Gauss-Manin connection in disguise: a search for new type of automorphic forms

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From ICERM Semester Program on "Computational Aspects of the Langlands Program" we read:

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> ...Despite its many successes, the Langlands program remains vague in many of its predictions, due in part to an absence of data to guide a precise formulation away from a few special cases.

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The basic idea in constructing ??? lies in a better understanding the differential equations of classical modular and automorphic forms and then trying to generalize such differential equations. Therefore, ??? does not start with Hermitian symmetric domains, action of arithmetic groups etc.

1. The following link in my homepage What is Gauss-Manin Connection in Disguise?
2. Article: Modular-type functions attached to mirror quintic Calabi-Yau varieties, Math. Zeit. 2015
3. Book: Gauss-Manin Connection in Disguise: Calabi-Yau modular forms
4. Article: Gauss-Manin Connection in Disguise: Calabi-Yau threefolds (Together with M. Alim, E. Scheidegger, S.T. Yau)
5. These slides are hyperlinked and can be found in my homepage.

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1. Physcist: What can you compute that I can't?
2. Mathematician: What can you prove that I can't?
3. Developing mathematical theories just for the sake of their beauties (and one's ego).

A classical example (due to Ramanujan, Chazy,...):

$$\begin{cases} \dot{E}_2 = \frac{1}{12}(E_2^2 - E_4) \\ \dot{E}_4 = \frac{1}{3}(E_2 E_4 - E_6) \\ \dot{E}_6 = \frac{1}{2}(E_2 E_6 - E_4^2) \end{cases} \quad \dot{*} = q\frac{\partial *}{\partial q} \tag{1}$$

It is satisfied by the Eisenstein series:

$$E_{2i}(q) := \left(1 + b_i \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2i-1}\right) q^n\right), \; i = 1, 2, 3. \tag{2}$$

$$(b_1, b_2, b_3) = (-24, 240, -504).$$

A non-classical example:

$$\begin{aligned}
\dot{t}_0 &= \tfrac{1}{t_5}(3750t_0^5 + t_0t_3 - 625t_4)\\
\dot{t}_1 &= \tfrac{1}{t_5}(-390625t_0^6 + 3125t_0^4t_1 + 390625t_0t_4 + t_1t_3)\\
\dot{t}_2 &= \tfrac{1}{t_5}(-5859375t_0^7 - 625t_0^5t_1 + 6250t_0^4t_2 + 5859375t_0^2t_4 + 625t_1t_4 + 2t_2t_3)\\
\dot{t}_3 &= \tfrac{1}{t_5}(-9765625t_0^8 - 625t_0^5t_2 + 9375t_0^4t_3 + 9765625t_0^3t_4 + 625t_2t_4 + 3t_3^2)\\
\dot{t}_4 &= \tfrac{1}{t_5}(15625t_0^4t_4 + 5t_3t_4)\\
\dot{t}_5 &= \tfrac{1}{t_5}(-625t_0^5t_6 + 9375t_0^4t_5 + 2t_3t_5 + 625t_4t_6)\\
\dot{t}_6 &= \tfrac{1}{t_5}(9375t_0^4t_6 - 3125t_0^3t_5 - 2t_2t_5 + 3t_3t_6)
\end{aligned}$$

$$\dot{*} = 5q\frac{\partial *}{\partial q}$$

$\frac{1}{24}t_0 = \frac{1}{120} + q + 175q^2 + 117625q^3 + 111784375q^4 +$
$126958105626^5 + 160715581780591q^6 +$
$218874699262438350q^7 + 314179164066791400375q^8 +$
$469234842365062637809375q^9 +$
$722875994952367766020759550q^{10} + O(q^{11})$

$\frac{-1}{750}t_1 = \frac{1}{30} + 3q + 930q^2 + 566375q^3 + 526770000q^4 +$
$592132503858q^5 + 745012928951258q^6 +$
$1010500474677945510q^7 + 1446287695614437271000q^8 +$
$2155340222852696651995625q^9 +$
$3314709711759484241245738380q^{10} + O(q^{11})$

$t_2 = \cdots$

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Book: Gauss-Manin Connection in Disguise: Calabi-Yau modular forms

Gauss-Manin connection in disguise: Why should one compute the periods of algebraic cycles?

