Humbert surfaces and the moduli of lattice polarized K3 surfaces

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Abstract

In this article we introduce a collection of partial differential equations in the moduli of lattice polarized K3 surfaces whose algebraic solutions are the loci of K3 surfaces with lattice polarizations of higher rank. In the special case of rank 17 polarization such loci encode the well-known Humbert surfaces. The differential equations treated in the present article are directly derived from the Gauss-Manin connection of families of lattice polarized K3 surfaces. We also introduce some techniques to calculate the Gauss-Manin connection with the presence of isolated singularities.

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¶We thank Jacob Lewis for many interesting conversations.

1 Introduction

Elliptic curves have two different types of analogue in the realm of surfaces: abelian surfaces possess a group structure, whereas K3 surfaces have trivial canonical bundle. An elliptic curve has a unique nonvanishing holomorphic one-form (up to an overall scaling), and a K3 surface has a unique nonvanishing holomorphic two-form. Unlike elliptic curves or abelian surfaces, K3 surfaces are simply connected.

The second cohomology $H^2(X, \mathbb{Z})$ of a K3 surface X is equipped with a lattice structure by means of the cup product. This lattice is isomorphic to

$$H \oplus H \oplus H \oplus E_8 \oplus E_8.$$

Here, H is the standard hyperbolic lattice, and E_8 is the unique even, unimodular, and negative definite lattice of rank 8. Specifying the lattice isomorphism determines X.

By the Lefschetz $(1, 1)$ theorem, the elements of the lattice $\mathrm{NS}(X) := H^2(X, \mathbb{Z}) \cap H^{1,1}$ are Poincaré dual to the Néron-Severi group of divisors in X. For an algebraic K3 surface, the signature of $\mathrm{NS}(X)$ must be $(1, a)$ for some $a \leq 19$; the *Picard rank* of the surface is $1 + a$. Let L be an even non-degenerate lattice of signature $(1, 19 - m),\ m \geq 0$. A lattice polarization on the K3 surface X is given by a primitive embedding

$$i : L \hookrightarrow \mathrm{NS(X)}.$$

whose image contains a pseudo-ample class, that is, a numerically effective class with positive self-intersection. We denote an L-polarized K3 surface by (X, i), or simply by X if the choice of polarization is clear. One may define lattice-polarized abelian surfaces in a similar fashion.

Let M be the rank-18 lattice $H \oplus E_8 \oplus E_8$, and let N be the lattice $H \oplus E_8 \oplus E_7$. In [CD07] and [CD12], Clingher and the first author investigated a correspondence between abelian surfaces and K3 surfaces polarized by M or N. The correspondence is clear for Hodge-theoretic reasons, and can be realized in an explicit geometric fashion; M-polarized K3 surfaces are in one-to-one correspondence with products of elliptic curves, while N-polarized K3 surfaces correspond to arbitrary abelian surfaces.

We study families of M- and N-polarized K3 surfaces. We say that a family $\mathcal{X} \to T$ of surfaces over an affine variety T is polarized by a lattice L if its fibers X_t are smooth and equipped with an L-polarization which varies continuously with t. We further assume that for a general fiber the image of the polarization is precisely $\mathrm{NS}(X)$. The locus of points in T such that the lattice polarization i is not surjective is a (typically infinite) union of irreducible varieties of codimension one. In this article we characterize such a locus using partial differential equations in T. Our first example is the case $X = E_1 \times E_2$, where $E_i,\ i = 1, 2$ are two elliptic curves. In this case a third order differential equation in the j-invariants of $E_i,\ i = 1, 2$ is tangent to modular curves; for treatments of this problem using the corresponding K3 surfaces, see [CDLW09] and §3. In Theorem 2.3, we show that we may use similar methods to characterize the failure of surjectivity for N-polarized K3 surfaces.

