The early history of Hodge theory and Hodge conjecture

Hossein Movasati

IMPA, Instituto de Matemática Pura e Aplicada, Brazil
www.impa.br/∼hossein/

Généralités.

1. Nous appellerons *surface hyperelliptique* toute surface telle que les coordonnées x, y, z d'un de ses points puissent s'exprimer par des fonctions uniformes de deux paramètres ayant quatre paires de périodes.

M. Appell a établi (¹), ce qu'on déduit d'ailleurs d'un théorème célèbre de MM. Poincaré et Picard, qu'on peut mettre x, y, z sous la forme

$$x = \frac{\theta_1(u, v)}{\theta_0(u, v)}, \qquad y = \frac{\theta_2(u, v)}{\theta_0(u, v)}, \qquad z = \frac{\theta_3(u, v)}{\theta_0(u, v)},$$

les fonctions θ_h étant uniformes, et vérifiant les relations

$$(1) \quad \left\{\begin{array}{l} \theta_h(u + 2\pi i, v) = \theta_h(u, v + 2\pi i) = \theta_h(u, v) \\ \theta_h(u + a, v + b) = e^{-mu+\alpha}\theta_h(u, v) \\ \theta_h(u + b, v + c) = e^{-mv+\beta}\theta_h(u, v). \end{array}\right\} \quad (h = 0, 1, 2, 3).$$

Figure: Pages 32 of G. Humbert (1859-1921), Théorie générale des surfaces hyperelliptiques, Journal de mathématiques pures et appliquées 4e série, tome 9 (1893), p. 29-170.

Riemann's theta series:

Let $a, b \in \mathbb{R}^g$ and $\mathbb{H}_g$ be the genus g Siegel domain. It is the set of $g \times g$ symmetric matrices τ over the complex numbers whose imaginary part is positive definite. The Riemann theta function $\theta \begin{bmatrix} a \\ b \end{bmatrix}$ with characteristics (a, b) is the following holomorphic function:

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} \quad : \quad \mathbb{C}^g \times \mathbb{H}_g \to \mathbb{C}$$

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) \quad := \quad \sum_{n \in \mathbb{Z}^g} \exp\left(\frac{1}{2}(n+a)^{\mathrm{t}} \cdot \tau \cdot (n+a) + (n+a)^{\mathrm{t}} \cdot (z+b) \right)$$

Let $\Lambda := \{\tau m + n \mid m, n \in \mathbb{Z}^g\} \cong \mathbb{Z}^{2g}$ and $\mathbb{C}^g/\Lambda$ be the corresponding complex compact torus.

Hyperelliptic surfaces:

Let us now consider $g = 2$. Let also $\theta_i, \quad i = 1, 2, 3, 4$ be a collection of linearly independent Riemann's theta functions with the same characteristics a, b. G. Humbert is talking about the image X of the map:

$$\mathbb{C}^2/\Lambda \to \mathbb{P}^3, \; z \mapsto [\theta_1; \theta_2; \theta_3; \theta_4]. \tag{1}$$

This is an algebraic surface with singularities.

Algebraic Geometry of Humbert and many others around 1900 was limited to affine varieties and so instead of the projective geometry notation, one was mainly interested on the surface

$$U := \left\{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = 0\right\} \subset X, \tag{2}$$

where f is the polynomial relation between the following quotient of theta series:

$$(x, y, z) = \left(\frac{\theta_2(z, \tau)}{\theta_1(z, \tau)}, \frac{\theta_3(z, \tau)}{\theta_1(z, \tau)}, \frac{\theta_4(z, \tau)}{\theta_1(z, \tau)}\right) \tag{3}$$

Why such a complicated example? Why not simply the Fermat surface:

$$X : x^d + y^d + z^d + w^d = 0.$$

Answer: At the time of Humbert and Picard we did know the topology of $\mathbb{C}^2/\Lambda$: It is the product of four circles. But we did not know the topology of a general surface and in particular Fermat surface!