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1. Article: Gauss-Manin connection in disguise: Noether-Lefschetz and Hodge loci
2. Book: A course in Hodge Theory: with emphasis on multilple integrals

These slides are hyperlinked and can be found in my homepage. Click on the link below to find an elementary problem.

An elementary problem

Hodge Conjecture:

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Let X be a smooth projective variety, m be an even number and $\delta \in H_m(X, \mathbb{Z})$ be a Hodge cycle, that is,

$$\int_\delta \omega = 0, \ \ \forall \text{ closed } (p, q)\text{-form in } X \text{ with } p > \frac{m}{2}$$

Hodge Conjecture:

Let X be a smooth projective variety, m be an even number and $\delta \in H_m(X, \mathbb{Z})$ be a Hodge cycle, that is,

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Then there is an algebraic cycle

$$Z = \sum n_i [Z_i], \ \ n_i \in \mathbb{Z}, \ \ \dim(Z_i) = \frac{m}{2}$$

such that $\delta = [Z] := \sum n_i [Z_i]$.

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Let $\{X_t\}_{t\in T}$ be a family of smooth projective varieties, m be an even number and let $Z_0 \subset X_0$ be a fixed algebraic cycle of dimension $\frac{m}{2}$. Let us assume that for a paprameter t the monodromy δ_t of $\delta_0 = [Z_0]$ is a Hodge cycle.

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Complete intersections inside hypersurfaces

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Let T be the parameter space of hypersurfaces of degree d in $\mathbb{P}^{m+1}$. Let also $X = X_t,\ t \in T$ be given by

$$X : f(x_0, x_1, \cdots, x_{m+1}) = 0.$$

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Fix integers $1 \leq d_1, d_2, \ldots d_{\frac{m}{2}+1} \leq d$ let $\check{T} \subset T$ be the parameter space of hypersurfaces with

$$f = f_1 g_1 + \cdots + f_{\frac{m}{2}+1} g_{\frac{m}{2}+1},\ \ \deg(f_i) = d_i,\ \deg(g_i) = d - d_i.$$

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Conjecture

Infinitesimal Hodge conjecture is true for pairs (X, Z) as above.

Theorem (Green, Voisin, 1991, $m = 2$, M., 2015, $m \geq 2$)

The infinitesimal Hodge cojecture is true for linear projective spaces inside hypersurfaces of degree d and dimension m with $d \geq 2 + \frac{4}{m}$.

This is the case $d_1 = d_2 = \cdots = d_{\frac{m}{2}+1} = 1$ in the previous slide.

Periods of Algebraic cycles

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Let

$$\omega_i := \mathrm{Residue}\left(\frac{x^i \cdot \sum_{j=0}^{m+1}(-1)^j x_j\, dx_0 \wedge \cdots \wedge \widehat{d} x_j \wedge \cdots \wedge dx_{m+1}}{f^{k+1}}\right)$$

with $k := \frac{m+2+\sum_{e=0}^{m+1} i_e}{d}$. We have $\omega_i \in H^m_{\mathrm{dR}}(X)$. After Griffiths 1970, we know that:

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Definition

A cycle $\delta \in H_m(X, \mathbb{Z})$ is a Hodge cycle if

$$\int_\delta \omega_i = 0, \;\; \forall i, \;\; k \leq \frac{m}{2}.$$

Let δ be a Hodge cycle. Let also

$$x_i := \frac{1}{(2\pi\sqrt{-1})^m}\int_\delta \omega_i, \quad \sum_{e=0}^{m+1} i_e = (\frac{m}{2}+1)d-(m+2)$$

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(Deligne 1970) If δ is the homology class of an algebraic cycle and X is defined over an algebraically closed field $\mathrm{k} \subset \mathbb{C}$ then $x_i \in \mathrm{k}$.

Proposition (Voisin 2002+M. 2015)

Let X_0 be the Fermat variety

$$X_0 : x_0^d + x_1^d + \cdots + x_{m+1}^d = 0.$$

and let $\delta \in H_m(X_0, \mathbb{Z})$ be a Hodge cycle. The rank of $[x_{i+j}]$ is the codimension of the tangent cone of the Hodge loci passing through $0 \in T$ and corresponding to $\delta \in H_m(X_0, \mathbb{Z})$.

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Hodge cycles of Fermat variety was extensively studied by Shioda around 1970.