

Eisenstein type series for Calabi-Yau varieties ¹

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Abstract

In this article we introduce an ordinary differential equation associated to the one parameter family of Calabi-Yau varieties which is mirror dual to the universal family of smooth quintic three folds. It is satisfied by seven functions written in the q -expansion form and the Yukawa coupling turns out to be rational in these functions. We prove that these functions are algebraically independent over the field of complex numbers, and hence, the algebra generated by such functions can be interpreted as the theory of (quasi) modular forms attached to the one parameter family of Calabi-Yau varieties. Our result is a reformulation and realization of a problem of Griffiths around seventies on the existence of automorphic functions for the moduli of polarized Hodge structures. It is a generalization of the Ramanujan differential equation satisfied by three Eisenstein series.

1 Introduction

Modular and quasi-modular forms as generating functions count very unexpected objects beyond the scope of analytic number theory. There are many examples for supporting this fact. The Shimura-Taniyama conjecture, now the modularity theorem, states that the generating function for counting \mathbb{F}_p -rational points of an elliptic curve over \mathbb{Z} for different primes p , is essentially a modular form (see [32] for the case of semi-stable elliptic curves and [5] for the case of all elliptic curves). Monstrous moonshine conjecture, now Borcherds theorem, relates the coefficients of the j -function with the representation dimensions of the monster group (see [4]). Counting ramified coverings of an elliptic curve with a fixed ramification data leads us to quasi-modular forms (see [9] and [18]).

In the context of Algebraic Geometry, the theory of modular forms is attached to elliptic curves and in a similar way the theory of Siegel and Hilbert modular forms is attached to polarized abelian varieties. One may dream of other modular form theories attached to other varieties of a fixed topological type. An attempt to formulate such theories was first done around seventies by P. Griffiths in the framework of Hodge structures, see [15]. However, such a formulation leads us to the notion of automorphic cohomology which has lost the generating function role of modular forms. Extending the algebra of any type of modular forms into an algebra, which is closed under canonical derivations, seems to be indispensable for further generalizations.

In 1991 there appeared the article [6] of Candelas, de la Ossa, Green and Parkes, in which they calculated in the framework of mirror symmetry a generating function, called the Yukawa coupling, which predicts the number of rational curves of a fixed degree in a generic quintic three fold. From mathematical point of view, the finiteness is still a conjecture carrying the name of Clemens (see [7]). Since then there was some effort

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to express the Yukawa coupling in terms of classical modular or quasi-modular forms, however, there was no success. The Yukawa coupling is calculated from the periods of a one parameter family of Calabi-Yau varieties and this suggests that there must be a theory of (quasi) modular forms attached to this family (this is also predicted in [1], page 2). The main aim of the present text is to realize the construction of such a theory.

Consider the following ordinary differential equation in seven variables $t_0, t_1, \dots, t_4, t_5, t_6$:

$$(1) \quad \begin{cases} \dot{t}_0 = \frac{1}{t_5}(\frac{6}{5}t_0^5 + \frac{1}{3125}t_0t_3 - \frac{1}{5}t_4) \\ \dot{t}_1 = \frac{1}{t_5}(-125t_0^6 + t_0^4t_1 + 125t_0t_4 + \frac{1}{3125}t_1t_3) \\ \dot{t}_2 = \frac{1}{t_5}(-1875t_0^7 - \frac{1}{5}t_0^5t_1 + 2t_0^4t_2 + 1875t_0^2t_4 + \frac{1}{5}t_1t_4 + \frac{2}{3125}t_2t_3) \\ \dot{t}_3 = \frac{1}{t_5}(-3125t_0^8 - \frac{1}{5}t_0^5t_2 + 3t_0^4t_3 + 3125t_0^3t_4 + \frac{1}{5}t_2t_4 + \frac{3}{3125}t_3^2) \\ \dot{t}_4 = \frac{1}{t_5}(5t_0^4t_4 + \frac{1}{625}t_3t_4) \\ \dot{t}_5 = \frac{t_6}{t_5} \\ \dot{t}_6 = (-\frac{72}{5}t_0^8 - \frac{24}{3125}t_0^4t_3 - \frac{3}{5}t_0^3t_4 - \frac{2}{1953125}t_3^2) + \frac{t_6}{t_5}(12t_0^4 + \frac{2}{625}t_3) \end{cases},$$

where

$$\dot{t} = 5q \frac{\partial t}{\partial q}.$$

We write each t_i as a formal power series in q , $t_i = \sum_{n=0}^{\infty} t_{i,n}q^n$ and substitute in the above differential equation and we see that it determines all the coefficients $t_{i,n}$ uniquely with the initial values:

$$(2) \quad t_{0,0} = \frac{1}{5}, \quad t_{0,1} = 24, \quad t_{4,0} = 0$$

and assuming that $t_{5,0} \neq 0$. After substitution we get the two possibilities $0, \frac{-1}{3125}$ for $t_{5,0}$, and $t_{i,n}$, $n \geq 2$ is given in terms of $t_{j,m}$, $j = 0, 1, \dots, 6$, $m < n$. See §17 for the first eleven coefficients of t_i 's. We calculate the expression $\frac{-(t_4 - t_0^5)^2}{625t_3^3}$ and write it in Lambert series form. It turns out that

$$\frac{-(t_4 - t_0^5)^2}{625t_3^3} = 5 + 2875 \frac{q}{1-q} + 609250 \cdot 2^2 \frac{q^2}{1-q^2} + \dots + n_d d^3 \frac{q^d}{1-q^d} + \dots.$$

Let W_ψ be the variety obtained by the resolution of singularities of the following quotient:

$$(3) \quad W_\psi := \{[x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^4 \mid x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0\}/G,$$

where G is the group

$$G := \{(\zeta_1, \zeta_2, \dots, \zeta_5) \mid \zeta_i^5 = 1, \zeta_1 \zeta_2 \zeta_3 \zeta_4 \zeta_5 = 1\}$$

acting in a canonical way. The family W_ψ is Calabi-Yau and it is mirror dual to the universal family of quintic varieties in \mathbb{P}^4 .

Theorem 1. *The quantity $\frac{-(t_4 - t_0^5)^2}{625t_3^3}$ is the Yukawa coupling associated to the family of Calabi-Yau varieties W_ψ .*

The q -expansion of the Yukawa coupling is calculated by Candelas, de la Ossa, Green, Parkes in [6], see also [24]. Using physical arguments they showed that n_d must be the number of degree d rational curves inside a generic quintic three fold. However, from mathematical point of view we have the Clemens conjecture which claims that there are

finite number of such curves for all $d \in \mathbb{N}$. This conjecture is established for $d \leq 9$ and remains open for d equal to 10 or bigger than it (see [17] and the references within there). The Gromov-Witten invariants N_d can be calculated using the well-known formula $N_d = \sum_{k|d} \frac{n_{d/k}}{k^3}$. The numbers n_d are called instanton numbers or BPS degeneracies. The \mathbb{C} -algebra generated by t_i 's can be considered as the theory of (quasi) modular forms attached to the family W_ψ . We prove:

Theorem 2. *The functions $t_i, i = 0, 1, \dots, 6$ are algebraically independent over \mathbb{C} , this means that there is no polynomial P in seven variables and with coefficients in \mathbb{C} such that $P(t_0, t_1, \dots, t_6) = 0$.*

Calculation of instanton numbers by our differential equation (1) or by using periods, see [6, 24], or by constructing moduli spaces of maps from curves to projective spaces, see [21], leads to the fact that they are rational numbers. It is conjectured that all n_d 's are integers (Gopakumar-Vafa conjecture, see [13]). Some partial results regarding this conjecture is established recently by Kontsevich-Schwarz-Vologodsky (see [22]) and Krattenthaler-Rivoal (see [23]).

All the quantities $t_i, i = 0, 1, \dots, 6$ and q can be written in terms of the periods of the family W_ψ . The differential form

$$\Omega = \frac{x_4 dx_0 \wedge dx_1 \wedge dx_2}{\frac{\partial Q}{\partial x_3}},$$

where Q is the defining polynomial of W_ψ , induces a holomorphic 3-form in W_ψ which we denote by the same letter Ω . Note that $5\psi\Omega$ is the standard choice of a holomorphic differential 3-form on W_ψ (see [6], p. 29). Let also $\delta_1, \delta_2, \delta_3, \delta_4$ be a particular basis of $H_3(W_\psi, \mathbb{Q})$ which will be explained in §11, and

$$x_{ij} = \frac{\partial^{j-1}}{\partial \psi^{j-1}} \int_{\delta_i} \Omega, \quad i, j = 1, 2, 3, 4.$$

Theorem 3. *The q -expansion of t_i 's are convergent and if we set $q = e^{2\pi i \frac{x_{21}}{x_{11}}}$ then*

$$\begin{aligned} t_0 &= a^{-3} \psi x_{11} \\ t_1 &= a^{-6} 625 x_{11} (5\psi^3 x_{12} + 5\psi^4 x_{13} + (\psi^5 - 1)x_{14}) \\ t_2 &= a^{-9} (-625) x_{11}^2 (5\psi^3 x_{11} + (\psi^5 - 1)x_{13}) \\ t_3 &= a^{-12} 625 x_{11}^3 (-5\psi^4 x_{11} + (\psi^5 - 1)x_{12}) \\ t_4 &= a^{-15} x_{11}^5 \\ t_5 &= a^{-11} \frac{-1}{5} (\psi^5 - 1) x_{11}^2 (x_{12} x_{21} - x_{11} x_{22}) \\ t_6 &= a^{-23} \frac{1}{25} (\psi^5 - 1) x_{11}^5 (-5\psi^4 x_{11} x_{12} x_{21} - 2(\psi^5 - 1) x_{12}^2 x_{21} - (\psi^5 - 1) x_{11} x_{13} x_{21} + \\ &\quad 5\psi^4 x_{11}^2 x_{22} + 2(\psi^5 - 1) x_{11} x_{12} x_{22} + (\psi^5 - 1) x_{11}^2 x_{23}), \end{aligned}$$

where $a = \frac{2\pi i}{5}$.

Once all the above quantities are given, using the Picard-Fuchs of x_{i1} 's, see (11), one can check easily that they satisfy the ordinary differential equation (1). However, how we have calculated them, and in particular moduli interpretation of $t_i, i = 0, 1, \dots, 6$, will be explained throughout the present text.

This work can be considered as a realization of a problem of Griffiths around 1970's on the automorphic form theory for the moduli of polarized Hodge structures, see [15]. In our case $H^3(W_\psi, \mathbb{C})$ is of dimension 4 and it carries a Hodge decomposition with Hodge numbers $h^{30} = h^{21} = h^{12} = h^{03} = 1$. As far as I know, this is the first case of automorphic function theory for families of varieties for which the corresponding Griffiths period domain is not Hermitian symmetric. It would be of interest to see how the results of this paper fit into the automorphic cohomology theory of Griffiths or vice versa.