The text is organized in the following way. In §2, we describe the relationship between a lattice polarization and the number of Picard-Fuchs equations making up the Gauss-Manin connection. As an example, we treat the classical case of a product of two elliptic curves in §3. In § 4, we recall the properties of the families of M- and N-polarized K3 surfaces studied in [CD07], [CDLW09], and [CD12]; the surfaces are realized as singular hypersurfaces in projective space. We review the Griffiths-Dwork method for computing Picard-Fuchs equations in § 5. The method fails to compute the full Gauss-Manin connection for the singular hypersurface realization of N-polarized K3 surfaces. However, we are able to extract Picard-Fuchs equations in different coordinate systems. In §6 we describe algorithms for calculating the full Gauss-Manin connection of the families of the M- and N-polarized K3 surface families. The techniques of this section are derived from [Mo11]. The main novelty is the calculation of the Gauss-Manin connection with the presence of isolated singularities. The result of the computations discussed in this section can be obtained from the third author's webpage.[Mo15]

2 Lattice polarizations and the Gauss-Manin connection

Let $\mathcal{X} \to T$ be a family of algebraic surfaces polarized by a lattice L of signature $(1, b - m - 3)$, where b is the second Betti number of the surfaces and T is an affine variety. Throughout the text, we work with the function ring R of T. For simplicity, we may take R to be any localization of the polynomial ring $\mathbb{Q}[a, b, c, \cdots]$. We denote by k the field of fractions of R; we may consider a family $\mathcal{X} \to T$ as a family of L-polarized surfaces over k. For an analytic space X and $x \in X$, (X, x) denotes a small neighborhood of x in X and $\mathcal{O}_{(X,x)}$ is the ring of germs of holomorphic functions in a neighborhood of x in X.

The relative algebraic de Rham cohomology $H^2_{\mathrm{dR}}(\mathcal{X}/T)$ (see [Gr66]) is a free R-module of finite rank. It carries the *Gauss-Manin connection* $\nabla : H^2_{\mathrm{dR}}(X) \to \Omega^1_T \otimes_{\mathsf{R}} H^2_{\mathrm{dR}}(\mathcal{X}/T)$, where Ω^1_T is the R-module of differential forms in R (cf. [KO68]). The elements in the image of the polarization $i : L \to H^2_{\mathrm{dR}}(\mathcal{X}/T)$ are constant sections of the connection, that is $\nabla(\alpha) = 0$ for any α in the image of i, so we have the induced connection

$$\nabla : H^2_{\mathrm{dR}}(\mathcal{X}/T)_i \to \Omega^1_T \otimes_{\mathsf{R}} H^2_{\mathrm{dR}}(\mathcal{X}/T)_i$$

which we denote again by ∇. Here $H^2_{\mathrm{dR}}(\mathcal{X}/T)_i = H^2_{\mathrm{dR}}(\mathcal{X}/T)/i(L)$. For any algebraic vector field v in T, we have $\nabla_v : H^2_{\mathrm{dR}}(X)_i \to H^2_{\mathrm{dR}}(X)_i$, so we can talk about the iteration $\nabla^i_v = \nabla_v \circ \nabla_v \circ \cdots \circ \nabla_v$, i-times.

In the topological context, the above algebraic connection can be viewed in the following way. The elements of $H^2_{\mathrm{dR}}(\mathcal{X}/T)_i$ are global sections of the cohomology bundle $\mathcal{H} := \bigcup_{t \in T} H^2(X_t, \mathbb{C})$, and the constant sections of ∇ are given by $\mathbb{C}$-linear combinations of sections with values in $\bigcup_{t \in T} H^2(X_t, \mathbb{Z})$. In this way, we can talk about ∇_v for any local analytic vector field v defined on some open set U in T. The connection ∇_v acts on holomorphic sections of $\mathcal{H}$ over U. We sometimes take a local holomorphic map $t : (\mathbb{C}^n, 0) \to T$ with $t_0 := t(0)$. In this case we denote by $(u_1, u_2, \ldots, u_n)$ a coordinate system in $(\mathbb{C}^n, 0)$ and by $\frac{\partial}{\partial u_i}$ the corresponding vector fields. The notation $\nabla_{\frac{\partial}{\partial u_i}}$ refers to the pull-back of the connection ∇ to $(\mathbb{C}^n, 0)$ and then its composition with the vector field $\frac{\partial}{\partial u_i}$. If the image of t is a subset of a subvariety S of T, then, by abuse of notation, we say that the local vector fields $\frac{\partial}{\partial u_i}$ are tangent to S around t_0.

