Gauss-Manin connection in disguise: a search for new type of automorphic forms

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> ...Despite its many successes, the Langlands program remains vague in many of its predictions, due in part to an absence of data to guide a precise formulation away from a few special cases.

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The basic idea in constructing ??? lies in a better understanding the differential equations of classical modular and automorphic forms and then trying to generalize such differential equations. Therefore, ??? does not start with Hermitian symmetric domains, action of arithmetic groups etc.

1. The following link in my homepage What is Gauss-Manin Connection in Disguise?
2. Article: Modular-type functions attached to mirror quintic Calabi-Yau varieties, Math. Zeit. 2015
3. Book: Gauss-Manin Connection in Disguise: Calabi-Yau modular forms
4. Article: Gauss-Manin Connection in Disguise: Calabi-Yau threefolds (Together with M. Alim, E. Scheidegger, S.T. Yau)
5. These slides are hyperlinked and can be found in my homepage.

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1. Physcist: What can you compute that I can't?
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3. Developing mathematical theories just for the sake of their beauties (and one's ego).

A classical example (due to Ramanujan, Chazy,...):

$$\begin{cases} \dot{E}_2 = \frac{1}{12}(E_2^2 - E_4) \\ \dot{E}_4 = \frac{1}{3}(E_2E_4 - E_6) \\ \dot{E}_6 = \frac{1}{2}(E_2E_6 - E_4^2) \end{cases} \quad \dot{*} = q\frac{\partial *}{\partial q} \tag{1}$$

It is satisfied by the Eisenstein series:

$$E_{2i}(q) := \left(1 + b_i \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2i-1}\right) q^n\right), \ i = 1, 2, 3. \tag{2}$$

$$(b_1, b_2, b_3) = (-24, 240, -504).$$

A non-classical example:

$$
\begin{aligned}
\dot{t}_0 &= \tfrac{1}{t_5}(3750t_0^5 + t_0t_3 - 625t_4)\\
\dot{t}_1 &= \tfrac{1}{t_5}(-390625t_0^6 + 3125t_0^4t_1 + 390625t_0t_4 + t_1t_3)\\
\dot{t}_2 &= \tfrac{1}{t_5}(-5859375t_0^7 - 625t_0^5t_1 + 6250t_0^4t_2 + 5859375t_0^2t_4 + 625t_1t_4 + 2t_2t_3)\\
\dot{t}_3 &= \tfrac{1}{t_5}(-9765625t_0^8 - 625t_0^5t_2 + 9375t_0^4t_3 + 9765625t_0^3t_4 + 625t_2t_4 + 3t_3^2)\\
\dot{t}_4 &= \tfrac{1}{t_5}(15625t_0^4t_4 + 5t_3t_4)\\
\dot{t}_5 &= \tfrac{1}{t_5}(-625t_0^5t_6 + 9375t_0^4t_5 + 2t_3t_5 + 625t_4t_6)\\
\dot{t}_6 &= \tfrac{1}{t_5}(9375t_0^4t_6 - 3125t_0^3t_5 - 2t_2t_5 + 3t_3t_6)
\end{aligned}
$$

$$\dot{*} = 5q\frac{\partial *}{\partial q}$$

$$\tfrac{1}{24}t_0 = \tfrac{1}{120} + q + 175q^2 + 117625q^3 + 111784375q^4 + 126958105626^5 + 160715581780591q^6 + 218874699262438350q^7 + 314179164066791400375q^8 + 469234842365062637809375q^9 + 722875994952367766020759550q^{10} + O(q^{11})$$

$$\tfrac{-1}{750}t_1 = \tfrac{1}{30} + 3q + 930q^2 + 566375q^3 + 526770000q^4 + 592132503858q^5 + 745012928951258q^6 + 1010500474677945510q^7 + 1446287695614437271000q^8 + 2155340222852696651995625q^9 + 3314709711759484241245738380q^{10} + O(q^{11})$$

$$t_2 = \cdots$$

$\cdots$

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Book: Gauss-Manin Connection in Disguise: Calabi-Yau modular forms