to go to higher dimensions.

The main aim of these books is to study the integrals:

$$\iint \frac{P(x,y,z)dxdy}{f'_z}, \quad P, f \in \mathbb{C}[x,y,z] \tag{1}$$

and in general the integration of any meromorphic differential form over ??.

The main aim of these books is to study the integrals:

$$\int\!\!\int \frac{P(x,y,z)dxdy}{f'_z}, \quad P, f \in \mathbb{C}[x,y,z] \tag{1}$$

and in general the integration of any meromorphic differential form over ??.In particular

$$\int \frac{Pdx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1}}{f^k}, \quad P, f \in \mathbb{C}[x_1, x_2, \ldots, x_{n+1}]. \tag{2}$$

Picard gave the name ’period’ to these integrals.

Picard gave the name ’period’ to these integrals.
“M. Picard a donné à ces integrales le nom de périodes; je ne saurais l’en blàmer puisque cette dénomination lui a permis d’exprimer dans un langage plus concis les intéressants résultats auxquels il est parvenu. Mais je crois qu’il serait fàcheux qu’elle s’introduisit définitivement dans la science et qu’elle serait propre à engendrer de nombreuses confusions", (H. Poincaré 1887).

For any 'smooth' projective variety X, and in particular a hypersurface in $\mathbb{P}^{n+1}$ given by $f = 0$:

For any ’smooth’ projective variety X, and in particular a hypersurface in $\mathbb{P}^{n+1}$ given by $f = 0$:

1. The study of integration domain was developed into, Poincaré’s Analysis Situs and later into singular (co)homology theories ($H_q(X, \mathbb{Z}), H^q(X, \mathbb{Z})$) and homological algebra.
2. The study of integrands were developed into de Rham cohomology ($H^q_{\mathrm{dR}}(X)$), and later to many cohomology theories in Algebraic Geometry.

Let us consider the projective space $\mathbb{P}^{n+1}$ with the coordinates $[x_0 : x_1 : \cdots : x_{n+1}]$, a smooth hypersurface in $\mathbb{P}^{n+1}$ given by $f = 0$, where f is a homogeneous polynomial of degree d in $x_0, x_1, \cdots, x_{n+1}$, and

Let us consider the projective space $\mathbb{P}^{n+1}$ with the coordinates $[x_0 : x_1 : \cdots : x_{n+1}]$, a smooth hypersurface in $\mathbb{P}^{n+1}$ given by $f = 0$, where f is a homogeneous polynomial of degree d in $x_0, x_1, \cdots, x_{n+1}$, and

$$\omega_i := \mathrm{Resi}\left(\frac{x^i \cdot \sum_{j=0}^{n+1} (-1)^j x_j \; dx_0 \wedge \cdots \wedge \widehat{d} x_j \wedge \cdots \wedge dx_{n+1}}{f^{k+1}}\right)$$

with $x^i = x_0^{i_0} \cdots x_{n+1}^{i_{n+1}},\ i_0, i_1, \ldots, i_{n+1} \in \mathbb{N}_0$, $k := \frac{n+2+\sum_{e=0}^{n+1} i_e}{d} \in \mathbb{N}$, and

$$\mathrm{Resi} : H_{\mathrm{dR}}^{n+1}(\mathbb{P}^{n+1} - X) \xrightarrow{\sim} H_{\mathrm{dR}}^{n}(X)_0$$

is the residue map.

After a theorem of Griffiths in 1970, we can introduce:

Definition

A topological cycle $\delta \in H_n(X, \mathbb{Z})$ *is called a Hodge cycle if*

$$\int_\delta \omega_i = 0, \quad \forall i \quad \textit{with} \quad \frac{n+2+\sum_{e=0}^{n+1} i_e}{d} \leq \frac{n}{2}.$$

Hodge Conjectures for hypersurfaces:

A cycle $\delta \in H_n(X, \mathbb{Z})$ is Hodge if and only if there is an algebraic cycle

$$Z := \sum_{i=1}^{s} a_i Z_i, \ \dim(Z_i) = \frac{n}{2}, \quad a_i \in \mathbb{Z}$$

and $a \in \mathbb{N}$ such that

$$a \cdot \delta = [Z].$$

Hodge Conjectures for hypersurfaces:

A cycle $\delta \in H_n(X, \mathbb{Z})$ is Hodge if and only if there is an algebraic cycle

$$Z := \sum_{i=1}^{s} a_i Z_i,\ \dim(Z_i) = \frac{n}{2},\ \ a_i \in \mathbb{Z}$$

and $a \in \mathbb{N}$ such that

$$a \cdot \delta = [Z].$$

If Picard were alive he would have formulated the Hodge conjecture in this way!

Hodge Conjectures for hypersurfaces:

A cycle $\delta \in H_n(X, \mathbb{Z})$ is Hodge if and only if there is an algebraic cycle

$$Z := \sum_{i=1}^{s} a_i Z_i, \ \dim(Z_i) = \frac{n}{2}, \ \ a_i \in \mathbb{Z}$$

and $a \in \mathbb{N}$ such that

$$a \cdot \delta = [Z].$$

If Picard were alive he would have formulated the Hodge conjecture in this way!
The first two chapters of my book search for the origin of the Hodge conjecture in the Picard-Simart Book.

Lefschetz $(1,1)$-theorem in its original format:

Lefschetz $(1, 1)$-theorem in its original format:

On an algebraic surface X a 2-dimensional homology cycle δ is the homology class of an algebraic curve if and only if

$$\int_{\delta} \omega = 0, \tag{3}$$

for all holomorphic differential 2-forms in X.

Lefschetz $(1,1)$-theorem in its original format:

On an algebraic surface X a 2-dimensional homology cycle δ is the homology class of an algebraic curve if and only if

$$\int_{\delta} \omega = 0, \quad (3)$$

for all holomorphic differential 2-forms in X.
For a smooth surface $X \subset \mathbb{P}^3$ the $\mathbb{Z}$-module of Hodge cycles, the Neron-Severi group and the Picard group are all the same.

Why is the Hodge conjecture difficult?

Why is the Hodge conjecture difficult?

“But the whole program [Grothendieck’s program on how to prove the Weil conjectures] relied on finding enough algebraic cycles on algebraic varieties, and on this question one has made essentially no progress since the 1970s.... For the proposed definition [of Grothendieck on a category of pure motives] to be viable, one needs the existence of “enough” algebraic cycles. On this question almost no progress has been made, (P. Deligne 2014)....la construction de cycles algébriques intéressants, les progrès ont été maigres, (P. Deligne 1994). "

Fermat varieties:

Fermat variety of dimension n and degree d:

$$X_n^d : \quad x_0^d + x_1^d + \cdots + x_{n+1}^d = 0. \tag{4}$$

Fermat surface of degree d:

$$X_2^d : \quad x^d + y^d + z^d + w^d = 0.$$

Well-known curves of the Fermat surface:

1. Linear curves:

$$x - \zeta_{2d} y = z - \zeta_{2d} w = 0.$$

Well-known curves of the Fermat surface:

1. Linear curves:

$$x - \zeta_{2d} y = z - \zeta_{2d} w = 0.$$

2. Aoki-Shioda curves for $3 \mid d$:

$$\begin{cases} x^{\frac{d}{3}} + y^{\frac{d}{3}} + z^{\frac{d}{3}} = 0, \\ w^3 - 3^{\frac{3}{d}} \zeta_{\frac{d}{3}} xyz = 0 \end{cases} \tag{5}$$

Well-known curves of the Fermat surface:

1. Linear curves:

$$x - \zeta_{2d} y = z - \zeta_{2d} w = 0.$$

2. Aoki-Shioda curves for $3 \mid d$:

$$\begin{cases} x^{\frac{d}{3}} + y^{\frac{d}{3}} + z^{\frac{d}{3}} = 0, \\ w^3 - 3^{\frac{3}{d}} \zeta_{\frac{d}{3}} xyz = 0 \end{cases} \tag{5}$$

Other curves can be produced using the automorphism group of the Fermat surface.