Here, I would like to say some words about the methods used in the present text and whether one can apply them to other families of varieties. We construct affine coordinates for the moduli of the variety W_ψ enhanced with elements in its third de Rham cohomology, see §3, §6 and §16. Such a moduli turns out to be of dimension seven and such coordinates, say t_i , $i = 0, 1, \dots, 6$, have certain automorphic properties with respect to the action of an algebraic group (the action of discrete groups in the classical theory of automorphic functions is replaced with the action of algebraic groups). We use the Picard-Fuchs equation of the periods of Ω and calculate the Gauss-Manin connection (see for instance [20]) of the universal family of Calabi-Yau varieties over the mentioned moduli space. The ordinary differential equation (1), seen as a vector field on the moduli space, has some nice properties with respect to the Gauss-Manin connection which determines it uniquely. A differential equation of type (1) can be introduced for other type of varieties, see [27], however, whether it has a particular solution with a rich enumerative geometry behind, depends strongly on some integral monodromy conditions, see §9, §8 and §11. For the moment I suspect that the methods introduced in this article can be generalized to arbitrary families of Calabi-Yau varieties and even to some other cases where the geometry is absent, see for instance the list of Calabi-Yau operators in [3, 31] and a table of mirror consistent monodromy representations in [10]. Since the theory of Siegel modular forms is well developed and in light of the recent work [8], see also the references within there, the case of K3 surfaces is quite promising. For local Calabi-Yau manifolds we need Siegel modular forms and this is explained in [1].

Here, I would like to discuss about the physics underlying the mathematical results and calculations of the present text. Higher genus topological string partition functions $F_g(t)$, $g \geq 1$ and their calculations play an important role in quantum field theory. Mirror symmetry conjecture leads to directly calculable predictions for the functions F_g for Calabi-Yau varieties using the periods of the mirror manifold. To understand better the importance of the quantities t_0, t_1, \dots, t_6 in quantum field theory, one has to understand the role of three classical Eisenstein series in the calculation of partition functions for elliptic curves (for a detailed discussion see [9]). The original definition of partition functions F_g in quantum Yang-Mills theory does not calculate them explicitly. In the case of elliptic curves M. R. Douglas in [11] finds explicit calculable formulas for F_g in terms of generalized theta functions. This leads in a natural way to the polynomial structure of partition functions in terms of three Eisenstein series. Since we have the Ramanujan differential equation between Eisenstein series, we conclude that derivatives of partition functions in the Kähler modulus have still such polynomial structures, and hence, they can be calculated in an effective way. In the case of mirror symmetry for quintic hypersurfaces the polynomial structure of partition functions are described by Yamaguchi and Yau in [33] (see also [2, 16] for further generalization of their results for other type of Calabi-Yau varieties). They show that partition functions are homogeneous polynomials in five quantities. A simple comparison of the period expressions in Theorem 3 and those of [33] end of §3.3, shows that these quantities can be computed explicitly in terms

of t_i , $i = 0, 1, \dots, 7$. In fact, Theorem 2 implies that five is the minimum number of generators for the partition functions. The particular appearance of t_5 in the differential equation (1) together with Theorem 3 imply that apart from the τ -frame, where in our case $\tau := \frac{x_{21}}{x_{11}}$, and ψ -frame in the mirror manifold, we have a y -frame which is given by $\frac{\partial \tau}{\partial y} = \frac{1}{t_6}$. Looking t_i , $i = 0, \dots, 4$'s as functions in y we have an ordinary differential equation in dimension five which is essentially the first five lines of (1) without the factor $\frac{1}{t_6}$. Using the result of Yamaguchi and Yau, this completely describes the polynomial and differential structure of the partition functions in the y -frame. I am not aware of any significance of the physical interpretation of the y -frame. Theorem 2 says that in order to understand the differential structure of partition functions in the τ -frame we need exactly seven functions. The ordinary differential equation in the y -frame will not result in the calculation of instanton type expansions of t_i 's as we have explained in the beginning of Introduction. The quantities t_5 and t_6 are necessary for such calculations.

The monodromy group Γ of the mirror manifold acts naturally and non trivially on the Kähler modulus τ and this leads to the modular properties of the partition functions with respect to Γ . From physical point of view this action is important and it describes the interchange of large area with small area which is a well-known example of the so called duality transformation in string theory (see [12]). In the case of elliptic curves this action is described in terms of the classical functional equations of Eisenstein series, and hence any polynomial of them (see [9]). In order to understand the modular properties of partition functions attached to quintic Calabi-Yau manifolds, we have to understand the modular properties of t_i 's. The modular properties of t_i 's are first described in terms of the action of an algebraic group, and then, using the period map this is translated into the modular properties of t_i 's with respect to Γ , see §17.

We have calculated the differential equation (1) and the first coefficients of t_i by Singular, see [14]. The reader who does not want to calculate everything by his own effort can obtain the corresponding Singular code from my web page.²

In the final steps of the present article Charles Doran informed me of his joint article [8] and the results obtained by Yamaguchi and Yau in [33]. Here, I would like to thank him for his interest. I would also like to thank the referee whose comments and suggestions improved the paper and who introduced me with the article [1]. Finally, sincere thanks goes to Satoshi Yamaguchi who informed me about further developments [2, 16] of his results with Prof. Yau.

2 Quasi-modular forms

The differential equation (1) is a generalization of the Ramanujan differential equation

$$(4) \quad \begin{cases} \dot{g}_1 = g_1^2 - \frac{1}{12}g_2 \\ \dot{g}_2 = 4g_1g_2 - 6g_3 \\ \dot{g}_3 = 6g_1g_3 - \frac{1}{3}g_2^2 \end{cases} \quad \dot{g} = 12q \frac{\partial g}{\partial q}$$

which is satisfied by the Eisenstein series:

$$(5) \quad g_i = a_k \left(1 + b_k \sum_{d=1}^{\infty} d^{2k-1} \frac{q^d}{1-q^d} \right), \quad k = 1, 2, 3,$$

²w3.impa.br/~hossein/manyfiles/singular-haftarticle.txt

where

$$(b_1, b_2, b_3) = (-24, 240, -504), \quad (a_1, a_2, a_3) = (1, 12, 8).$$

We have calculated (1) using the Gauss-Manin connection of the family W_ψ which is essentially the Picard-Fuchs differential equation of the holomorphic differential form of the family W_ψ . This is done in a similar way as we calculate (4) from the Gauss-Manin connection of a family of elliptic curves, see [25, 26]. The general theory of differential equations of type (1) and (4) is developed in [27]. Relations between the Gauss-Manin connection and Eisenstein series appear in the appendix of [19]. Let g_1, g_2, g_3 be the Eisenstein series (5). The \mathbb{C} -algebra $\mathbb{C}[g_1, g_2, g_3]$ is freely generated by g_1, g_2, g_3 . With $\deg(g_i) = i, i = 1, 2, 3$, its homogeneous pieces are quasi-modular forms over $\mathrm{SL}(2, \mathbb{Z})$. It can be shown that any other quasi-modular form for subgroups of $\mathrm{SL}(2, \mathbb{Z})$ with finite index, is in the algebraic closure of $\mathbb{C}(g_1, g_2, g_3)$.

3 Moduli space, I

In the affine coordinates $x_0 = 1$, the variety W_ψ is given by:

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 \mid f = 0\}/G,$$

where

$$f = -z - x_1^5 - x_2^5 - x_3^5 - x_4^5 + 5x_1x_2x_3x_4$$

and we have introduced a new parameter $z := \psi^{-5}$. We also use $W_{1,z}$ to denote the variety W_ψ . For $z = 0, 1, \infty$ the variety $W_{1,z}$ is singular and for all others it is a smooth variety of complex dimension 3. From now on, by $W_{1,z}$ we mean a smooth one. Up to constant there is a unique holomorphic three form on $W_{1,z}$ which is given by

$$\eta = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{df}.$$

Note that the pair $(W_{1,z}, 5\eta)$ is isomorphic to $(W_\psi, 5\psi\Omega)$, with Ω as in the Introduction. The later is in [6] p. 29. The third de Rham cohomology of $W_{1,z}$, namely $H_{\mathrm{dR}}^3(W_{1,z})$, carries a Hodge decomposition with Hodge numbers $h^{30} = h^{21} = h^{12} = h^{03} = 1$. By Serre duality $H^2(W_{1,z}, \Omega^1) \cong H^1(W_{1,z}, \Theta)$, where Ω^1 (rep. Θ) is the sheaf of holomorphic differential 1-forms (resp. vector fields) on $W_{1,z}$. Since $h^{21} = \dim_{\mathbb{C}} H^2(W_{1,z}, \Omega^1) = 1$, the deformation space of $W_{1,z}$ is one dimensional. This means that $W_{1,z}$ can be deformed only through the parameter z . In fact z is the classifying function of such varieties. Note that the finite values of z does not cover the smooth variety $W_\psi, \psi = 0$.

Let us take the polynomial ring $\mathbb{C}[t_0, t_4]$ in two variables t_0, t_4 (the variables t_1, t_2 and t_3 will appear later). It can be seen easily that the moduli S of the pairs (W, ω) , where W is as above and ω is a holomorphic differential form on W , is isomorphic to

$$S \cong \mathbb{C}^2 \setminus \{(t_0^5 - t_4)t_4 = 0\},$$

where we send the pair $(W_{1,z}, a\eta)$ to $(t_0, t_4) := (a^{-1}, za^{-5})$. The multiplicative group $G_m := \mathbb{C}^*$ acts on S by:

$$(W, \omega) \bullet k = (W, k^{-1}\omega), \quad k \in G_m, \quad (W, \omega) \in S.$$

In coordinates (t_0, t_4) this corresponds to

$$(6) \quad (t_0, t_4) \bullet k = (kt_0, k^5t_4), \quad (t_0, t_4) \in S, \quad k \in G_m.$$

