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Computing a period / Taylor series
Feb. 2017, IMPA.

$$\boxed{(k-1)!\int_{\delta_t} \operatorname{Resi}\left(\frac{x^{\beta}\Omega}{f^k}\right)} = \boxed{\sum_{a:I_d\to \mathbb{N}_0} \frac{1}{a!} D_{\beta+a^*}\, P_{\overline{\beta+a^*}} \cdot t^{a}}$$

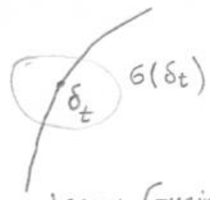
(1) $\Omega = \sum (-1)^j x_j\, dx_0\wedge\ldots\wedge\widehat{dx_j}\wedge\ldots\wedge dx_{n+1}$, $x^{\beta} := x_0^{\beta_0} x_1^{\beta_1}\cdots x_{n+1}^{\beta_{n+1}}$

$f_t = x_0^d + x_1^d + \cdots + x_{n+1}^d - \sum_{\alpha\in I_d} t_\alpha\cdot x^\alpha$, $\{x^\alpha \mid \alpha\in I_d\}$ is a finite set of monomials of degree d.

$k := \sum_{i=0}^{n+1} \frac{\beta_i+1}{d} \in \mathbb{N}$, $X_t : \mathbb{P}(f_t=0) \subseteq \mathbb{P}^{n+1}$, $t=(t_\alpha, \alpha\in I_d)\in (T,0)$

$\frac{x^\beta\Omega}{f^k} \in H^{n+1}_{dR}(\mathbb{P}^{n+1}\setminus X_t)$, $\operatorname{Resi}: H^{n+1}_{dR}(\mathbb{P}^{n+1}\setminus X_t) \longrightarrow H^{n}_{dR}(X_t)$

$\delta_t \in H_n(X_t, \mathbb{Z})$. $2\pi i\int_{\delta_t}\operatorname{Resi}(\omega) = \int_{\sigma(\delta_t)} \omega$



If you don't know Resi:

If you don't know homology:

Thom-Leray-Gysin map

put sx_0^d insted of x_0^d and take the affine chart $x_0=1$

$\Delta_t := \{(x_1, x_2, \ldots x_{n+1}) \in \mathbb{R}^{n+1} \mid f_t(x) \leqslant 0\}$, $s\in\mathbb{R}^-$, t small, this is a bounded region in $\mathbb{R}^{n+1}$ ($t=0$, $d=2$, n-dimensional sphere). $\delta_t := \partial\Delta_t$

$h(t) :=$ Volume $(\Delta_t) = \int_{\Delta_t} dx_1\wedge dx_2\wedge\ldots\wedge dx_{n+1}$, $h_\beta(t) = \int_{\Delta_t} x^\beta dx_1\wedge\ldots dx_{n+1}$

$$\left.\frac{\partial h(t)}{\partial s}\right|_{s=-1} = R_\beta(t).$$

(2) $t^a := \prod_{\alpha\in I_d} t_\alpha^{a_\alpha}$, $a! = \prod_{\alpha\in I_d}(a_\alpha!)$ $a^* = \sum_{\alpha\in I_d} a_\alpha\cdot\alpha$. $|a| = \sum_{\alpha\in I_d} a_\alpha$

$D_{\check\beta} := \left(k+|a|-1-\sum_{i=0}^{n+1}\left[\frac{\check\beta_i+1}{d}\right]\right)! \prod_{i=1}^{n+1}\left(\left\{\frac{\check\beta_i+1}{d}\right\}\right)_{\left[\frac{\beta_i+1}{d}\right]}$, $\{x\} = x-[x]$.

$(x)_y = x(x+1)\cdots(x+y-1)$ Pochhammer symbol.

$P_{\check\beta} := \int_{\delta_0}\operatorname{Resi}\left(\frac{x^{\check\beta}\Omega}{(x_0^d+x_1^d+\cdots+x_{n+1}^d)^{\check k}}\right)$, $\check k := \sum\frac{\check\beta_i+1}{d} \in \mathbb{N}$.

for $\beta \in \mathbb{N}_0^{n+2}$, $\bar{\beta} \in \mathbb{N}^{n+2}$ is defined by

$$0 \leqslant \bar{\beta}_i \leqslant m_i - 1 \qquad \beta_i \equiv_d \bar{\beta}_i .$$

we can compute $P_\beta^\vee$'s: $\quad P_\beta^\vee \in \mathbb{Q}(\zeta_d) \dfrac{\Gamma(\frac{\beta_1+1}{d})\Gamma(\frac{\beta_2+1}{d})\cdots\Gamma(\frac{\beta_{n+1}+1}{d})}{\Gamma\left(\sum_{i=1}^{n+1} \frac{\beta_i+1}{d}\right)}$.

Hodge & Noether-Lefschetz locus:

$$= \left\langle P_\beta(t),\ k := \sum_{i=0}^{n+1} \frac{\beta_i+1}{d} \in \mathbb{N},\ \leqslant \frac{n}{2} \right\rangle \subseteq \mathcal{O}_{T,0}$$

$V :=$ the zero set of this ideal $\quad V := \mathrm{Spec}\left(\mathcal{O}_{T,0}/\text{this ideal} \right)$

We may assume $0 \in T \overset{def}{\Leftrightarrow} \delta_0$ is a Hodge cycle of the Fermat variety.

Theorem (Deligne-Cattani-Kaplan, 198?, Deligne 1980) V is an algebraic variety defined over an abelian extension of $\mathbb{Q}(\zeta_d)$.

The linear part of P_β's: Consider

$$P_\beta : \qquad \sum_{i=0}^{n+1} \frac{\beta_i+1}{d} = \frac{n}{2}+1 .$$

the set of such β's call it $I_{(\frac{n}{2}+1)d-n-2}$. $I_{\frac{n}{2}d-n-2}$ the set of β with $\sum \frac{\beta_i+1}{d} \in \mathbb{N}_0$, $\leq \frac{n}{2}$.

The matrix

$$\left[P_{i+j} \right]_{i \in I_d,\ j \in I_{\frac{n}{2}d-n-2}}$$

Theorem: (Griffith-Harris-Carlson-Green, 1980, Voisin 2002, Movasati 2015)

$\ker\ [P_{i+j}] \simeq$ Zariski tangent space of the analytic scheme V at 0.

Theorem (Noether's theorem): For a generic surface $X \subseteq \mathbb{P}^3$, any curve in X is an intersection of X with another surface; In other words, $\mathrm{Pic}(X) \simeq \mathbb{Z}\,\mathcal{O}(1)$.

Let $1 \leqslant d_1 \leqslant d_2 \leqslant \cdots d_{\frac{n}{2}+1} \leqslant \frac{d}{2}$, $\underline{d} = (d_1, d_2, \dots d_{\frac{n}{2}+1})$

$T_{\underline{d}}$:= space of hypersurfaces of the form $f = f_1 g_1 + f_2 g_2 + \cdots + f_{\frac{n}{2}+1} g_{\frac{n}{2}+1}$

$\deg(f_i) = d_i$, $\deg(g_i) = d - d_i$

The case $d = (1, 1, \dots 1)$, this parametrizes $(\mathbb{P}^{\frac{n}{2}} \subseteq X_t)$.

Theorem (Conjecture): irred branchs of $T_{\underline{d}}$ in $(T, 0)$ are

1. components of the Hodge locus.
2. Such components are reduced and smooth.