We wish to use ∇ to characterize loci of T where the polarization $i : L \to \mathrm{NS}(X_t)$ is not surjective. Let ω be a meromorphic differential 2-form on $\mathcal{X}$ which restricted to $X_t,\ t \in T$, is holomorphic everywhere. The restriction gives us an element in $H^2_{\mathrm{dR}}(X)_i$ which we denote again by ω.

Theorem 2.1. *Let $S \subset T$ be an algebraic subset of codimension one in T. If for all $t_0 \in S$ the polarization $i : L \to \mathrm{NS}(X_{t_0})$ is not surjective, then for all $t_0 \in S$ and any local vector field $\frac{\partial}{\partial u}$ tangent to S in a neighborhood of t_0, the $\mathbb{C}$-vector space generated by*

$$\nabla^i_{\frac{\partial}{\partial u}}\omega,\ i = 0, 1, 2, \ldots,$$

in $H^2_{\mathrm{dR}}(X_{t_0})_i$ has dimension strictly less than $m + 2$.

Proof. Let Γ be the Poincaré dual of the image of L in $H^2_{\mathrm{dR}}(X_{t_0})$. Then Γ is a primitive sublattice of $H_2(X_{t_0}, \mathbb{Z})$. We define

$$H_2(X_{t_0}, \mathbb{Z})_i := H_2(X_{t_0}, \mathbb{Z})/\Gamma.$$

The hypothesis of the theorem implies that there is a continuous family of cycles $0 \neq \delta_s \in H_2(X_s, \mathbb{Z})_i,\ s \in (S, t_0)$ such that the integral $\int_{\delta_s} \omega$ is identically zero. If $\frac{\partial}{\partial u}$ is a local vector field tangent to (S, t_0) then we have

$$0 = \frac{\partial^i}{\partial u^i} \int_{\delta_{t(u)}} \omega = \int_{\delta_{t(u)}} \nabla^i_{\frac{\partial}{\partial u}}\omega,\ i = 0, 1, \ldots$$

Thus, the integration of all $\nabla^i_{\frac{\partial}{\partial u}}\omega,\ i = 0, 1, 2, \ldots$ over $\delta_{t(u)}$ is identically zero, so they cannot generate the whole $\mathbb{C}$-vector space $H^2_{\mathrm{dR}}(X_{t(u)})_i$. Note that $t(0) = t_0$. □

2.1 The differential rank-jump property

It is natural to ask whether the converse of Theorem 2.1 holds: can we use the dimension of the vector space $\nabla^i_{\frac{\partial}{\partial u}}\omega$ to detect whether a lattice polarization is surjective?

Definition 2.2. *Let $\mathcal{X} \to T$ be a family of algebraic surfaces polarized by a lattice L of signature $(1, b - m - 3)$, where b is the second Betti number of the surfaces. Suppose $\mathcal{X}$ has the property that if $S \subset T$ is an algebraic subset of codimension one such that for all $t_0 \in S$ and any local vector field $\frac{\partial}{\partial u}$ tangent to S in a neighborhood of t_0, the $\mathbb{C}$-vector space generated by*

$$\nabla^i_{\frac{\partial}{\partial u}}\omega,\ i = 0, 1, 2, \ldots,$$

in $H^2_{\mathrm{dR}}(X_{t_0})_i$ has dimension strictly less than $m+2$, then the polarization $i : L \to \mathrm{NS}(X_{t_0})$ is not surjective for all $t_0 \in S$. Then we say $\mathcal{X}$ has the differential rank-jump *property.*

Theorem 2.3. *Let $\mathcal{X} \to T$ be a family of rank 17 N-polarized K3 surfaces which has open image in the moduli space of N-polarized K3 surfaces. Then $\mathcal{X}$ has the differential rank-jump property.*

We prove Theorem 2.3 in §4.2. We now describe conditions that will guarantee a family $\mathcal{X} \to T$ has the differential rank-jump property. Let us take a codimension one irreducible subvariety S of T and assume that for any local vector field $\frac{\partial}{\partial u}$ tangent to (S, t_0), the $\mathbb{C}$-vector space generated by

$$\nabla^i_{\frac{\partial}{\partial u}} \omega, \ i = 0, 1, 2, \dots,$$

in $H^2_{\mathrm{dR}}(X_{t_0})_i$ has dimension strictly less than $m+2$. By the same argument as in the proof of Corollary 2.6, for any collection of local vector fields $\frac{\partial}{\partial u_i}, \ i = 1, 2, \cdots, n$ all tangent to (S, t_0), the $\mathbb{C}$-vector space generated by (2) in $H^2_{\mathrm{dR}}(X_{t_0})_i$ is also of dimension strictly less than $m + 2$.