We denote by (W_{t_0, t_4}, ω_1) the pair $(W_{1, \frac{t_4}{t_0^5}, t_0^{-1}\eta})$. The one parameter family $W_{1, z}$ (resp. W_ψ) can be recovered by putting $t_0 = 1$ and $t_4 = z$ (resp. $t_0 = \psi$ and $t_4 = 1$). In fact, the pair (W_{t_0, t_4}, ω_1) in the affine chart $x_0 = 1$ is given by:

$$(7) \quad \left(\{f_{t_0, t_4}(x) = 0\} / G, \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{df_{t_0, t_4}} \right),$$

where

$$f_{t_0, t_4} := -t_4 - x_1^5 - x_2^5 - x_3^5 - x_4^5 + 5t_0 x_1 x_2 x_3 x_4.$$

4 Gauss-Manin connection, I

We would like to calculate the Gauss-Manin connection

$$\nabla : H_{\text{dR}}^3(W/S) \rightarrow \Omega_S^1 \otimes_{\mathcal{O}_S} H_{\text{dR}}^3(W/S).$$

of the two parameter proper family of varieties W_{t_0, t_4} , $(t_0, t_4) \in S$. By abuse of notation we use $\frac{\partial}{\partial t_i}$, $i = 0, 4$ instead of $\nabla \frac{\partial}{\partial t_i}$. We calculate ∇ with respect to the basis

$$\omega_i = \frac{\partial^{i-1}}{\partial t_0^{i-1}}(\omega_1), \quad i = 1, 2, 3, 4$$

of global sections of $H_{\text{dR}}^3(W/S)$. For this purpose we return back to the one parameter case. We set $t_0 = 1$ and $t_4 = z$ and calculate the Picard-Fuchs equation of η with respect to the parameter z :

$$\frac{\partial^4 \eta}{\partial z^4} = \sum_{i=1}^4 a_i(z) \frac{\partial^{i-1} \eta}{\partial z^{i-1}} \quad \text{modulo relatively exact forms.}$$

This is in fact the linear differential equation

$$(8) \quad I'''' = \frac{-24}{625z^4 - 625z^3} I + \frac{-24z + 5}{5z^4 - 5z^3} I' + \frac{-72z + 35}{5z^3 - 5z^2} I'' + \frac{-8z + 6}{z^2 - z} I'''$$

which is calculated in [6], see also [27] for some algorithms which calculate such differential equations. It is satisfied by the periods $I(z) = \int_{\delta_z} \eta$, $\delta \in H_3(W_{1, z}, \mathbb{Q})$ of the differential form η on the the family $W_{1, z}$. In the basis $\frac{\partial^i \eta}{\partial z^i}$, $i = 0, 1, 2, 3$ the Gauss-Manin connection matrix has the form

$$(9) \quad A(z) dz := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1(z) & a_2(z) & a_3(z) & a_4(z) \end{pmatrix} dz.$$

Now, consider the identity map

$$g : W_{(t_0, t_4)} \rightarrow W_{1, z},$$

which satisfies $g^* \eta = t_0 \omega_1$. Under this map

$$\frac{\partial}{\partial z} = \frac{-1}{5} \frac{t_0^6}{t_4} \frac{\partial}{\partial t_0} \left(= t_0^5 \frac{\partial}{\partial t_4} \right).$$

From these two equalities we obtain a matrix $S = S(t_0, t_4)$ such that

$$\left[\eta, \frac{\partial \eta}{\partial z}, \frac{\partial^2 \eta}{\partial z^2}, \frac{\partial^3 \eta}{\partial z^3}\right]^t = S^{-1}[\omega_1, \omega_2, \omega_3, \omega_4]^t,$$

where t denotes the transpose of matrices, and the Gauss-Manin connection in the basis ω_i , $i = 1, 2, 3, 4$ is:

$$\left(dS + S \cdot A\left(\frac{t_4}{t_0}\right) \cdot d\left(\frac{t_4}{t_0}\right)\right) \cdot S^{-1}$$

which is the following matrix after doing explicit calculations:

$$(10) \quad \begin{pmatrix} -\frac{1}{5t_4} dt_4 & dt_0 + \frac{-t_0}{5t_4} dt_4 & 0 & 0 \\ 0 & \frac{-2}{5t_4} dt_4 & dt_0 + \frac{-t_0}{5t_4} dt_4 & 0 \\ 0 & 0 & \frac{-3}{5t_4} dt_4 & dt_0 + \frac{-t_0}{5t_4} dt_4 \\ \frac{-t_0}{t_0^5 - t_4} dt_0 + \frac{t_0^2}{5t_0^5 t_4 - 5t_4^2} dt_4 & \frac{-15t_0^2}{t_0^5 - t_4} dt_0 + \frac{3t_0^3}{t_0^3 t_4 - t_4^2} dt_4 & \frac{-25t_0^3}{t_0^5 - t_4} dt_0 + \frac{5t_0^4}{t_0^3 t_4 - t_4^2} dt_4 & \frac{-10t_0^4}{t_0^5 - t_4} dt_0 + \frac{6t_0^5 + 4t_4}{5t_0^5 t_4 - 5t_4^2} dt_4 \end{pmatrix}$$

From the above matrix or directly from (8) one can check that the periods x_{i1} , $i = 1, 2, 3, 4$ in the Introduction satisfy the Picard-Fuch equation:

$$(11) \quad I'''' = \frac{-\psi}{\psi^5 - 1} I + \frac{-15\psi^2}{\psi^5 - 1} I' + \frac{-25\psi^3}{\psi^5 - 1} I'' + \frac{-10\psi^4}{\psi^5 - 1} I''', \quad ' = \frac{\partial}{\partial \psi}.$$

5 Intersection form and Hodge filtration

For $\omega, \alpha \in H_{\text{dR}}^3(W_{t_0, t_4})$ let

$$\langle \omega, \alpha \rangle := \frac{1}{(2\pi i)^3} \int_{W_{t_0, t_4}} \omega \cup \alpha.$$

This is Poincaré dual to the intersection form in $H_3(W_{t_0, t_4}, \mathbb{Q})$. In $H_{\text{dR}}^3(W_{t_0, t_4})$ we have the Hodge filtration

$$\{0\} = F^4 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H_{\text{dR}}^3(W_{t_0, t_4}), \quad \dim_{\mathbb{C}}(F^i) = 4 - i.$$

There is a relation between the Hodge filtration and the intersection form which is given by the following collection of equalities:

$$\langle F^i, F^j \rangle = 0, \quad i + j \geq 4.$$

The Griffiths transversality is a property combining the Gauss-Manin connection and the Hodge filtration. It says that the Gauss-Manin connection sends F^i to $\Omega_S^1 \otimes F^{i-1}$ for $i = 1, 2, 3$. Using this we conclude that:

$$\omega_i \in F^{4-i}, \quad i = 1, 2, 3, 4.$$

Proposition 1. *The intersection form in the basis ω_i is:*

$$[\langle \omega_i, \omega_j \rangle] = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{625}(t_4 - t_0^5)^{-1} \\ 0 & 0 & -\frac{1}{625}(t_4 - t_0^5)^{-1} & -\frac{1}{125}t_0^4(t_4 - t_0^5)^{-2} \\ 0 & \frac{1}{625}(t_4 - t_0^5)^{-1} & 0 & \frac{1}{125}t_0^3(t_4 - t_0^5)^{-2} \\ -\frac{1}{625}(t_4 - t_0^5)^{-1} & \frac{1}{125}t_0^4(t_4 - t_0^5)^{-2} & -\frac{1}{125}t_0^3(t_4 - t_0^5)^{-2} & 0 \end{pmatrix}$$

Proof. Let Ω be the differential form ω_1 with restricted parameters $t_0 = \psi$ and $t_4 = 1$. We have $\langle 5\psi\Omega, \frac{\partial^3(5\psi\Omega)}{\partial^3\psi} \rangle = \frac{1}{5^2} \frac{\psi^2}{1-\psi^5}$ (see [6], (4.6)). From this we get:

$$(12) \quad \langle \omega_1, \omega_4 \rangle = 5^{-4} \frac{1}{t_4 - t_0^5}.$$

The corresponding calculations are as follows: In (t_0, t_4) coordinates we have $\psi = t_0 t_4^{-\frac{1}{5}}$ and $\frac{\partial}{\partial \psi} = t_4^{\frac{1}{5}} \frac{\partial}{\partial t_0}$ and

$$\begin{aligned} \langle \psi\Omega, \frac{\partial^3\psi\tilde{\eta}}{\partial^3\psi} \rangle &= \langle t_0\omega_1, (t_4^{\frac{1}{5}} \frac{\partial}{\partial t_0})^{(3)}(t_0\omega_1) \rangle \\ &= \langle t_0\omega_1, t_0 t_4^{\frac{3}{5}} \omega_4 \rangle = t_0^2 t_4^{\frac{3}{5}} \langle \omega_1, \omega_4 \rangle. \end{aligned}$$

From another side $\frac{1}{5^2} \frac{\psi^2}{1-\psi^5} = \frac{1}{5^2} \frac{t_0^2 t_4^{\frac{3}{5}}}{t_4 - t_0^5}$.

We make the derivation of the equalities $\langle \omega_1, \omega_3 \rangle = 0$ and (12) with respect to t_0 and use the Picard-Fuchs equation of ω_1 with respect to the parameter t_0 and with t_4 fixed:

$$\frac{\partial \omega_4}{\partial t_0} = M_{41}\omega_1 + M_{42}\omega_2 + M_{43}\omega_3 + M_{44}\omega_4$$

Here, M_{ij} is the (i, j) -entry of (10) after setting $dt_4 = 0$, $dt_0 = 1$. We get

$$\langle \omega_2, \omega_3 \rangle = -\langle \omega_1, \omega_4 \rangle, \quad \langle \omega_2, \omega_4 \rangle = \frac{\partial \langle \omega_1, \omega_4 \rangle}{\partial t_0} - M_{44} \langle \omega_1, \omega_4 \rangle$$

Derivating further the second equality we get:

$$\langle \omega_3, \omega_4 \rangle = \frac{\partial \langle \omega_2, \omega_4 \rangle}{\partial t_0} - M_{43} \langle \omega_2, \omega_3 \rangle - M_{44} \langle \omega_2, \omega_4 \rangle.$$

□

6 Moduli space, II

Let T be the moduli of pairs (W, ω) , where W is a Calabi-Yau variety as before and $\omega \in H_{\text{dR}}^3(W) \setminus F^1$ and F^1 is the biggest non trivial piece of the Hodge filtration of $H_{\text{dR}}^3(W)$. In this section, we construct good affine coordinates for the moduli space T .

Let G_m be the multiplicative group $(\mathbb{C} - \{0\}, \cdot)$ and let G_a be the additive group $(\mathbb{C}, +)$. Both these algebraic groups act on the moduli spaces T :

$$(W, \omega) \bullet k = (W, k\omega), \quad k \in G_m, \quad (W, \omega) \in T,$$

$$(W, \omega) \bullet k = (W, \omega + k\omega'), \quad k \in G_a, \quad (W, \omega) \in T,$$

where ω' is uniquely determined by $\langle \omega', \omega \rangle = 1$, $\omega' \in F^3$. We would like to have affine coordinates $(t_0, t_1, t_2, t_3, t_4)$ for T such that;

1. We have a canonical map

$$\pi : T \rightarrow S, \quad (W, \omega) \mapsto (W, \omega'),$$

where ω' is determined uniquely by $\langle \omega', \omega \rangle = 1$, $\omega' \in F^3$. In terms of the coordinates t_i 's it is just the projection on t_0, t_4 coordinates.