Theorem (Schuett, Shioda, van Luijk, 2010)

If $d \leq 100$ and $(d, 6) = 1$, then the Neron-Severi group of X is generated by lines on X.

Theorem (Schuett, Shioda, van Luijk, 2010)

If $d \leq 100$ *and* $(d, 6) = 1$*, then the Neron-Severi group of* X *is generated by lines on* X*.*

Theorem (Degtyarev, 2015)

For the above affirmation we only need the hypothesis $d \leq 4$ *or* $(d, 6) = 1$*.*

Theorem (Schuett, Shioda, van Luijk, 2010)

If $d \leq 100$ and $(d, 6) = 1$, then the Neron-Severi group of X is generated by lines on X.

Theorem (Degtyarev, 2015)

For the above affirmation we only need the hypothesis $d \leq 4$ or $(d, 6) = 1$.

Theorem (Aoki 1988)

For $d \leq 11$ lines together with Aoki-Shioda curves generate the Neron-Severi group of X over $\mathbb{Q}$.

The lines and the Aoki-Shioda curves are not enough to generate the Neron-Severi group of

$$X_2^{12}: \quad x^{12} + y^{12} + z^{12} + w^{12} = 0$$

The lines and the Aoki-Shioda curves are not enough to generate the Neron-Severi group of

$$X_2^{12}: \quad x^{12} + y^{12} + z^{12} + w^{12} = 0$$

In the literature we do not have a complete list of algebraic curves which generate the Neron-Severi group of X_2^{12}, (N. Aoki, in a private letter has told me that he will soon publish an article on this!)

Periods of Hodge/algebraic cycles:

For a Hodge cycle $\delta \in H_n(X, \mathbb{Z})_0$, we define its periods

$$\mathsf{p}_i := \frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}} \int_\delta \omega_i, \qquad (6)$$

$$\sum_{e=0}^{n+1} i_e = (\frac{n}{2}+1)d - (n+2).$$

Periods of Hodge/algebraic cycles:

For a Hodge cycle $\delta \in H_n(X, \mathbb{Z})_0$, we define its periods

$$\mathsf{p}_i := \frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}} \int_\delta \omega_i, \tag{6}$$

$$\sum_{e=0}^{n+1} i_e = (\frac{n}{2} + 1)d - (n+2).$$

Theorem (P. Deligne 1982)

If X and the algebraic cycle Z are defined over a field $\mathsf{k} \subset \mathbb{C}$ and $\delta = [Z]$ then

$$\mathsf{p}_i \in \mathsf{k}.$$

If X is the Fermat variety then p_i's are in an abelian extension of $\mathbb{Q}(\zeta_d)$.

Periods of Hodge/algebraic cycles:

For a Hodge cycle $\delta \in H_n(X, \mathbb{Z})_0$, we define its periods

$$\mathrm{p}_i := \frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}} \int_{\delta} \omega_i, \tag{6}$$

$$\sum_{e=0}^{n+1} i_e = (\frac{n}{2} + 1)d - (n+2).$$

Theorem (P. Deligne 1982)

If X and the algebraic cycle Z are defined over a field $\mathrm{k} \subset \mathbb{C}$ and $\delta = [Z]$ then

$$\mathrm{p}_i \in \mathrm{k}.$$

If X is the Fermat variety then p_i's are in an abelian extension of $\mathbb{Q}(\zeta_d)$.

This observation naturally leads us to the notion of absolute Hodge cycles due to P. Deligne.

Let us consider

$$X_t : x_0^d + x_1^d + \cdots + x_{n+1}^d - \sum_{j \in I_d} t_j x^j = 0, \tag{7}$$

where

$$I_d := \{j \in \mathbb{N}_0^{n+1}, \ \sum_{e=0}^{n+1} j_e = d, \ 0 \leq j_e \leq d-2\}.$$

Here $t = (t_j, \ j \in I_d) \in \mathsf{T} := \mathbb{C}^{\# I_d}$ and

$$X_0 = X_n^d.$$

Hodge and Noether-Lefschetz locus:

Let $\delta_t \in H_n(X_t, \mathbb{Z}),\ \ t \in (\mathsf{T}, 0)$ be a continous family of cycles and δ_0 be a Hodge cycle. Let also $\omega_1, \omega_2, \ldots, \omega_a$ be as in the definition of a Hodge cycle.