Let $n = \dim_{\mathbb{C}} T$ and $(u_1, u_2, \dots, u_{n-1})$ be a coordinate system around a smooth point t of S. Let us choose a basis $\tilde{\Omega} := [\omega_1, \omega_2, \dots, \omega_r]^{\mathsf{t}}, \ r < m+2, \ \omega_1 = \omega$ for the $\mathcal{O}_{(S,t_0)}$-module generated by (2) and write the Gauss-Manin connection in this basis:

$$\nabla \tilde{\Omega} = A \otimes_{\mathcal{O}_{(S,t)}} \tilde{\Omega},$$

where A is a matrix with entries which are differential 1-forms in (S, t_0). A fundamental system of solutions for this system is given by the integration of $\tilde{\Omega}$ over n linearly independent continuous family of cycles with $\mathbb{C}$ coefficients. These are $\mathbb{C}$-linear combination of continuous family of cycles in $H_2(X_t, \mathbb{Z})_i$. It follows that the $\mathbb{C}$-vector space spanned by the periods $\int_{\delta_s} \omega, \ \delta_s \in H_2(X_s, \mathbb{Z})_i, \ s \in (S, t_0)$ has dimension strictly less than $m + 2$ and so there are constants $a_i \in \mathbb{C}, \ i = 1, \dots, m+2$ such that

$$\int_{\sum a_i \delta_{i,s}} \omega = \sum a_i \int_{\delta_{i,s}} \omega = 0, \quad s \in (S, t_0).$$

where $\delta_{i,s} \in H_2(X_s, \mathbb{Z})_i, \quad i = 1, 2, \dots, m+2, \ s \in (S, t_0)$ is a continuous family of cycles which form a basis for the $\mathbb{Z}$-module $H_2(X_s, \mathbb{Z})_i$. Let $\delta_s := \sum a_i \delta_{i,s}$.

Definition 2.4. Let $\mathcal{X} \to T$ be a proper smooth family of algebraic K3 surfaces or abelian surfaces. We say that the family is *perfect* if the following condition is satisfied: For a continuous family of cycles $\delta_s \in H_2(X_s, \mathbb{C}), \ s \in (T, t_0)$, if the locus of parameters s such that $\int_{\delta_s} \omega = 0$ is a part of a codimension one algebraic set, then δ_s up to multiplication by a constant is in $H_2(X_s, \mathbb{Z})$.

Perfect families have the differential rank-jump property.

2.2 The period domain and perfect families

Let $V_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module of rank $m+2$, and let $\psi_{\mathbb{Z}} : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$ be a non-degenerate (not necessarily unimodular) symmetric bilinear form on $V_{\mathbb{Z}}$. The *period domain* determined by $\psi_{\mathbb{Z}}$ is

$$D := \mathbb{P}(\{\omega \in V_{\mathbb{C}} \mid \psi_{\mathbb{C}}(\omega, \omega) = 0, \ \psi_{\mathbb{C}}(\omega, \bar{\omega}) > 0\})$$

The group $\Gamma_{\mathbb{Z}} := \mathrm{Aut}(V_{\mathbb{Z}}, \psi_{\mathbb{Z}})$ acts from the left on D. The quotient $\Gamma \backslash D$ is the moduli of polarized Hodge structures of type $1, m, 1$ with the polarization $\psi_{\mathbb{Z}}$.