2. With respect to the action of G_m , t_i 's behave as below:

$$t_i \bullet k = k^{i+1}t_i, \quad i = 0, 1, \dots, 4.$$

3. With respect to the action of G_a , t_i 's behave as below:

$$t_i \bullet k = t_i, \quad i = 0, 2, 3, 4, \quad k \in G_a,$$

$$t_1 \bullet k = t_1 + k, \quad k \in G_a.$$

In order to construct t_i 's we take the family W_{t_0, t_4} as before and three new variable t_1, t_2, t_3 . One can verify easily that

$$\{(t_0, t_1, t_2, t_3, t_4) \in \mathbb{C}^5 \mid t_4(t_4 - t_0^5) \neq 0\} \cong T,$$

$$(t_0, t_1, t_2, t_3, t_4) \mapsto (W_{t_0, t_{n+1}}, \omega),$$

where

$$(13) \quad \omega = t_1\omega_1 + t_2\omega_2 + t_3\omega_3 + \frac{\omega_4}{\langle \omega_1, \omega_4 \rangle}.$$

7 Gauss-Manin connection, II

For the five parameter family $W_t, t := (t_0, t_1, t_2, t_3, t_4) \in T$, we calculate the differential forms α_i , $i = 1, 2, 3, 4$ in T which are defined by the equality:

$$\nabla\omega = \sum_{i=1}^4 \alpha_i \otimes \omega_i,$$

where ω is defined in (13), and we check that the $\mathbb{Q}(t)$ vector space spanned by α_i is exactly of dimension 4 and so up to multiplication by a rational function in $\mathbb{Q}(t)$ there is a unique vector field Ra which satisfies

$$(14) \quad \alpha_i(\text{Ra}) = 0, \quad i = 1, 2, 3, 4$$

or equivalently $\nabla_{\text{Ra}}\omega = 0$. We calculate this vector field and get the following expression:

$$\begin{aligned} \text{Ra} = & \left(\frac{6}{5}t_0^5 + \frac{1}{3125}t_0t_3 - \frac{1}{5}t_4 \right) \frac{\partial}{\partial t_0} + \left(-125t_0^6 + t_0^4t_1 + 125t_0t_4 + \frac{1}{3125}t_1t_3 \right) \frac{\partial}{\partial t_1} \\ & + \left(-1875t_0^7 - \frac{1}{5}t_0^5t_1 + 2t_0^4t_2 + 1875t_0^2t_4 + \frac{1}{5}t_1t_4 + \frac{2}{3125}t_2t_3 \right) \frac{\partial}{\partial t_2} + \\ & \left(-3125t_0^8 - \frac{1}{5}t_0^5t_2 + 3t_0^4t_3 + 3125t_0^3t_4 + \frac{1}{5}t_2t_4 + \frac{3}{3125}t_3^2 \right) \frac{\partial}{\partial t_3} + \left(5t_0^4t_4 + \frac{1}{625}t_3t_4 \right) \frac{\partial}{\partial t_4}. \end{aligned}$$

This appears in the first five lines of the ordinary differential equation (1). The other pieces of this differential equation has to do with the fact that the choice of Ra is not unique. Let

$$\alpha := \frac{t_0dt_4 - 5t_4dt_0}{(t_4 - t_0^5)t_4}.$$

The vector field Ra turns to be unique after putting the condition

$$(15) \quad \alpha(\text{Ra}) = 1$$

We have calculated Ra from (15) and (14). The choice of α up to multiplication by a rational function is canonical (see bellow). However, choosing such a rational function does not seem to be canonical.

Proposition 2. *There is a unique basis $\tilde{\omega}_i$, $i = 1, 2, 3, 4$ of $H_{\text{dR}}^3(W_t)$, $t \in T$ such that*

1. *It is compatible with the Hodge filtration, i.e. $\tilde{\omega}_i \in F^{4-i} \setminus F^{5-i}$.*
2. *$\tilde{\omega}_4 = \omega$ and $\langle \tilde{\omega}_1, \tilde{\omega}_4 \rangle = 1$.*
3. *The Gauss-Manin connection matrix A of the family $W \rightarrow T$ in the mentioned basis is of the form*

$$A = \begin{pmatrix} * & \alpha & 0 & 0 \\ * & * & \alpha & 0 \\ * & * & * & b_4\alpha \\ * & * & * & * \end{pmatrix}$$

and

$$A(\text{Ra}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & b_2 & b_3 & b_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $b_2, b_3, b_4 \in \mathbb{C}[t]$.

Our proof of the above proposition is algorithmic and in fact we calculate b_i 's

$$b_2 = -\frac{72}{5}t_0^8 - \frac{24}{3125}t_0^4t_3 - \frac{3}{5}t_0^3t_4 - \frac{2}{1953125}t_3^2$$

$$b_3 = 12t_0^4 + \frac{2}{625}t_3, \quad b_4 = -\frac{1}{5^7}(t_0^5 - t_4)^2$$

and $\tilde{\omega}_i$'s:

$$\tilde{\omega}_1 = \omega_1, \quad \tilde{\omega}_2 = (-t_0^4 - \frac{1}{3125}t_3)\omega_1 + (\frac{1}{5}t_0^5 - \frac{1}{5}t_4)\omega_2, \quad \tilde{\omega}_4 = \omega$$

$$\tilde{\omega}_3 := (-\frac{14}{5}t_0^8 + \frac{1}{15625}t_0^5t_2 - \frac{1}{625}t_0^4t_3 - \frac{1}{5}t_0^3t_4 - \frac{1}{15625}t_2t_4 - \frac{2}{9765625}t_3^2)\omega_1 +$$

$$(\frac{3}{5}t_0^9 + \frac{2}{15625}t_0^5t_3 - \frac{3}{5}t_0^4t_4 - \frac{2}{15625}t_3t_4)\omega_2 + (\frac{1}{25}t_0^{10} - \frac{2}{25}t_0^5t_4 + \frac{1}{25}t_4^2)\omega_3.$$

The polynomials b_2 and b_3 appear in the last line of the ordinary differential equation (1).

Proof. The equalities in the second item and $\tilde{\omega}_1 \in F^3$ determine both $\tilde{\omega}_1 = \omega_1, \tilde{\omega}_4 = \omega$ uniquely. We first take the 3-forms $\tilde{\omega}_i = \omega_i$, $i = 2, 3$ as in the previous section and write the Gauss-Manin connection of the five parameter family of Calabi-Yau varieties W_t , $t \in T$ in the basis $\tilde{\omega}_i$, $i = 1, 2, 3, 4$:

$$\nabla[\tilde{\omega}_i]_{4 \times 1} = [\alpha_{ij}]_{4 \times 4}[\tilde{\omega}_i]_{4 \times 1}.$$

We explain how to modify $\tilde{\omega}_2$ and $\tilde{\omega}_3$ and get the basis in the announcement of the proposition. Let R be the $\mathbb{Q}(t)$ vector space generated by $\alpha_{4,i}$, $i = 1, 2, \dots, 4$. It does not

depend on the choice of the basis $\tilde{\omega}_i$ and we already mentioned that it is of dimension 4. If we replace $\tilde{\omega}_2$ by $\tilde{\omega}_2 + a\tilde{\omega}_1$ then α_{11} is replaced by $\alpha_{11} - a\alpha_{12}$. Modulo R the space of differential forms on T is one dimensional and since $\alpha_{12} \notin R$, we choose a in such a way that $\alpha_{11} - a\alpha_{12} \in R$. We do this and so we can assume that $\alpha_{11} \in R$. The result of our calculations shows that α_{12} is a multiple of $t_0 dt_4 - 5t_4 dt_0$. We replace ω_2 by $r\tilde{\omega}_2$ with some $r \in \mathbb{Q}(t)$ and get the desired form for α_{12} . We repeat the same procedure for $\tilde{\omega}_3$. In this step we replace $\tilde{\omega}_3$ by $r_3\tilde{\omega}_3 + r_2\tilde{\omega}_2 + r_1\tilde{\omega}_1$ with some $r_1, r_2, r_3 \in \mathbb{Q}(t)$. \square

8 Polynomial Relations between periods

We take a basis $\delta_1, \delta_2, \delta_3, \delta_4 \in H_3(W_{t_0, t_4}, \mathbb{Q})$ such that the intersection form in this basis is given by:

$$(16) \quad \Psi := [\langle \delta_i, \delta_j \rangle] = \begin{pmatrix} 0 & 0 & 0 & -\frac{6}{5} \\ 0 & 0 & \frac{2}{5} & 0 \\ 0 & -\frac{2}{5} & 0 & 2 \\ \frac{6}{5} & 0 & -2 & 0 \end{pmatrix}.$$

It is also convenient to use the basis $[\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3, \tilde{\delta}_4] = [\delta_1, \delta_2, \delta_3, \delta_4]\Psi^{-1}$. In this basis the intersection form is $[\langle \tilde{\delta}_i, \tilde{\delta}_j \rangle] = \Psi^{-t}$. Let ω_i , $i = 1, 2, 3, 4$ be the basis of the de Rham cohomology $H_{\text{dR}}^3(W_{t_0, t_4})$ constructed in §4 and let $\tilde{\delta}_i^p \in H^3(W_{t_0, t_4}, \mathbb{Q})$ be the Poincaré dual of $\tilde{\delta}_i$, that is, it is defined by the property $\int_{\delta} \tilde{\delta}_i^p = \langle \delta, \tilde{\delta}_i \rangle$ for all $\delta \in H_3(W_{t_0, t_4}, \mathbb{Q})$. If we write ω_i in terms of $\tilde{\delta}_i^p$ what we get is:

$$[\omega_1, \omega_2, \omega_3, \omega_4] = [\tilde{\delta}_1^p, \tilde{\delta}_2^p, \tilde{\delta}_3^p, \tilde{\delta}_4^p] \left[\int_{\delta_i} \omega_j \right]$$

that is, the coefficients of the base change matrix are the periods of ω_i 's over δ_i 's and not $\tilde{\delta}_i$'s. The matrix $[\int_{\delta_i} \omega_j]$ is called the period matrix associated to the basis ω_i of $H_{\text{dR}}^3(W_{t_0, t_4})$ and the basis δ_i of $H_3(W, \mathbb{Q})$. We have

$$(17) \quad [\langle \omega_i, \omega_j \rangle] = \left[\int_{\delta_i} \omega_j \right]^t \Psi^{-t} \left[\int_{\delta_i} \omega_j \right].$$

