Differential Equations and Arithmetic

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Hilbert's sixteen problem:

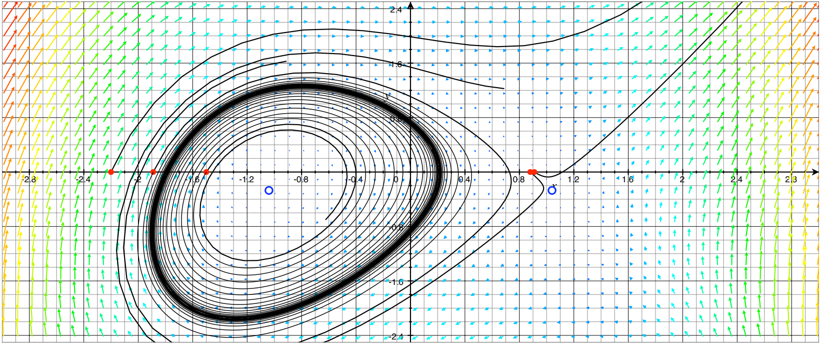

Figure: A limit cycle crossing $(x, y) \sim (-1.79, 0)$.

$$\mathcal{F} : \begin{cases} \dot{x} = 2y + \frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + 0.9y \end{cases}, \epsilon \in (\mathbb{R}, 0).$$

Center Problem:

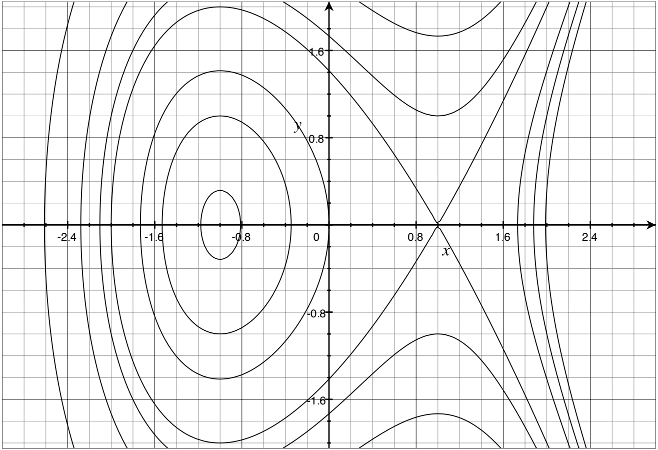

Figure: $f := y^2 - x^3 + 3x$, $f - t = 0$, $t = -1.9, -1, 0, 2, 3, 5, 10$

$$\begin{cases} \dot{x} = f_y \\ \dot{y} = -f_x \end{cases} .$$

1. Painlevé VI differential equation
2. Heun differential equation
3. Picard-Fuchs differential equations
4. Gauss hypergeometric equation and continued fractions
5. Darboux and Halphen differential equations
6. Bershadsky-Cecotti-Ooguri-Vafa anomaly equation
7. $\cdots$

Gauss-Manin connection:[1]

Let $P(x) := 4(x-t_1)^3 + t_2(x-t_1) + t_3$. We have

$$\begin{pmatrix} d\left(\int \frac{dx}{\sqrt{P(x)}}\right) \\ d\left(\int \frac{xdx}{\sqrt{P(x)}}\right) \end{pmatrix} =$$

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where

$$\Delta := 27t_3^2 - t_2^3, \ \alpha := 3t_3dt_2 - 2t_2dt_3.$$

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The above data is the Gauss-Manin connection of the family of elliptic curves $y^2 = P(x)$ before the invention of cohomology theories (before 1900).

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We are looking for parameters $t_1(\tau), t_2(\tau), t_3(\tau)$ such that

$$\begin{aligned} \int \frac{dx}{\sqrt{P(x)}} &= a\tau + b \\ \int \frac{xdx}{\sqrt{P(x)}} &= -b \end{aligned}$$

where a, b are two constants which do not depend on τ. This happens if and only if t_1, t_2, t_3 satisfies the differential equation

Ramanujan differential equation:[2]

$$\mathrm{R}: \begin{cases} \dot{t}_1 = t_1^2 - \frac{1}{12} t_2 \\ \dot{t}_2 = 4 t_1 t_2 - 6 t_3 \\ \dot{t}_3 = 6 t_1 t_3 - \frac{1}{3} t_2^2 \end{cases}, \quad \dot{} := \frac{\partial}{\partial \tau}$$

[2] Srinivasa Ramanujan (1987-1920)

Halphen property:[3]

If t_1, t_2, t_3 is a solution of the Ramanujan differential equation then for any element in

$$\mathrm{SL}(2,\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{C},\ ad - bc = 1 \right\}$$

the functions

$$(c\tau + d)^{-2} t_1 \left(\frac{a\tau + b}{c\tau + d} \right) - c(cz + d)^{-1},$$

$$(cz + d)^{-4} t_2 \left(\frac{a\tau + b}{c\tau + d} \right), (cz + d)^{-6} t_2 \left(\frac{a\tau + b}{c\tau + d} \right)$$

are also coordinates of a solution of R.