Hodge and Noether-Lefschetz locus:

Let $\delta_t \in H_n(X_t, \mathbb{Z}), \quad t \in (\mathsf{T}, 0)$ be a continous family of cycles and δ_0 be a Hodge cycle. Let also $\omega_1, \omega_2, \dots, \omega_a$ be as in the definition of a Hodge cycle.

Definition

The (analytic) Hodge locus passing through 0 and corresponding to δ is the following local analytic scheme:

$$V_{\delta_0} := \mathrm{Spec}\left(\mathcal{O}_{\mathsf{T},0} \Big/ \left\langle \int_{\delta_t} \omega_1, \int_{\delta_t} \omega_2, \cdots, \int_{\delta_t} \omega_a \right\rangle \right) \subset \mathrm{Spec}(\mathcal{O}_{\mathsf{T},0}). \tag{8}$$

The underlying analytic variety is:

$$\left\{ t \in (\mathsf{T}, 0) \;\middle|\; \int_{\delta_t} \omega_1 = \int_{\delta_t} \omega_2 = \cdots = \int_{\delta_t} \omega_a = 0 \right\}. \tag{9}$$

In the two dimensional case, that is $\dim(X_t) = 2$, Hodge locus is usually called Noether-Lefschetz locus.

Theorem

The Zariski tangent space of the Hodge locus V_{δ_0} passing through the Fermat point $0 \in \mathsf{T}$ and corresponding to the Hodge cycle δ_0 is given by $\ker([\mathsf{p}_{i+j}]^{\mathsf{t}})$.

Theorem

The Zariski tangent space of the Hodge locus V_{δ_0} passing through the Fermat point $0 \in \mathsf{T}$ and corresponding to the Hodge cycle δ_0 is given by $\ker([\mathsf{p}_{i+j}]^{\mathsf{t}})$.

The origin of the above theorem lies in the so called infinitesimal variation of Hodge structures (IVHS) of families of projective varieties developed by P. Griffiths and his collaborators in 1983. C. Voisin in her book in 2003 relates IVHS with the Zariski tangent space of the Hodge locus. For Fermat varieties one has to do further computations (done by the author in 2015) to get this Theorem.

Theorem (M. 2015)

For any smooth hypersurface of degree d and dimension n in a Zariski neighborhood of the Fermat variety with $d \geq 2 + \frac{4}{n}$ and a linear projective space $\mathbb{P}^{\frac{n}{2}} \subset X$, deformations of $\mathbb{P}^{\frac{n}{2}}$ as an algebraic cycle and Hodge cycle are the same.

Theorem (M. 2015)

For any smooth hypersurface of degree d and dimension n in a Zariski neighborhood of the Fermat variety with $d \geq 2 + \frac{4}{n}$ and a linear projective space $\mathbb{P}^{\frac{n}{2}} \subset X$, deformations of $\mathbb{P}^{\frac{n}{2}}$ as an algebraic cycle and Hodge cycle are the same.

The proof of the above theorem reduces to the following: for $\mathbb{P}^{\frac{n}{2}} \subset X^d_n$ and

$$\mathsf{p}_i := \frac{1}{(2\pi\sqrt{-1})^{\frac{n}{2}}} \int_{\mathbb{P}^{\frac{n}{2}}} \omega_i, \tag{10}$$

with

$$\sum_{e=0}^{n+1} i_e = (\frac{n}{2} + 1)d - (n+2).$$

show that

$$\mathrm{rank}([\mathsf{p}_{i+j}]) = \binom{\frac{n}{2} + d}{d} - (\frac{n}{2} + 1)^2. \tag{11}$$

For further reading see my book:

A course in Hodge theory: with emphasis on multiple integrals

http://w3.impa.br/ hossein/myarticles/hodgetheory.pdf