Let (X, i) be an L-polarized surface. Because L is non-degenerate, $i(L) \oplus i(L)^{\perp}$ has finite index in $H^2(X, \mathbb{Z})$. The restriction of the cup product to $i(L)^{\perp}$ determines a non-degenerate lattice of signature $(2, m)$. Let $\psi_{\mathbb{Z}}$ be the associated bilinear form. The holomorphic 2-form $\omega \in H^2_{\mathrm{dR}}(X) = H^2(X, \mathbb{C})$ lies in $i(L)^{\perp} \otimes \mathbb{C}$. Therefore, we have a *period map*

$$p : \mathcal{M} \to \Gamma_{\mathbb{Z}} \backslash D$$

where $\mathcal{M}$ is the coarse moduli space of L-polarized surfaces. For K3 surfaces with pseudo-ample polarizations, the Torelli problem is true and p is a biholomorphism of analytic spaces (see [Do96]).

We need explicit affine coordinates on D. Let $\delta_1, \delta_2, \ldots, \delta_{m+2}$ be a basis of the $\mathbb{Z}$-module $V_{\mathbb{Z}}$, and let $\Psi_0 = [\psi_{\mathbb{Z}}(\delta_i, \delta_j)]$. For $\omega \in V_{\mathbb{C}}$, let

$$\omega = \sum_{i=1}^{m+2} x_i \delta_i,\ x_i \in \mathbb{C}.$$

We have

$$\psi_{\mathbb{C}}(\omega, \omega) = x\Psi_0 x^{\mathsf{t}},\ \psi_{\mathbb{C}}(\omega, \bar{\omega}) = x\Psi_0 \bar{x}^{\mathsf{t}} \text{ where } x = [x_1, x_2, \ldots, x_{m+2}],$$

and so

$$D = \{[x] \in \mathbb{P}^{m+1} \mid x\Psi_0 x^{\mathsf{t}} = 0,\ x\Psi_0 \bar{x}^{\mathsf{t}} > 0\ \}.$$

If we view $[x] \in \mathbb{P}^{m+2}$ as a $(m+2) \times 1$ matrix, then the group

$$\Gamma_{\mathbb{Z}} := \{A \in \mathrm{GL}(m+2, \mathbb{Z}) \mid A^{\mathsf{t}}\Psi_0 A = \Psi_0\}$$

acts on D from the left by matrix multiplication.

We say L-polarized surfaces have a *perfect moduli space* if the following condition is satisfied:

Condition 2.5. *Let $c = [c_1 : c_2 : \ldots : c_{m+2}] \in \mathbb{P}^{m+1}_{\mathbb{C}}$. The set $\{x \in D \mid \sum_{i=1}^{m+2} c_i x_i = 0\}$ induces an analytic subvariety in $\Gamma\backslash D$ if and only if $c \in \mathbb{P}^{m+1}_{\mathbb{Q}}$.*

Families with perfect moduli spaces satisfy the differential rank-drop property. We prove that N-polarized K3 surfaces of rank 17 have perfect moduli spaces in §4.3.

2.3 Rank jumps and partial differential equations

Let us choose a basis $\omega_i,\ i = 1, 2, \ldots, m+2,\ \omega_1 = \omega$ for the R-module $H^2_{\mathrm{dR}}(\mathcal{X}/T)_i$. The Gauss-Manin connection matrix A with entries in Ω^1_T is determined uniquely by the equality

$$\nabla(\Omega) = A \otimes_{\mathsf{R}} \Omega,\ \Omega := [\omega_1, \omega_2, \ldots, \omega_{m+2}]^{\mathsf{t}}.$$

(See § 5, § 6, and [Mo11] for techniques to compute A.) From now on we assume that T is of dimension m. Let us consider a local holomorphic map $t : (\mathbb{C}^{m-1}, 0) \to (T, t_0)$. We denote by $(u_1, u_2, u_3, \ldots, u_{m-1})$ the coordinate system in $(\mathbb{C}^{m-1}, 0)$ and by $\frac{\partial}{\partial u_i},\ i = 1, 2, \ldots, m-1$ the corresponding local vector fields. For simplicity we write

$$\alpha_{u_i} := \nabla_{\frac{\partial}{\partial u_i}} \alpha,\ \alpha \in H^2_{\mathrm{dR}}(X/T)_i,$$

$$f_{u_i} := \frac{\partial f}{\partial u_i},\ f \in \mathsf{k}.$$

For any word x, y in the u_i, let $R^{x,y}$ be the $(m+2) \times (m+2)$ matrix satisfying

$$\begin{pmatrix} \omega \\ \omega_{u_1} \\ \vdots \\ \omega_{u_{m-1}} \\ \omega_y \\ \omega_x \end{pmatrix} = R^{x,y} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{m+2} \end{pmatrix}.$$

