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Why should one compute periods of algebraic cycles?

Math. Olympoid problem: $m, d \in \mathbb{N}$, for $N \in \mathbb{N}$ let

$$I_N := \{(i_0, \dots, i_{m+1}) \in \mathbb{Z}^{m+2} \mid 0 \leq i_e \leq d-2,\ i_0+i_1+i_2+\dots+i_{m+1}=N\}$$

Assume that m is even and $d \geq 2+\frac{4}{m}$.

Consider a colection of numbers x_i indexed by $i \in I_{(\frac{m}{2}+1)d-m-2}$

For any other i which is not in this set $x_i := 0$

Let $[x_{i+j}]$ be a matrix whose rows and columns are indexed by

$$i \in I_{\frac{m}{2}d-m-2},\ j \in I_d$$

and in its (i,j)-entry we have x_{i+j}. Prove if

$$\boxed{\operatorname{Rank}[x_{i+j}] < \binom{\frac{m}{2}+d}{d} - \left(\frac{m}{2}+1\right)^2}$$

then all x_i's are zero.

$\# I_d = \binom{d+m+1}{m+1} - (m+2)^2$.

If you get bothered of my talk and like challenging problems, try this one.

Infinitesimal Hodge conjecture: X_0 a projective variety of dim m, $Z \subseteq X_0$ a subvariety of dim $\frac{m}{2}$. We have

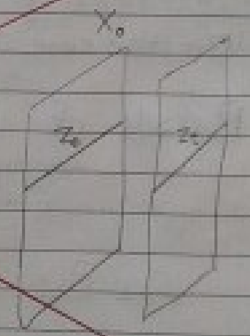
$$\delta_0 = [Z_0] \in H_m(X_0, \mathbb{Z})$$

Now, let put X_0 inside a family, $\{X_t\}_{t \in T}$

$$\delta_t \in H_m(X_t, \mathbb{Z})$$

If δ_t is a Hodge cycle then there is a deformation

$Z_t \subseteq X_t$ of Z_0 such that $\delta_t = [Z_t]$.



In general IHC must be wrong.

- Gauss-Manin connection in disguise: Noether-Lefschetz and Hodge Loci
- Topics in Hodge theory with emphasis on multiple integrals (Book).

Definition $\delta \in H_m(X, \mathbb{Z})$ is a Hodge cycle if

$$\int_\delta \omega = 0 \quad \text{for all closed } (p,q)\text{-forms } p > \frac{m}{2},\ p+q=m$$

It is very unfortunate not to find the version of Hodge cycle. It doesn't need Hodge decomposition

Traceback H.C: Picard-Simart Book on multiple integrals. Just Picard rank (Picard group) is remained.

Main example: T: parameter space of hypersurfaces of degree d in $\mathbb{P}^{m+1}$, $X_t: f_t(x_0, x_1, \dots, x_{m+1}) = 0$

let $d = d_1 + d - d_1 = d_2 + d - d_2 = \dots = d_{\frac{m}{2}+1} + d - d_{\frac{m}{2}+1}$

$\tilde{T} \subseteq T$ corresponding to polynomials

$\| $

$T(d_1, d_2, \dots d_{\frac{m}{2}+1})$. $f_t = f_1 g_1 + f_2 g_2 + \dots + f_{\frac{m}{2}+1} g_{\frac{m}{2}+1}$

f_i a homo. poly of degree d_i

g_i " " " $d - d_i$

$Z_t \subseteq X_t$ is given by $f_1 = f_2 = \dots = f_{\frac{m}{2}+1} = 0$.

IHC: If the monodromy $\delta_t \in H_m(X_t, \mathbb{Z})$ of $\delta_0 [Z_0] \in H_m(X_0, \mathbb{Z})$ is a Hodge cycle then $t \in \tilde{T}$ and $\delta_t = [Z_t]$.

Theorem (Green-Voisin 1990, $m=2$, M. $m \geq 2$). The above conjecture is true for $d_1 = d_2 = \dots = d_{\frac{m}{2}+1} = 1$)

In this case $Z_t \simeq \mathbb{P}^{\frac{m}{2}} \hookrightarrow X_t \subseteq \mathbb{P}^{m+1}$.

