Gauss-Manin connection in disguise: Why should one compute the periods of algebraic cycles?

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1. Article: Gauss-Manin connection in disguise: Noether-Lefschetz and Hodge loci
2. Book: A course in Hodge Theory: with emphasis on multilple integrals

These slides are hyperlinked and can be found in my homepage. Click on the link below to find an elementary problem.

An elementary problem

Hodge Conjecture:

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Let X be a smooth projective variety, m be an even number and $\delta \in H_m(X, \mathbb{Z})$ be a Hodge cycle, that is,

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Then there is an algebraic cycle

$$Z = \sum n_i [Z_i], \ \ n_i \in \mathbb{Z}, \ \ \dim(Z_i) = \frac{m}{2}$$

such that $\delta = [Z] := \sum n_i [Z_i]$.

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Complete intersections inside hypersurfaces

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Let T be the parameter space of hypersurfaces of degree d in $\mathbb{P}^{m+1}$. Let also $X = X_t,\ t \in T$ be given by

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Fix integers $1 \leq d_1, d_2, \ldots d_{\frac{m}{2}+1} \leq d$ let $\check{T} \subset T$ be the parameter space of hypersurfaces with

$$f = f_1 g_1 + \cdots + f_{\frac{m}{2}+1} g_{\frac{m}{2}+1},\ \ \deg(f_i) = d_i,\ \deg(g_i) = d - d_i.$$

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Conjecture

Infinitesimal Hodge conjecture is true for pairs (X, Z) as above.

Theorem (Green, Voisin, 1991, $m = 2$, M., 2015, $m \geq 2$)

The infinitesimal Hodge cojecture is true for linear projective spaces inside hypersurfaces of degree d and dimension m with $d \geq 2 + \frac{4}{m}$.

This is the case $d_1 = d_2 = \cdots = d_{\frac{m}{2}+1} = 1$ in the previous slide.

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$$\omega_i := \text{Residue}\left(\frac{x^i \cdot \sum_{j=0}^{m+1}(-1)^j x_j \, dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{m+1}}{f^{k+1}}\right)$$

with $k := \frac{m+2+\sum_{e=0}^{m+1} i_e}{d}$. We have $\omega_i \in H^m_{\mathrm{dR}}(X)$. After Griffiths 1970, we know that:

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Definition

A cycle $\delta \in H_m(X, \mathbb{Z})$ is a Hodge cycle if

$$\int_\delta \omega_i = 0, \ \ \forall i, \ \ k \leq \frac{m}{2}.$$

Let δ be a Hodge cycle. Let also

$$x_i := \frac{1}{(2\pi\sqrt{-1})^m}\int_\delta \omega_i, \quad \sum_{e=0}^{m+1} i_e = (\frac{m}{2}+1)d-(m+2)$$

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(Deligne 1970) If δ is the homology class of an algebraic cycle and X is defined over an algebraically closed field $\mathrm{k} \subset \mathbb{C}$ then $x_i \in \mathrm{k}$.

Proposition (Voisin 2002+M. 2015)

Let X_0 be the Fermat variety

$$X_0 : x_0^d + x_1^d + \cdots + x_{m+1}^d = 0.$$

and let $\delta \in H_m(X_0, \mathbb{Z})$ be a Hodge cycle. The rank of $[x_{i+j}]$ is the codimension of the tangent cone of the Hodge loci passing through $0 \in T$ and corresponding to $\delta \in H_m(X_0, \mathbb{Z})$.

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Hodge cycles of Fermat variety was extensively studied by Shioda around 1970.