Let us analyze the matrix $R^{x,y}$ in more detail. We take a collection $x_i,\ i=1,2,\ldots,n$, of n regular functions on T forming a quasi-affine coordinate system on T, that is, the map $(x_1,x_2,\cdots,x_n):T\to\mathbb{C}^n$ is an embedding of T as a quasi-affine subvariety of $\mathbb{C}^n$. Any regular function on T can be written as a polynomial in x and so the R-module Ω^1_T is generated by $dx_i,\ i=1,2,3,\ldots,n$. This implies that the Gauss-Manin connection matrix A can be written as

$$A=\sum_{i=1}^m A_i dx_i,\ \ A_i\in \mathrm{Mat}(m+2,\mathsf{R}).$$

Since the Gauss-Manin connection is integrable we have $dA=A\wedge A$ and so

$$\frac{\partial A_i}{\partial x_j}+A_iA_j=\frac{\partial A_j}{\partial x_i}+A_jA_i.$$

Let

$$\begin{aligned}
A^{u_1} &:= \sum_{i=1}^n A_i x_{i,u_1}\\
A^{u_1u_2} &:= A^{u_1}A^{u_2}+(A^{u_1})_{u_2}=(\sum_{i=1}^n A_ix_{i,u_1})(\sum_{i=1}^n A_ix_{i,u_2})+\sum_{i=1}^n (A_i)_{u_2}x_{i,u_1}+\sum_{i=1}^n A_ix_{i,u_1u_2}\\
&= \sum_{i=1}^n\sum_{j=1}^n (A_iA_j+\frac{\partial A_i}{\partial x_j})x_{i,u_1}x_{j,u_2}+\sum_{i=1}^n A_ix_{i,u_1u_2}\\
A^{u_1u_2u_3} &:= A^{u_1u_2}A^{u_3}+(A^{u_1u_2})_{u_3}=\cdots
\end{aligned}$$

and so on. These are uniquely determined by the equalities $\Omega_{u_1}=A^{u_1}\Omega,\ \Omega_{u_1u_2}=A^{u_1u_2}\Omega$ and $\Omega_{u_1u_2u_3}=A^{u_1u_2u_3}\Omega$. Now, the first row of $R^{x,y}$ is just $[1,0,0,\ldots,0]$ and the i-th row, $2\le i\le m$, of $R^{x,y}$ is the first row of the matrix A^{u_i}. The $(m+1)$-th and $(m+2)$-th row of $R^{x,y}$ are respectively the first row of A^y and A^x.

Corollary 2.6. *Let $\mathcal{X}\to T$ be a family of L-polarized algebraic surfaces with the differential rank-jump property, and let $S\subset T$ be an algebraic subset of codimension one in T. Then for all $t_0\in S$ the polarization $i:L\to \mathrm{NS}(X_{t_0})$ is not surjective if and only if for x and $t:(\mathbb{C}^{m-1},0)\to(S,t_0)$ as above we have the following collection of partial differential equations:*

$$\det(R^{x,y})=0,\ \forall x \textit{ of length 2, } y \textit{ of length 2 or 3} \tag{1}$$

For the proof of the reverse direction of the above corollary, we need the following lemma.

Lemma 2.7. *Let $t:(\mathbb{C}^m,0)\to(T,t_0)$ be a coordinates system around a point $t_0\in T$. The differential forms $\omega,\omega_{u_i},\ i=1,2,\ldots,m$ are linearly independent for a generic point in the image of t.*

Proof. If the assertion is not true, then we have a meromorphic vector field V in $(\mathbb{C}^m,0)$ such that $\nabla_V\omega=\omega$. Let $\gamma(s)\in(\mathbb{C}^m,0),\ s\in(\mathbb{C},0)$ be a solution of V. Integrating $\nabla_V\omega=\omega$ over topological two-cycles of $X_{\gamma(s)}$, we conclude that the periods $\int_{\delta_{\gamma(s)}}\omega$ are all of the form $c_\delta e^s$, where c_δ is a constant depending only on δ. We conclude that $\frac{\omega}{e^s}$ has constant periods which is in contradiction with the local Torelli problem for K3 surfaces and the fact that t is a coordinates system. □