Taking the determinant of this equality we can calculate $\det([\int_{\delta_i} \omega_j])$ up to sign:

$$(18) \quad \det(\text{pm}) = \frac{12}{5^{10}} \frac{1}{(t_4 - t_0^5)^2}.$$

There is another effective way to calculate this determinant without the sign ambiguity. For simplicity, we use the restricted parameters $t_4 = 1$ and $t_0 = \psi$ and the notation $x_{ij} := \int_{\delta_i} \omega_j$ as in the Introduction. Proposition 1 and the equality (17) gives us 6 non

trivial relations between x_{ij} 's:

$$\begin{aligned}
0 &= -\frac{25}{6}x_{12}x_{21} + \frac{25}{6}x_{11}x_{22} + \frac{5}{2}x_{22}x_{31} - \frac{5}{2}x_{21}x_{32} - \frac{5}{6}x_{12}x_{41} + \frac{5}{6}x_{11}x_{42} \\
0 &= -\frac{25}{6}x_{13}x_{21} + \frac{25}{6}x_{11}x_{23} + \frac{5}{2}x_{23}x_{31} - \frac{5}{2}x_{21}x_{33} - \frac{5}{6}x_{13}x_{41} + \frac{5}{6}x_{11}x_{43} \\
0 &= -\frac{25}{6}x_{14}x_{21} + \frac{25}{6}x_{11}x_{24} + \frac{5}{2}x_{24}x_{31} - \frac{5}{2}x_{21}x_{34} - \frac{5}{6}x_{14}x_{41} + \frac{5}{6}x_{11}x_{44} - \frac{1}{625(\psi^5 - 1)} \\
0 &= -\frac{25}{6}x_{13}x_{22} + \frac{25}{6}x_{12}x_{23} + \frac{5}{2}x_{23}x_{32} - \frac{5}{2}x_{22}x_{33} - \frac{5}{6}x_{13}x_{42} + \frac{5}{6}x_{12}x_{43} + \frac{1}{625(\psi^5 - 1)} \\
0 &= -\frac{25}{6}x_{14}x_{22} + \frac{25}{6}x_{12}x_{24} + \frac{5}{2}x_{24}x_{32} - \frac{5}{2}x_{22}x_{34} - \frac{5}{6}x_{14}x_{42} + \frac{5}{6}x_{12}x_{44} - \frac{\psi^4}{125(\psi^5 - 1)^2} \\
0 &= -\frac{25}{6}x_{14}x_{23} + \frac{25}{6}x_{13}x_{24} + \frac{5}{2}x_{24}x_{33} - \frac{5}{2}x_{23}x_{34} - \frac{5}{6}x_{14}x_{43} + \frac{5}{6}x_{13}x_{44} + \frac{\psi^3}{125(\psi^5 - 1)^2}.
\end{aligned}$$

These equalities correspond to the entries (1, 2), (1, 3), (1, 4), (2, 3), (2, 4) and (3, 4) of (17). In the ideal of $\mathbb{Q}(\psi)[x_{ij}]$, $i, j = 1, 2, 3, 4$ generated by the polynomials $f_{12}, f_{13}, f_{14}, f_{23}, f_{2,4}, f_{34}$ in the right hand side of the above equalities the polynomial $\det([x_{ij}])$ is reduced to the right hand side of (18). For instance, Singular check this immediately (see [14]). Let y_{ij} be indeterminate variables, $R = \mathbb{C}(\psi)[y_{ij}]$, $i, j = 1, 2, 3, 4$ and

$$I := \{f \in R \mid f(x_{ij}) = 0\}.$$

Proposition 3. *The ideal I is generated by $f_{12}, f_{13}, f_{14}, f_{23}, f_{2,4}, f_{34}$.*

Proof. Let E be the differential field over $F = \mathbb{C}(\psi)$ generated by x_{ij} 's. Note that the matrix $[x_{ij}]$ is the fundamental system of the linear differential equation:

$$\frac{\partial}{\partial \psi}[x_{ij}] = [x_{ij}]B(\psi)^t,$$

where $B(\psi)$ is obtained from the matrix (10) by putting $dt_0 = 1$, $dt_4 = 0$, $t_0 = \psi$, $t_4 = 1$. The homology group $H_3(W_\psi, \mathbb{Q})$ has a symplectic basis and hence the monodromy group of W_ψ is a subgroup of $\mathrm{Sp}(4, \mathbb{Z})$. Consequently, the differential Galois group $G(E/F)$ is an algebraic subgroup of $\mathrm{Sp}(4, \mathbb{C})$ and it contains a maximal unipotent matrix which is the monodromy around $z = 0$. By a result of Saxl and Seitz, see [29], we have $G(E/F) = \mathrm{Sp}(4, \mathbb{C})$. Therefore, $\dim G(E/F) = 10$ which is the transcendental degree of the field E over F (see [30]). \square

9 A leaf of Ra

The solutions of the the vector field Ra in the moduli space T are the locus of parameters such that all the periods of ω are constant. We want to choose a solution of Ra and write it in an explicit form. We proceed as follows:

Let $\tilde{\delta}_i, \delta_i$ $i = 1, 2, 3, 4$ be two basis of $H_3(W_t, \mathbb{Q})$ as in §8 and let $C_{4 \times 1} = [c_1, c_2, c_3, c_4]^t$ be given by the equality

$$[\langle \tilde{\delta}_i, \tilde{\delta}_j \rangle]C = [1, 0, 0, 0]^t.$$

and so $C = [0, 0, 0, -\frac{6}{5}]^t$. We are interested on the loci L of parameters $s \in T$ such that

$$(19) \quad \int_{\delta_i} \omega = c_i, \quad i = 1, 2, 3, 4.$$

We will write each coordinate of s in terms of periods: first we note that, on this locus we have

$$\int_{\delta_1} \omega_1 = 1$$

because

$$\begin{aligned} 1 &= \langle \omega_1, \omega \rangle = \sum_{i,j} \langle \tilde{\delta}_i, \tilde{\delta}_j \rangle \int_{\delta_i} \omega_1 \int_{\delta_j} \omega \\ &= \left[\int_{\delta_1} \omega_1, \dots, \int_{\delta_4} \omega_1 \right] [\langle \tilde{\delta}_i, \tilde{\delta}_j \rangle] C = \int_{\delta_1} \omega_1. \end{aligned}$$

By our choice ω_1 does not depend on t_1, t_2 and t_3 . Therefore, the locus of parameters s in T such that $\int_{\delta_1} \omega_1 = 1$ is given by

$$(20) \quad (s_0, s_4) = (t_0, t_4) \bullet \int_{\delta_1} = (t_0 \int_{\delta_1} \omega_1, t_4 (\int_{\delta_1} \omega_1)^5)$$

with arbitrary s_1, s_2, s_3 . This is because for $k = (\int_{\delta_1} \omega_1)^{-1}$, we have $\int_{\delta_1} k\omega_1 = 1$ and under the identification $(t_0, t_4) \mapsto (W_{t_0, t_4}, \omega_1)$, the pair $(t_0, t_4) \bullet k^{-1}$ is mapped to $(W_{t_0, t_4}, k\omega_1)$. To find s_1, s_2, s_3 parameters we proceed as follows: we know that $\omega = s_1\omega_1 + s_2\omega_2 + s_3\omega_3 + \frac{\omega_4}{\langle \omega_1, \omega_4 \rangle}$. This together with (19) and (12) imply that

$$\left[\int_{\delta_i} \omega_j \right]_{4 \times 4} [s_1, s_2, s_3, 625(s_4 - s_0^5)]^t = C$$

which gives formulas for s_1, s_2, s_3 in terms of periods. Let us write all these in terms of the periods of the one parameter family W_ψ . Recall the notation x_{ij} in the Introduction. We have $s_0 = \psi x_{11}$, $s_1 = x_{11}^5$ and

$$\int_{\delta_i} \omega_j = x_{11}^{-j} x_{ij}.$$

Note that in the above equality the cycle δ_i lives in $W_{\psi x_{11}, x_{11}^5}$. We restrict s_i 's to $t_0 = \psi$, $t_4 = 1$, we use the equality (18) and we get:

$$\begin{aligned} s_k &= -\frac{6}{5} \frac{(-1)^{4+k} \det[x_{11}^{-j} x_{ij}]_{i,j=1,2,3,4, i \neq 4, j \neq k}}{\det[x_{11}^{-j} x_{ij}]} \\ &= -\frac{6}{5} \frac{5^{10}}{12} (1 - \psi^5)^2 (-1)^{4+k} x_{11}^k \det[x_{ij}]_{i,j=1,2,3,4, i \neq 4, j \neq k} \quad k = 1, 2, 3, \end{aligned}$$

Modulo the ideal I in §8 the expressions for s_k 's can be reduced to to the shorter expressions in the right hand side of the equalities in Theorem 3. In the left hand side we have written t_i instead of s_i . We also get the relation

$$625x_{11}^5(1 - \psi^5) = -\frac{6}{5} \frac{5^{10}}{12} (1 - \psi^5)^2 x_{11}^4 \det[x_{ij}]_{i,j=1,2,3}.$$

The function $\psi \rightarrow s(\psi) := (s_0(\psi), s_1(\psi), \dots, s_4(\psi))$ is tangent to the vector field Ra but it is not its solution. In order to get a solution, one has to make a change of variable in ψ .

10 The parametrization

Let $\tilde{\omega}_i$, $i = 1, 2, 3, 4$ be the basis of the de Rham cohomology of W_t , $t \in T$ constructed in Proposition 2. We consider the period map:

$$\text{pm} : T \rightarrow \text{Mat}(4), \quad t \mapsto \left[\int_{\delta_i} \tilde{\omega}_j \right]_{4 \times 4},$$

where $\text{Mat}(4)$ is the set of 4×4 matrices. By our construction of $\tilde{\omega}_i$, its image is of dimension 5 and so it is an embedding in some open neighborhood U of a point $p \in L$ in T . We restrict its inverse $s = (s_0, s_1, s_2, s_3, s_4)$ to $\text{pm}(L)$, where L is defined in §9. Note that a point in $\text{pm}(L)$ is of the form:

$$P = \begin{pmatrix} 1 & p_{12} & p_{13} & 0 \\ \tau & p_{22} & p_{23} & 0 \\ p_{31} & p_{32} & p_{33} & 0 \\ p_{41} & p_{42} & p_{43} & -\frac{6}{5} \end{pmatrix}.$$

We consider s_0, s_1, s_2, s_3, s_4 and all the quantities p_{ij} as functions of τ and set $\dot{a} = \frac{\partial a}{\partial \tau}$. This is our derivation in (1). Note that τ as a function in ψ is given by:

$$\tau = \frac{\int_{\delta_2} \Omega}{\int_{\delta_1} \Omega}.$$

We have $\dot{s}(\tau) = x(\tau) \cdot \text{Ra}(s(\tau))$ for some holomorphic function x in $U \cap L$, because Ra is tangent to the locus L and s is a local parametrization of L . Let A be the Gauss-Manin connection matrix of the family W_t , $t \in T$ in the basis $\tilde{\omega}_i$, $i = 1, 2, 3, 4$. We have $d(\text{pm}) = \text{pm} \cdot A^t$, from which it follows

$$\begin{pmatrix} 0 & \dot{p}_{12} & \dot{p}_{13} & 0 \\ 1 & \dot{p}_{22} & \dot{p}_{23} & 0 \\ \dot{p}_{31} & \dot{p}_{32} & \dot{p}_{33} & 0 \\ \dot{p}_{41} & \dot{p}_{42} & \dot{p}_{43} & 0 \end{pmatrix} = \begin{pmatrix} 1 & p_{12} & p_{13} & 0 \\ \tau & p_{22} & p_{23} & 0 \\ p_{31} & p_{32} & p_{33} & 0 \\ p_{41} & p_{42} & p_{43} & -\frac{6}{5} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & x \cdot b_2(s) & 0 \\ 0 & x & x \cdot b_3(s) & 0 \\ 0 & 0 & x \cdot b_4(s) & 0 \end{pmatrix}.$$