No more hyperelliptic surface:

In modern language Humbert and Picard studied singular models for abelian surfaces: A purely algebraic approach to the topic of equations for abelian varieties was introduced by Mumford in a series of articles 1966, which in turn, are based on many works of Baily, Cartier, Igusa, Siegel, Weil. It does not seem to me that the contributions of Humbert and Picard as founders of the subject is acknowledged in the modern treatment of equations defining abelian varieties.

The modern meaning of hyperelliptic surface seems to appear first in the work of Bombieri and Mumford in 1977. In a private letter Mumford explains this: “I believe Enrico and I knew that the word “hyperelliptic" had been used classically as a name for abelian surfaces but we felt that this usage was no longer followed, i.e. after Weil’s books, the term “abelian varieties" had taken precedence. So the word “hyperelliptic" seemed to be a reasonable term for this other class of surfaces."

To close personal recollections, let me tell you what made me turn with all possible vigor to topology. From the ρ_0 formula of Picard, applied to a hyperelliptic surface Φ (topologically the product of 4 circles) I had come to believe that the second Betti number $R_2(\Phi) = 5$, whereas clearly $R_2(\Phi) = 6$. What was wrong? After considerable time it dawned upon me that Picard only dealt with *finite* 2-cycles, the only useful cycles for calculating periods of certain double integrals. Missing link? The cycle at infinity, that is the plane section of the surface at infinity. This drew my attention to cycles carried by an algebraic curve, that is to *algebraic* cycles, and $\cdots$ the harpoon was in!

Figure: S. Lefschetz, A page of mathematical autobiography. Bull. Amer. Math. Soc., 74:854–879, 1968.

Lefschetz puzzle:

Let us redefine X to be the abelian surface $\mathbb{C}^2/\Lambda$, let Y be the subvariety of X which is the zero locus of θ_1 (the curve at infinity) and $U := X \setminus Y$. From the long exact sequence of $U \subset X$ we get

$$\begin{array}{ccccc} H_2(U) & \hookrightarrow & H_2(X) & \to & H_0(Y) \cong \mathbb{Z} \\ 5 & & 6 & & 1 \end{array}$$

The second map is the intersection with Y. The homology class of Y in X is a one dimensional subspace of $H_2(X)$ and $1 = 6 - 5$.

Algebraic de Rham cohomology:

Picard proves that the the second algebraic de Rham cohomology of U is of dimension 5:

$$\dim H^2_{\mathrm{dR}}(U) = 5.$$

Algebraic de Rham cohomology:

Picard proves that the the second algebraic de Rham cohomology of U is of dimension 5:

$$\dim H^2_{\mathrm{dR}}(U) = 5.$$

The algebraic de Rham cohomology is mainly attributed to A. Grothendieck (1966). Grothendieck's contribution is just a final remark on a creation of a mathematical object! At the time Picard computed this, neither de Rham nor Grothendicek was born!

Lefschetz $(1,1)$-theorem in its original format:

On an algebraic surface X a 2-dimensional homology cycle δ is the homology class of an algebraic curve if and only if

$$\int_\delta \omega = 0, \tag{4}$$

for all holomorphic differential 2-forms in X.

Lefschetz $(1,1)$-theorem in its original format:

On an algebraic surface X a 2-dimensional homology cycle δ is the homology class of an algebraic curve if and only if

$$\int_{\delta} \omega = 0, \tag{4}$$

for all holomorphic differential 2-forms in X.

Definition

A cycle $\delta \in H_2(X)$ with the above property is called a Hodge cycle.

The Hodge conjecture is just the generalization of the above theorem in higher dimensions.
For a smooth surface $X \subset \mathbb{P}^3$ the $\mathbb{Z}$-module of Hodge cycles, the Neron-Severi group and the Picard group are all the same.