[3] George Henri Halphen (1844-1889)

Transcendency of solutions vs. transcendency of numbers:

(M. 2008) For any point $t \in \mathbb{C}^3 \backslash \{27t_3^2 - t_2^3 = 0\}$ and L_t the solution of R through t, the set $\bar{\mathbb{Q}}^3 \cap L_t$ is either empty or it has only one element. In other words, every solution contains at most one point with algebraic coordinates.

Write each t_i as a formal power series in $q := e^{2\pi i\tau}$, $t_i = \sum_{n=0}^{\infty} t_{i,n} q^n$ and substitute in the above differential equation. We will get a recursion.

$$\dot{t} = 12q\frac{\partial t}{\partial q}$$

Initial values

$$t_{1,0} = 1, \; t_{1,1} = -24$$

Eisenstein series:

After calculating some coefficients and consulting the encyclopedia of integer sequences we may conjecture that t_i's are well-known Eisenstein series:

$$t_i = a_i \Big(1 + b_i \sum_{n=1}^{\infty} \sigma_{2i-1}(n) q^n\Big), \quad i = 1, 2, 3, \tag{1}$$

where

$$(b_1, b_2, b_3) = (-24, 240, -504), \quad (a_1, a_2, a_3) = (1, 12, 8).$$

$$\sigma_k(n) := \sum_{d|n} d^k.$$

Modular forms:

1. Monstrous moonshine conjecture:

$$j = 1728\frac{E_4^3}{E_4^3 - E_6^2} = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

Modular forms:

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   Let $\eta(q) := \frac{E_4^3 - E_6^2}{1728} = q\prod_{i=1}^{\infty}(1 - q^n)^{24}$.
2. Arithmetic modularity theorem:

   $$\eta(q)^2\eta(q^{11})^{12} = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10}$$

   attached to the elliptic curve:

   $$E : y^2 + y = x^3 - x^2$$

A generalization of Ramanujan differential equation:

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$$
\begin{cases}
\dot{t}_0 = \frac{1}{t_5}(6 \cdot 5^4 t_0^5 + t_0 t_3 - 5^4 t_4) \\
\dot{t}_1 = \frac{1}{t_5}(-5^8 t_0^6 + 5^5 t_0^4 t_1 + 5^8 t_0 t_4 + t_1 t_3) \\
\dot{t}_2 = \frac{1}{t_5}(-3 \cdot 5^9 t_0^7 - 5^4 t_0^5 t_1 + 2 \cdot 5^5 t_0^4 t_2 + 3 \cdot 5^9 t_0^2 t_4 + 5^4 t_1 t_4 + 2 t_2 t_3) \\
\dot{t}_3 = \frac{1}{t_5}(-5^{10} t_0^8 - 5^4 t_0^5 t_2 + 3 \cdot 5^5 t_0^4 t_3 + 5^{10} t_0^3 t_4 + 5^4 t_2 t_4 + 3 t_3^2) \\
\dot{t}_4 = \frac{1}{t_5}(5^6 t_0^4 t_4 + 5 t_3 t_4) \\
\dot{t}_5 = \frac{1}{t_5}(-5^4 t_0^5 t_6 + 3 \cdot 5^5 t_0^4 t_5 + 2 t_3 t_5 + 5^4 t_4 t_6) \\
\dot{t}_6 = \frac{1}{t_5}(3 \cdot 5^5 t_0^4 t_6 - 5^5 t_0^3 t_5 - 2 t_2 t_5 + 3 t_3 t_6)
\end{cases}
$$

Book: Gauss-Manin connection in disguise: Calabi-Yau modular forms, Surveys of Modern Mathematics, International Press, Boston. http://w3.impa.br/∼hossein/myarticles/GMCD-MQCY3.pdf, 170 pages, 2016.