Here we have used the particular form of A in Proposition 2. The equalities corresponding to the entries $(1, i)$, $i \geq 2$ together with the fact that $x \neq 0, b_4(s) \neq 0$ imply that $p_{12} = p_{13} = 0$. The equality for the entry $(2, 1)$ implies that $x = \frac{1}{p_{22}}$. Using these, we have

$$(21) \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & \dot{p}_{22} & \dot{p}_{23} & 0 \\ \dot{p}_{31} & \dot{p}_{32} & \dot{p}_{33} & 0 \\ \dot{p}_{41} & \dot{p}_{42} & \dot{p}_{43} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tau & p_{22} & p_{23} & 0 \\ p_{31} & p_{32} & p_{33} & 0 \\ p_{41} & p_{42} & p_{43} & -\frac{6}{5} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{p_{22}} & 0 & \frac{b_2(s)}{p_{22}} & 0 \\ 0 & \frac{1}{p_{22}} & \frac{b_3(s)}{p_{22}} & 0 \\ 0 & 0 & \frac{b_4(s)}{p_{22}} & 0 \end{pmatrix}.$$

11 Periods

Four linearly independent solutions of (8) are given by $\psi_0, \psi_1, \psi_2, \psi_3$, where

$$(22) \quad \sum_{i=0}^3 \psi_i(\tilde{z}) \epsilon^i + O(\epsilon^4) = \sum_{n=0}^{\infty} \frac{(1+5\epsilon)(2+5\epsilon) \cdots (5n+5\epsilon)}{((1+\epsilon)(2+\epsilon) \cdots (n+\epsilon))^5} \tilde{z}^{n+\epsilon}, \quad \tilde{z} = \frac{z}{5^5}$$

see for instance [21]. In fact, there are four topological cycles $\delta_1, \delta_2, \delta_3, \delta_4 \in H_3(W_z, \mathbb{Q})$ such that

$$\int_{\delta_i} \eta = \frac{(2\pi i)^{4-i}}{5^4} (i-1)! \psi_{i-1}.$$

Performing the monodromy of (22) around $z = 0$, we get the same expression multiplied with $e^{2\pi i \epsilon}$. Therefore, the monodromy $\tilde{\psi}_i$ of ψ_i is given according to the equalities:

$$\begin{aligned} \tilde{\psi}_0 &= \psi_0, \quad \tilde{\psi}_1 = (2\pi i)\psi_0 + \psi_1, \quad \tilde{\psi}_2 = \frac{(2\pi i)^2}{2!}\psi_0 + (2\pi i)\psi_1 + \psi_2, \\ \tilde{\psi}_3 &= \frac{(2\pi i)^3}{3!}\psi_0 + \frac{(2\pi i)^2}{2!}\psi_1 + (2\pi i)\psi_2 + \psi_3. \end{aligned}$$

This implies that the topological monodromy, which acts on $H_3(W_{1,z}, \mathbb{Q})$, in the basis δ_i , $i = 1, 2, 3, 4$ is given by

$$(23) \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

Further, the intersection form in this basis is Ψ in (16), and the monodromy around the other singularity is

$$\begin{pmatrix} 1 & -\frac{25}{6} & 0 & -\frac{5}{6} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

see for instance [31], page 5. In fact in [31] the authors have considered the basis $C[\delta_1, \delta_2, \delta_3, \delta_4]^t$, where

$$C = \begin{pmatrix} 0 & \frac{25}{6} & 0 & \frac{5}{6} \\ \frac{25}{6} & 0 & \frac{5}{2} & 0 \\ 0 & 5 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{pmatrix}.$$

Note that in the mentioned reference when the authors say that with respect to a basis $\delta_1, \delta_2, \delta_3, \delta_4$ of a vector space, a linear map is given by the matrix T then the action of the linear map on δ_i is the i -th coordinate of $[\delta_1, \delta_2, \delta_3, \delta_4]T$ and not $T[\delta_1, \delta_2, \delta_3, \delta_4]^t$. Define

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tau & 1 & 0 & 0 \\ \tau^2 & 2\tau & 2 & 0 \\ \tau^3 & 3\tau^2 & 6\tau & 6 \end{pmatrix}.$$

Note that

$$D = Z^{-1}\dot{Z} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and under the monodromy M , τ goes to $\tau + 1$ and Z goes to MZ . Therefore

$$Q = Z^{-1}P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p_{22} & p_{23} & 0 \\ \frac{1}{2}(p_{31} - \tau^2) & \frac{1}{2}p_{32} - \tau p_{22} & \frac{1}{2}p_{33} - \tau p_{23} & 0 \\ \frac{1}{3}\tau^3 - \frac{1}{2}\tau p_{31} + \frac{1}{6}p_{41} & \frac{1}{2}\tau^2 p_{22} - \frac{1}{2}\tau p_{32} + \frac{1}{6}p_{42} & \frac{1}{2}\tau^2 p_{23} - \frac{1}{2}\tau p_{33} + \frac{1}{6}p_{43} & -\frac{1}{5} \end{pmatrix}$$

is invariant under the monodromy around 0. The differential equation of P is given in (21) which we write it in the form $\dot{P} = \frac{1}{p_{22}}P \cdot A(\text{Ra})^t$. From this we calculate the differential equation of Q ;

$$\dot{Q} = -Z^{-1}\dot{Z}Z^{-1}P + Z^{-1}\dot{P} = -DQ + \frac{1}{q_{22}}QA(\text{Ra})^t =$$

$$\frac{1}{q_{22}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q_{23} & q_{22}b_2 + q_{23}b_3 & 0 \\ q_{32} & -q_{22}^2 + q_{33} & q_{32}b_2 + q_{33}b_3 - q_{22}q_{23} & 0 \\ -q_{22}q_{31} + q_{42} & -q_{22}q_{32} + q_{43} & q_{42}b_2 + q_{43}b_3 - \frac{1}{5}b_4 - q_{22}q_{33} & 0 \end{pmatrix}.$$

Let us use the new notation $s_5 = q_{22}$ and $s_6 = q_{23}$. The first five lines of our differential equation (1) is just $\dot{s} = \frac{1}{s_5}\text{Ra}(s)$ and the next two lines correspond to the equalities of (2, 2) and (2, 3) entries of the above matrices. Note that in (1) we have used the notation t_i instead of s_i .

12 Calculating q -expansions

All the quantities s_i are invariant under the monodromy M around $z = 0$. This implies that they are invariant under the transformation $\tau \rightarrow \tau + 1$. Therefore, all s_i 's can be written in terms of the new variable $q = e^{2\pi i\tau}$. In order to calculate all these q -expansions, it is enough to restrict to the case $t_0 = 1, t_1 = t_2 = t_3 = 0, t_4 = z$. We want to write

$$s_0 = \int_{\delta_1} \eta, \quad s_4 = z \left(\int_{\delta_1} \eta \right)^5$$

in terms of q . Calculating ψ_0 and ψ_1 from the formula (22) we get:

$$\psi_0 = \sum_{m=0}^{\infty} \frac{(5m)!}{(m!)^5} z^m$$

$$\psi_1 = \ln(\tilde{z})\psi_0(\tilde{z}) + 5\tilde{\psi}_1(\tilde{z}), \quad \tilde{\psi}_1 := \sum_{m=1}^{\infty} \frac{(5m)!}{(m!)^5} \left(\sum_{k=m+1}^{5m} \frac{1}{k} \right) \tilde{z}^m$$

and so

$$q = e^{2\pi i \frac{\int_{\delta_2} \eta}{\int_{\delta_1} \eta}} = \tilde{z} e^{5 \frac{\tilde{\psi}_1(\tilde{z})}{\psi_0(\tilde{z})}}.$$

By comparing few coefficients of \tilde{z}^i and we get

$$(24) \quad s_0 = \int_{\delta_1} \eta = \frac{1}{5} \left(\frac{2\pi i}{5} \right)^3 \psi_0 = \frac{1}{5} \left(\frac{2\pi i}{5} \right)^3 (1 + 5!q + 21000q^2 + \dots)$$

$$(25) \quad s_4 = z \left(\int_{\delta_1} \eta \right)^5 = 5^5 \left(\frac{1}{5} \left(\frac{2\pi i}{5} \right)^3 \right)^5 \tilde{z} \psi_0^5 = \left(\frac{2\pi i}{5} \right)^{15} (0 + q - 170q^2 + \dots).$$

In the differential equation (1), we consider the weights

$$(26) \quad \deg(t_i) = 3(i+1), \quad i = 0, 1, \dots, 4, \quad \deg(t_5) = 11, \quad \deg(t_6) = 23.$$

In this way in its right hand side we have homogeneous rational functions of degree 4, 7, 10, 13, 16, 12, 24 which is compatible with the left hand side if we assume that the derivation increases the degree by one. We have $\frac{\partial}{\partial \tau} = (\frac{2\pi i}{5})5q\frac{\partial}{\partial q}$ and so $(\frac{2\pi i}{5})^{-\deg(t_i)}s_i$, $i = 0, 1, \dots, 6$ is the solution presented in the Introduction. The initial values (2) in the Introduction are taken from the equalities (24) and (25). In the literature, see for instance [21, 28], we find also the equalities:

$$q_{31} = \frac{1}{2}(p_{31} - \tau^2) = \frac{1}{2}\left(\frac{\int_{\delta_3} \eta}{\int_{\delta_1} \eta} - \left(\frac{\int_{\delta_2} \eta}{\int_{\delta_1} \eta}\right)^2\right) = \frac{1}{(2\pi i)^2}\left(\frac{\psi_2}{\psi_0} - \frac{1}{2}\left(\frac{\psi_1}{\psi_0}\right)^2\right) =$$

$$\frac{1}{(2\pi i)^2} \frac{1}{5} \left(\sum_{n=1}^{\infty} \left(\sum_{d|n} n_d d^3 \right) \frac{q^n}{n^2} \right),$$

$$q_{14} = \frac{1}{3}\tau^3 - \frac{1}{2}\tau p_{31} + \frac{1}{6}p_{41} = \frac{1}{(2\pi i)^3} \left(\frac{1}{3} \left(\frac{\psi_1}{\psi_0} \right)^3 - \frac{\psi_1}{\psi_0} \frac{\psi_2}{\psi_0} + \frac{\psi_3}{\psi_0} \right) = \frac{2}{5} \frac{1}{(2\pi i)^3} \left(\sum_{n=1}^{\infty} \left(\sum_{d|n} n_d d^3 \right) \frac{q^n}{n^3} \right),$$

where n_d are as explained in the Introduction.