Proof of Corollary 2.6. Let us first prove the forward direction. Note that if $\frac{\partial}{\partial u_i}$, $i = 1, 2, \ldots, k$, is a collection of local vector fields tangent to S then the $\mathbb{C}$-vector space generated by

$$\omega_x, \quad \text{where } x \text{ is any word in } u_i \ i = 1, 2, \ldots, k \tag{2}$$

in $H^2_{\mathrm{dR}}(X_{t(u)})_i$ is also of dimension strictly less than $m+2$. This follows by taking the vector field $\sum_{i=1}^k a_i \frac{\partial}{\partial u_i}$, where a_i's are unknown constants, and applying Theorem 2.1. Take $k = m-1$. We conclude that $\omega, \omega_{u_1}, \cdots, \omega_{u_{m-1}}, \omega_y, \omega_x$ are linearly independent in $H^2_{\mathrm{dR}}(X_{t(u)})_i$, whereas $\omega_1, \omega_2, \ldots, \omega_{m+2}$ form a basis for $H^2_{\mathrm{dR}}(X_{t(u)})_i$. It follows that the determinant of the matrix $R^{x,y}$ is identically zero.

Now, let us prove the reverse direction. Let V be the vector space generated by $\omega, \omega_i, \ i = 1, 2, \ldots, m-1$. By Lemma 2.7 V is of dimension m. If all the second derivatives $\omega_{u_i u_j}$ are in V then by further derivations of the corresponding equalities we conclude that V is closed under all derivations and so for all $t_0 \in S$ and any local vector field $\frac{\partial}{\partial u}$ tangent to S in a neighborhood of t_0, the $\mathbb{C}$-vector space generated by $\nabla^i_{\frac{\partial}{\partial u}} \omega, \ i = 0, 1, 2, \ldots$, in $H^2_{\mathrm{dR}}(X_{t_0})_i$ has dimension strictly less than $m+2$. Using the converse of Theorem 2.1, the proof is finished. In a similar way, if at least one of the second derivatives, say $\omega_{u_1 u_1}$ is not in V, then by our hypothesis (3), all other second derivatives and third derivatives are in the vector space generated by V and $\omega_{u_1 u_1}$. Taking further derivatives of the corresponding equalities, we obtain the hypothesis of the converse of Theorem 2.1. $\square$

2.4 A remark on the number of partial differential equations in Corollary 2.6

Assume that $\mathcal{X} \to T$ has the differential rank-jump property and $\omega_{u_1 u_1}$ is linearly independent with $\omega, \omega_{u_i}, \ i = 1, 2, \ldots, m-1$. Then in Corollary 2.6 we can reduce the number of partial differential equations to

$$\det(R^{x, u_1 u_1}) = 0, \ x = u_1 u_i, 2 \le i \le m-1, \ u_i u_j, \ 2 \le i \le j \le m-1, \tag{3}$$

$$\det(R^{x, u_1 u_1}) = 0, \ x = u_1 u_1 u_i, \ 1 \le i \le m-1. \tag{4}$$

Note that we have in total $\frac{(m-1)(m+2)}{2} - 1$ partial differential equations. Of these equations, $m-1$ are third-order and the rest are second order. Let us take a coordinate system $(u_1, u_2, \ldots, u_{m-1})$ around t_0 for S such that $u_1 = u$. From the partial differential equations (3) and (4) it follows that the $\mathbb{C}$-vector space generated by $\omega, \omega_{u_1}, \cdots, \omega_{u_{m-1}}, \omega_{u_1 u_1}, \omega_x$ has dimension less than $m+2$; by our hypothesis, the first $m+1$ differential forms are linear independent. We conclude that ω_x is in the vector space V generated by $\omega, \omega_{u_1}, \cdots, \omega_{u_{m-1}}, \omega_{u_1 u_1}$. Therefore, the vector space V is closed under all derivations $\frac{\partial}{\partial u_i}$. In particular, the $\mathbb{C}$-vector space generated by $\nabla^i_{\frac{\partial}{\partial u}} \omega, \ i = 0, 1, 2, \ldots$, in $H^2_{\mathrm{dR}}(X_{t_0})_i$ has dimension strictly less than $m+2$.