Periods of algebraic cycles: In general one needs Atiyah-Hodge-Grothendieck algebraic de Rham cohomology

For hypersurfaces, the story is easier (Griffiths ~1970).

$$\Omega = \frac{P(x) \overbrace{\sum x_i \, dx_0 \wedge \widehat{dx_i} \wedge \dots dx_{m+1}}^{\eta}}{f^k} \in H^{m+1}_{dR}(\mathbb{P}^{m+1} \setminus X)$$

$P(x)$ homog. of deg x such that ω is inv. under $\mathbb{C}^*$-action.

The residue map

$$\mathrm{Res}: H^{m+1}_{dR}(\mathbb{P}^{m+1} \setminus X) \xrightarrow{\sim} H^m_{dR}(X)_0 \hookrightarrow H^m_{dR}(X)$$

let $x^i = x_0^{i_0} \cdots x_{m+1}^{i_{m+1}}$, $i \in I$, be a monomial basis of $\mathbb{C}[x] / \langle \frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_{m+1}} \rangle$.

$$\omega_i := \mathrm{Res}\left(\frac{x^i \eta}{f^k} \right) \in H^m_{dR}(X)$$

$$k_i = \frac{\sum i_e + (m+2)}{d} < \frac{(m+2)(d-1)}{d} < m+2$$

Periods of algebraic cycles: (Deligne) $\frac{1}{(2\pi i)^{\frac{m}{2}}} \int_{[Z]} \omega_i \in$ the base field, where X, Z, ω_i are defined.

Hodge cycle: $\delta \in H_m(X, \mathbb{Z})$ is a Hodge cycle if

$$\int_\delta \omega_i = 0 \quad \forall i \qquad k_i = \frac{\sum_{e=0}^{m+2} i_e + m + 2}{d} \leq \frac{m}{2}$$

This follows from Griffiths description of the Hodge filtration of $H^m_{dR}(X)$.

Period vector of a Hodge cycle / Algebraic cycle

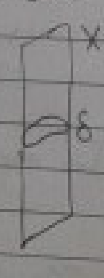
$$x_i := \frac{1}{(2\pi i)^{2m}} \int_\delta \omega_i \qquad \sum_{e=0}^{m+2} i_e = \left(\frac{m}{2}+1\right) d - (m+2) = \left(\frac{m}{2}+1\right)(d-2).$$

From now on: $X_0 =$ Fermat variety $x_0^d + x_1^d + \dots + x_{m+1}^d = 0$.

The monomial basis: $x^i = x_0^{i_0} \cdots x_{m+1}^{i_{m+1}}$ $\quad 0 \leq i_e \leq d-2$

Theo. $\mathrm{Rank}\,[x_{i+j}] =$ codimension of the tangent cone of the Hodge loci passing through 0 and corresp. to $\delta \in H_m(X_0, \mathbb{Z})$

(Partially is stated in Voisin's book II)



let us take $\mathbb{P}^{\frac{m}{2}} \hookrightarrow X_0 \subset \mathbb{P}^{m+1}$, linear projective space

$$x_i := \int_{[\mathbb{P}^{\frac{m}{2}}]} \omega_i$$

Elementary problem $\Rightarrow$ $\mathrm{Rank}([x_{i+j}]) \geqslant \binom{\frac{m}{2}+d}{d} - (\frac{m}{2}+1)^2$

$$\parallel$$

$$\mathrm{codim}\,(T(\underbrace{1,1,1,\dots,1}_{\frac{m}{2}+1}))$$

From another side $\mathrm{Ran}([x_{i+j}]) \leqslant$ ..

because this is the codimension of the tangent cone

How about other complete intersections?

For that we need to compute x_i's!

Fermat sextic fourfold $X_0: x_0^6 + x_1^6 + \dots + x_5^6 = 0$.

$\dim T(d_1,d_2,d_3)$	(d_1,d_2,d_3)
141	(3,3,3)
71	(3,3,1)
122	(3,3,2)
37	(3,1,1)
106	(3,2,2)
62	(3,1,2)
19	(1,1,1)
32	(1,1,2)
54	(1,2,2)
92	(2,2,2)

(*)

Hodge numbers of $H^4_{dR}(X_0)$

1 426 1752 426 1

This must follow from Shioda 1979?

Proposition: The $\mathbb{Q}$-vector space of Hodge cycles is of dim 1752.

(Shioda-Aoki 1980). H.C is true in this case.

Conj: In each case the rank $[x_{i+j}]$ is given by (*)

Theorem? If one choose a random Hodge cycle in $H_4(X_0, \mathbb{Z})$ then the corresponding $[x_{i+j}]$ satisfies

Rank $[x_{i+j}] = 426$.