13 Proof of Theorem 3

The proof of the equalities for t_0, t_1, t_3, t_4 is done in §9. In §7 we have calculated $\tilde{\omega}_2, \tilde{\omega}_3$ in terms of ω_2 and ω_3 . In §10 and §11 we have defined

$$s_5 = p_{22} = q_{22} = \int_{\delta_2} \tilde{\omega}_2, \quad s_6 = p_{23} = q_{23} = \int_{\delta_2} \tilde{\omega}_3.$$

Using $\int_{\delta_i} \omega_j = x_{11}^{-j} x_{ij}$ we get the expressions for s_5, s_6 in Theorem 3. Note that for simplicity in Theorem 3 we have again used the notation t_i instead of $s_i a^{-\deg(t_i)}$, where $a = \frac{2\pi i}{5}$ and $\deg(t_i)$ is defined in (26).

14 Proof of Theorem 1

The Yukawa coupling $k_{\tau\tau\tau}$ is a quantity attached to the family of Calabi-Yau varieties $W_{1,z}$. It can be written in terms of periods:

$$k_{\tau\tau\tau} = \frac{-5^{-4}a^6}{\left(z\frac{\partial \tau}{\partial z}\right)^3(z-1)\left(\int_{\delta_1} \eta\right)^2},$$

where $\tau = \frac{\int_{\delta_2} \eta}{\int_{\delta_1} \eta}$ and $a = \frac{2\pi i}{5}$, see for instance [24] page 258. In [6] the authors have calculated the q -expansion of the Yukawa coupling and they have reached to spectacular predictions presented in Introduction. Let us calculate the Yukawa coupling in terms of

our auxiliary quantities s_i . We use the notation $t_i = s_i a^{-\deg(t_i)}$.

$$\begin{aligned}
k_{\tau\tau\tau} &= \frac{-5^{-4}a^6}{\left(\frac{t_4}{t_0^3}\right)^3 \left(\frac{\partial\left(\frac{t_4}{t_0^5}\right)}{\partial\tau}\right)^{-3} \left(\frac{t_4}{t_0^5} - 1\right)(a^3t_0)^2} = \frac{-5^{-4} \left(\frac{t_4}{t_0^5}\right)^3}{\left(\frac{t_4}{t_0^5}\right)^3 \left(\frac{t_4}{t_0^5} - 1\right)t_0^2} = \frac{-5^{-4}(t_0t_4 - 5t_0^2t_4)^3t_0^{12}}{t_4^3(t_4 - t_0)} \\
&= \frac{-5^{-4}(t_0(5t_0^4t_4 + \frac{1}{625}t_3t_4) - 5(\frac{6}{5}t_0^5 + \frac{1}{3125}t_0t_3 - \frac{1}{5}t_4)t_4)^3t_0^{12}}{t_5^3t_4^3(t_4 - t_0)} \\
&= \frac{-5^{-4}(t_4 - t_0^5)^2}{t_5^3}
\end{aligned}$$

Theorem 1 is proved.

15 Proof of Theorem 2

First, we note that if there is a polynomial relation with coefficients in \mathbb{C} between $t_i, i = 0, 1, \dots, 6$ (as power series in $q = e^{2\pi i\tau}$ and hence as functions in τ) then the same is true if we change the variable τ by some function in another variable. In particular, we put $\tau = \frac{x_{21}}{x_{11}}$ and obtain t_i 's in terms of periods. Now, it is enough to prove that the period expressions in Theorem 3 are algebraically independent over \mathbb{C} . Using Proposition 3, it is enough to prove that the variety induced by the ideal $\tilde{I} = \langle t_i - k_i, i = 0, 1, \dots, 6 \rangle + I \subset k[y_{ij}, i, j = 1, 2, 3, 4]$ is of dimension $16 - 6 - 7 = 3$. Here k_i 's are arbitrary parameters, I is the ideal in §8, $k = \mathbb{C}(k_i, i = 0, 1, \dots, 6)$ and in the expressions of t_i we have written y_{ij} instead of x_{ij} . This can be done by any software in commutative algebra (see for instance [14]).

16 Moduli space, III

In this section we introduce moduli interpretation for t_5 and t_6 . Let $\tilde{R}a$ be the vector field in \mathbb{C}^7 corresponding to (1) and let $\tilde{\omega}_i, i = 1, 2, 3, 4$ be the differential forms calculated in Proposition 2. Consider $t_i, i = 0, 1, 2, \dots, 6$ as unknown parameters. We define a new basis $\hat{\omega}_i, i = 1, 2, 3, 4$ of $H_{\text{dR}}^3(W_{t_0, t_4})$:

$$\hat{\omega}_1 = \tilde{\omega}_1, \hat{\omega}_2 = \frac{1}{t_5}\tilde{\omega}_2, \hat{\omega}_3 = \frac{5^7}{(t_4 - t_0^5)^2}(-t_6\tilde{\omega}_2 + t_5\tilde{\omega}_3), \hat{\omega}_4 = \tilde{\omega}_4.$$

The intersection form in the basis $\hat{\omega}_i, i = 1, 2, 3, 4$ is a constant matrix and in fact it is:

$$(27) \quad \Phi := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

The Gauss-Manin connection composed with $\tilde{\text{Ra}}$ has also the form:

$$\nabla_{\tilde{\text{Ra}}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{(t_4 - t_0^5)^2}{5^7 t_5^3} & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It is interesting that the Yukawa coupling appears as the only non constant term in the above matrix. Let X be the moduli of pairs $(W, [\alpha_1, \alpha_2, \alpha_3, \alpha_4])$, where W is a Calabi-Yau variety as before, $\alpha_i \in F^{4-i} \setminus F^{5-i}$, $F^i \subset H_{\text{dR}}^3(W)$ is the i -th piece of the Hodge filtration, α_i 's form a basis of $H_{\text{dR}}^3(W)$ and the intersection form in α_i 's is given by the matrix (27). We have the isomorphism

$$(28) \quad \{t \in \mathbb{C}^7 \mid t_5 t_4 (t_4 - t_0^5) \neq 0\} \cong X$$

$$t \mapsto (W_{t_0, t_4}, [\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3, \hat{\omega}_4])$$

which gives the full moduli interpretation of all t_i 's.

17 A conjecture and concluding remarks

A conjecture: We have calculated the first eleven coefficients of

$$(29) \quad \frac{1}{24}t_0, \frac{-1}{750}t_1, \frac{-1}{50}t_2, \frac{-1}{5}t_3, -t_4, 25t_5, 15625t_6$$

in the differential equation (1).

	q^0	q^1	q^2	q^3	q^4	q^5	q^6
$\frac{1}{24}t_0$	$\frac{1}{120}$	1	175	117625	111784375	126958105626	160715581780591
$\frac{-1}{750}t_1$	$\frac{1}{30}$	3	930	566375	526770000	592132503858	745012928951258
$\frac{-1}{50}t_2$	$\frac{7}{10}$	107	50390	29007975	26014527500	28743493632402	35790559257796542
$\frac{-1}{5}t_3$	$\frac{6}{5}$	71	188330	100324275	86097977000	93009679497426	114266677893238146
$-t_4$	0	-1	170	41475	32183000	32678171250	38612049889554
$25t_5$	$-\frac{1}{125}$	15	938	587805	525369650	577718296190	716515428667010
$15625t_6$	0	-15	26249	3512835	2527019900	2381349669050	2699403828169815
q^7	q^8			q^9			q^{10}
218874699262438350	314179164066791400375			469234842365062637809375			722875994952367766020759550
1010500474677945510	1446287695614437271000			2155340222852696651995625			3314709711759484241245738380
48205845153859479030	68647453506412345755300			101912303698877609329100625			156263153250677320910779548340
152527823430305901510	215812408812642816943200			318839967257572460805706125			487033977592346076373921829980
50189141795178390	69660564113425804800			101431587084669781525125			153189681044166218779637500
962043316960737646	1366589803139580122090			2024744003173189934886225			3099476777084481347731347688
3414337117855753978	4647615139046603293280			6668975996587015549602975			9957519516309695103093241870

We have also calculated the Yukawa coupling $\frac{-(t_4 - t_0^5)^2}{625t_5^3}$. The numbers n_d in the Introduction are given by:

5, 2875, 609250, 317206375, 242467530000, 229305888887625, 248249742118022000, 295091050570845659250, 375632160937476603550000, 503840510416985243645106250, 704288164978454686113488249750

Based on these calculations we may conjecture:

Conjecture 1. All q -expansions of

$$\frac{1}{24}t_0 - \frac{1}{120}, \frac{-1}{750}t_1 - \frac{1}{30}, \frac{-1}{50}t_2 - \frac{7}{10}, \frac{-1}{5}t_3 - \frac{6}{5}, -t_4, 25t_5 + \frac{1}{125}, 15625t_6$$

have positive integer coefficients.

We have verified the conjecture for the coefficients of $q^i, i \leq 50$ (see the author's web page)³. The rational numbers which appear in (29) are chosen in such a way that the coefficients $t_{i,n}, n = 1, 2, \dots, 10$ become positive integers and for each fixed i they are relatively prime. Writing the series t_i as Lambert series $a_0 + \sum_{d=1}^{\infty} a_d \frac{q^d}{1-q^d}$ does not help for understanding the structure of $t_{i,n}$. It is not possible to factor out some potential of d from a_d 's for each t_i . One should probably take out a polynomial in q from t_i and then try to understand the nature of the sequences. The on-line encyclopedia of integer sequences does not recognize the integer sequences of t_0, t_1, \dots, t_6 . This supports the fact that the general formula for t_i 's or any interpretation of them is not yet known.

Is there a Calabi-Yau monster?: The parameter $j_* = z^{-1} = \frac{t_0^5}{t_4}$ classifies the Calabi-Yau varieties of type (3), that is, each such Calabi-Yau variety is represented exactly by one value of j_* and two such Calabi-Yau varieties are isomorphic if and only if the corresponding j_* values are equal. This is similar to the case of elliptic curves which are classified by the classical j -function (see §2). We have calculated also the q -expansions of j_* :

$$\begin{aligned} 3125 \cdot j_* &= \frac{1}{q} + 770 + 421375q + 274007500q^2 + 236982309375q^3 + 251719793608904q^4 \\ &+ 304471121626588125q^5 + 401431674714748714500q^6 + 562487442070502650877500q^7 \\ &+ 824572505123979141773850000q^8 + 1013472859153384775272872409691q^9 + O(q^{10}) \end{aligned}$$

The coefficient 3125 is chosen in such a way that all the coefficients of $q^i, i \leq 9$ in $3125 \cdot j_*$ are integer and all together are relatively prime. Note that the moduli parameter j_* in our case has two cusps ∞ and 1, that is, for these values of j_* we have singular fibers. Our q -expansion is written around the cusp ∞ .