Let X be the Fermat surface given by

$$U : x^d + y^d + z^d = 1.$$

in the affine coordinates. Nowadays we know that

$$\dim H_2(U) = (d-1)^3$$

$$\dim H_2(X) = (d-1)^3 - (d-1)^2 + (d-1) + 1$$

Let X be the Fermat surface given by

$$U : x^d + y^d + z^d = 1.$$

in the affine coordinates. Nowadays we know that

$$\dim H_2(U) = (d-1)^3$$

$$\dim H_2(X) = (d-1)^3 - (d-1)^2 + (d-1) + 1$$

Lefschetz theorem in this case is: A 2-dimensional homology cycle δ in X is the homology class of an algebraic curve if and only if

$$\int_\delta x^{\beta_1} y^{\beta_2} z^{\beta_3} (zdxdy - ydxdz + xdydz) = 0, \qquad (5)$$

for all $\beta_1, \beta_2, \beta_3 \in \mathbb{N} \cup \{0\}$ with $\beta_1 + \beta_2 + \beta_3 < d - 3$.

Computational Hodge conjecture:

Given a Hodge cycle $\delta \in H_2(X, \mathbb{Z})$, construct an algebraic cycle $[Z] = \sum Z_i$ such that Z is homologous to δ.
This is still a challenging problem in dimension two where we have Lefschetz's theorem.

Well-known curves of the Fermat surface:

1. Linear curves:

$$x - \zeta_{2d} y = z - \zeta_{2d} w = 0.$$

Well-known curves of the Fermat surface:

1. Linear curves:

$$x - \zeta_{2d} y = z - \zeta_{2d} w = 0.$$

2. Aoki-Shioda curves for $3 \mid d$:

$$\begin{cases} x^{\frac{d}{3}} + y^{\frac{d}{3}} + z^{\frac{d}{3}} = 0, \\ w^3 - 3^{\frac{3}{d}} \zeta_{\frac{d}{3}} xyz = 0 \end{cases} \tag{6}$$

Other curves can be produced using the automorphism group of the Fermat surface.

Theorem (Schuett, Shioda, van Luijk, 2010)

If $d \leq 100$ and $(d, 6) = 1$, then the Neron-Severi group of X is generated by lines on X.

Theorem (Schuett, Shioda, van Luijk, 2010)

If $d \leq 100$ *and* $(d, 6) = 1$*, then the Neron-Severi group of* X *is generated by lines on* X*.*

Theorem (Degtyarev, 2015)

For the above affirmation we only need the hypothesis $d \leq 4$ *or* $(d, 6) = 1$*.*

Theorem (Schuett, Shioda, van Luijk, 2010)

If $d \leq 100$ and $(d, 6) = 1$, then the Neron-Severi group of X is generated by lines on X.

Theorem (Degtyarev, 2015)

For the above affirmation we only need the hypothesis $d \leq 4$ or $(d, 6) = 1$.

Theorem (Aoki 1988)

For $d \leq 11$ lines together with Aoki-Shioda curves generate the Neron-Severi group of X.

The lines and the Aoki-Shioda curves are not enough to generate the Neron-Severi group of

$$x^{12} + y^{12} + z^{12} + w^{12} = 0$$

The lines and the Aoki-Shioda curves are not enough to generate the Neron-Severi group of

$$x^{12} + y^{12} + z^{12} + w^{12} = 0$$

In 2010 Aoki and Shioda produced other curves to find the generators of the Neron-Severi group of X.

“But the whole program [Grothendieck’s program on how to prove the Weil conjectures] relied on finding enough algebraic cycles on algebraic varieties, and on this question one has made essentially no progress since the 1970s.... For the proposed definition [of Grothendieck on a category of pure motives] to be viable, one needs the existence of “enough” algebraic cycles. On this question almost no progress has been made, (P. Deligne 2014)....la construction de cycles algébriques intéressants, les progrès ont été maigres, (P. Deligne 1994). "

For further reading see my book:

A course in Hodge theory: with emphasis on multiple integrals

http://w3.impa.br/ hossein/myarticles/hodgetheory.pdf