All the beautiful history behind the interpretation of the coefficients of the classical j -function of elliptic curves, monster group, monstrous moonshine conjecture and Borcherds proof, may indicate us another fascinating mathematics behind the q -expansion of the j_* -function of the varieties (3).

Symplectic basis: The basis $\hat{\delta}_i, i = 1, 2, 3, 4$ given by $[\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_4]^t = C[\delta_1, \delta_2, \delta_3, \delta_4]^t$, where

$$C := \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -\frac{5}{2} & -\frac{5}{2} & 0 \\ 0 & \frac{25}{6} & 0 & \frac{5}{6} \end{pmatrix},$$

is the symplectic basis of $H_3(W_\psi, \mathbb{Q})$, that is, the intersection matrix in this basis is given by:

$$[\langle \hat{\delta}_i, \hat{\delta}_j \rangle]_{4 \times 4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

³<http://w3.impa.br/~hossein/manyfiles/calculation-haftarticle.txt>

In this basis the monodromy group Γ of the family W_ψ is generated by

$$\hat{T} := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{pmatrix}, \quad \hat{S} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In a private communication, Prof. Duco van Straten informed me of the following conjecture: the group Γ has infinite index in $\mathrm{Sp}(4, \mathbb{Z})$.

Modular properties of t_i 's: Recall the notation in §16. There is an algebraic group which acts on the right hand side of the isomorphism (28). It corresponds to the base change in $\hat{\omega}_i$, $i = 1, 2, 3, 4$ such that the new basis is still compatible with the Hodge filtration and we have still the intersection matrix (27):

$$G := \{g = [g_{ij}]_{4 \times 4} \in \mathrm{GL}(4, \mathbb{C}) \mid g_{ij} = 0, \text{ for } j < i \text{ and } g^t \Phi g = \Phi\},$$

$$(W, [\alpha_1, \alpha_2, \alpha_3, \alpha_4]) \bullet g = (W, [\alpha_1, \alpha_2, \alpha_3, \alpha_4]g).$$

Therefore, we have the action of G on the t -space, where $t = (t_0, t_1, \dots, t_6)$, from the right which we denote it again by \bullet . The period map $t \mapsto [\int_{\hat{\delta}_i} \hat{\omega}_j]$ written in the symplectic basis gives us the modular properties of t_i with respect to the monodromy group $\Gamma = \langle \hat{T}, \hat{S} \rangle$ as follows. We regard t_i 's as the coordinates of the inverse of the period map restricted to the matrices:

$$(30) \quad P(\tau) := \begin{pmatrix} \tau & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 \\ x_{41} & -\tau x_{32} + x_{31} & -\tau & 1 \end{pmatrix},$$

where

$$x_{31} = \frac{1}{2}(5(\tau + \tau^2) + \frac{1}{(2\pi i)^2} (\sum_{n=1}^{\infty} (\sum_{d|n} n_d d^3) \frac{e^{2\pi i \tau n}}{n^2})),$$

$$x_{32} = x'_{31}, \quad x'_{41} = x_{31} - \tau x'_{31}.$$

(recall that in this paragraph we are using the symplectic basis $\hat{\delta}_i$). For an element A in the monodromy group Γ and τ fixed, there is a unique element $g(\tau, A)$ of G and a matrix $P(\tilde{\tau})$ of the form (30) such that

$$A \cdot P(\tau) = P(\tilde{\tau}) \cdot g(\tau, A)$$

and so

$$t(\tau) = t(A \cdot P(\tau)) = t(P(\tilde{\tau})) \bullet g(\tau, A).$$

This is the modular functional equation of t . The explicit calculation of these modular properties and the full definition of a modular or a quasi-modular form will be done in the forthcoming articles. In the case of elliptic curves, the same procedure as above gives us the modularity of the Eisenstein series g_2 and g_3 and the functional equation of g_1 with an anomalous shift under $\mathrm{SL}(2, \mathbb{Z})$ (see [25]). In our context, two different phenomena appear: first we do not have a functional equation for individual t_i , but we have functional equations involving all t_i 's. Second, we have functional equations involving x_{31} and its derivatives.

References

- [1] Mina Aganagic, Vincent Bouchard, and Albrecht Klemm. Topological strings and (almost) modular forms. *Comm. Math. Phys.*, 277(3):771–819, 2008, arXiv:hep-th/0607100v2.
- [2] Murad Alim and Jean Dominique Länge. Polynomial structure of the (open) topological string partition function. *J. High Energy Phys.*, (10):045, 13, 2007, arXiv:0708.2886v2[hep-th].
- [3] Gert Almkvist and Wadim Zudilin. Differential equations, mirror maps and zeta values. In *Mirror symmetry. V*, volume 38 of *AMS/IP Stud. Adv. Math.*, pages 481–515. Amer. Math. Soc., Providence, RI, 2006, arXiv:math/0402386v2[math.NT].
- [4] Richard E. Borcherds. Monstrous moonshine and monstrous Lie superalgebras. *Invent. Math.*, 109(2):405–444, 1992.
- [5] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. On the modularity of elliptic curves over \mathbf{Q} : wild 3-adic exercises. *J. Amer. Math. Soc.*, 14(4):843–939 (electronic), 2001.
- [6] Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nuclear Phys. B*, 359(1):21–74, 1991.
- [7] Herbert Clemens. Curves on generic hypersurfaces. *Ann. Sci. École Norm. Sup. (4)*, 19(4):629–636, 1986.
- [8] Adrian Clingher and Charles F. Doran. Lattice polarized K3 surfaces and Siegel modular forms. 2010. arXiv:1004.3503v1[math.AG].
- [9] Robbert Dijkgraaf. Mirror symmetry and elliptic curves. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 149–163. Birkhäuser Boston, Boston, MA, 1995.
- [10] Charles F. Doran and John W. Morgan. Mirror symmetry and integral variations of Hodge structure underlying one-parameter families of Calabi-Yau threefolds. In *Mirror symmetry. V*, volume 38 of *AMS/IP Stud. Adv. Math.*, pages 517–537. Amer. Math. Soc., Providence, RI, 2006, arXiv:math/0505272v1[math.AG].
- [11] Michael R. Douglas. Conformal field theory techniques in large N Yang-Mills theory. In *Quantum field theory and string theory (Cargèse, 1993)*, volume 328 of *NATO Adv. Sci. Inst. Ser. B Phys.*, pages 119–135. Plenum, New York, 1995, arXiv:hep-th/9311130v1.
- [12] A. Giveon, M. Porrati, and E. Rabinovici. Target space duality in string theory. *Phys. Rep.*, 244(2-3):77–202, 1994, arXiv:hep-th/9401139v1.
- [13] Rajesh Gopakumar and Cumrun Vafa. M-theory and topological strings–II. 1998, arXiv:hep-th/9812127v1.
- [14] G.-M. Greuel, G. Pfister, and H. Schönemann. SINGULAR 2.0. A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, 2001. <http://www.singular.uni-kl.de>.

- [15] Phillip A. Griffiths. Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems. *Bull. Amer. Math. Soc.*, 76:228–296, 1970.
- [16] Shinobu Hosono. BCOV ring and holomorphic anomaly equation. In *New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008)*, volume 59 of *Adv. Stud. Pure Math.*, pages 79–110. Math. Soc. Japan, Tokyo, 2010, arXiv:0810.4795v2 [math.AG].
- [17] Trygve Johnsen and Steven L. Kleiman. Toward Clemens’ conjecture in degrees between 10 and 24. *Serdica Math. J.*, 23(2):131–142, 1997. Conference on Geometry and Mathematical Physics (Zlatograd, 1995), arXiv:alg-geom/9601024v2.
- [18] Masanobu Kaneko and Don Zagier. A generalized Jacobi theta function and quasi-modular forms. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 165–172. Birkhäuser Boston, Boston, MA, 1995.
- [19] Nicholas M. Katz. p -adic properties of modular schemes and modular forms. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 69–190. Lecture Notes in Mathematics, Vol. 350. Springer, Berlin, 1973.
- [20] Nicholas M. Katz and Tadao Oda. On the differentiation of de Rham cohomology classes with respect to parameters. *J. Math. Kyoto Univ.*, 8:199–213, 1968.
- [21] Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139, Basel, 1995. Birkhäuser.
- [22] Maxim Kontsevich, Albert Schwarz, and Vadim Vologodsky. Integrality of instanton numbers and p -adic B-model. *Phys. Lett. B*, 637(1-2):97–101, 2006, arXiv:hep-th/0603106v3.
- [23] Christian Krattenthaler and Tanguy Rivoal. On the integrality of the Taylor coefficients of mirror maps. *Duke Math. J.*, 151(2):175–218, 2010, arXiv:0907.2577v2[math.NT].
- [24] David R. Morrison. Picard-Fuchs equations and mirror maps for hypersurfaces. In *Essays on mirror manifolds*, pages 241–264. Int. Press, Hong Kong, 1992. arXiv:alg-geom/9202026v1.
- [25] Hossein Movasati. On differential modular forms and some analytic relations between Eisenstein series. *Ramanujan J.*, 17(1):53–76, 2008. arXiv:math/0610861v1[math.NT].
- [26] Hossein Movasati. On elliptic modular foliations. *Indag. Math. (N.S.)*, 19(2):263–286, 2008. arXiv:0806.3926v1[math.AG].
- [27] Hossein Movasati. Modular foliations and periods of hypersurfaces. *Preprint*, 2009.
- [28] Rahul Pandharipande. Rational curves on hypersurfaces (after A. Givental). *Astérisque*, (252):Exp. No. 848, 5, 307–340, 1998. Séminaire Bourbaki. Vol. 1997/98.

- [29] Jan Saxl and Gary M. Seitz. Subgroups of algebraic groups containing regular unipotent elements. *J. London Math. Soc. (2)*, 55(2):370–386, 1997.
- [30] Marius van der Put and Michael F. Singer. *Galois theory of linear differential equations*, volume 328 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2003.
- [31] Christian van Enckevort and Duco van Straten. Monodromy calculations of fourth order equations of Calabi-Yau type. In *Mirror symmetry. V*, volume 38 of *AMS/IP Stud. Adv. Math.*, pages 539–559. Amer. Math. Soc., Providence, RI, 2006. arXiv:math/0412539v2[math.AG].
- [32] Andrew Wiles. Modular elliptic curves and Fermat’s last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.
- [33] Satoshi Yamaguchi and Shing-Tung Yau. Topological string partition functions as polynomials. *J. High Energy Phys.*, (7):047, 20 pp. (electronic), 2004, arXiv:hep-th/0406078v2.