3 Product of two elliptic curves

As an example, we consider the Gauss-Manin connection of the family of abelian surfaces which are products of elliptic curves. Let $X = E_1 \times E_2$ be a product of two elliptic curves and let the polarization be given by the divisor $E_1 \times \{p_2\} + \{p_1\} \times E_2$. In this case $m = 2$

and we consider each E_k parametrized by the classical j-invariant $j_k \in \mathbb{P}^1$. In Weierstrass coordinates we have:

$$E_k\colon y^2 + xy - x^3 + \frac{36}{j_k - 1728}x + \frac{1}{j_k - 1728} = 0,\ j_k \neq 0, 1728,\ k = 1, 2$$

We consider $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ as the compactification of the moduli of elliptic curves with a cusp ∞. Therefore, $\mathbb{P}^1 \times \mathbb{P}^1$ is the compactification of the moduli of pairs of elliptic curves.

3.1 Gauss-Manin connection

Let f be the defining polynomial of $E = E_k,\ k = 1, 2$. We calculate the Gauss-Manin connection of E in the basis $[\alpha_k, \omega_k]^{\mathsf{t}} = [\frac{dx\wedge dy}{df},\ \frac{dx\wedge dy}{df}]^{\mathsf{t}}$

$$\nabla_{\frac{\partial}{\partial j}} \begin{pmatrix} \alpha_k \\ \omega_k \end{pmatrix} = A \cdot \begin{pmatrix} \alpha_k \\ \omega_k \end{pmatrix}$$

where

$$A = \frac{1}{j(j-1728)} \begin{pmatrix} -432 & -60 \\ -(j-1728) & 432 \end{pmatrix}.$$

Let $A_k,\ k = 1, 2$ be two copies of A corresponding to $j_k,\ k = 1, 2$. We have

$$H^2_{\mathrm{dR}}(E_1 \times E_2)_i = H^1_{\mathrm{dR}}(E_1) \otimes_{\mathbb{C}} H^1_{\mathrm{dR}}(E_2)$$

for which we choose the basis $[\tilde{\omega}_k]_{k=1,2,3,4} := [\alpha_1 \otimes, \alpha_2, \alpha_1 \otimes \omega_2, \omega_1 \otimes \alpha_2, \omega_1 \otimes \omega_2]^{\mathsf{t}}$. In this basis the Gauss-Manin connection matrix is given by:

$$A = \begin{pmatrix} (A_1)_{11} & 0 & (A)_{12} & 0 \\ 0 & (A_1)_{11} & 0 & (A_1)_{12} \\ (A_1)_{21} & 0 & (A_1)_{22} & 0 \\ 0 & (A_1)_{21} & 0 & (A_1)_{22} \end{pmatrix} dj_1 + \begin{pmatrix} (A_2)_{11} & (A_2)_{12} & 0 & 0 \\ (A_2)_{21} & (A_2)_{22} & 0 & 0 \\ 0 & 0 & (A_2)_{21} & (A_2)_{22} \\ 0 & 0 & (A_2)_{21} & (A_2)_{22} \end{pmatrix} dj_2$$

3.2 The box equation

After simplifying the differential equation $\det(R^{uuu}) = 0$, where

$$\begin{pmatrix} \omega \\ \omega_u \\ \omega_{uu} \\ \omega_{uuu} \end{pmatrix} = R^{uuu,uu} \begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \tilde{\omega}_3 \\ \tilde{\omega}_4 \end{pmatrix}$$

and $\omega = \tilde{\omega}_1 = \alpha_1 \otimes \alpha_2$, we obtain the *box equation*

$$\square(j_1) = \square(j_2) \tag{5}$$

where

$$\square(j(u)) = j'(u)^2 \frac{36j(u)^2 - 41j(u) + 32}{144(j(u)-1)^2 j(u)^2} + \frac{1}{2}\{j(u), u\}$$

and

$$\{j(u), u\} = \frac{2j'(u)j'''(u) - 3j''(u)^2}{2j'(u)^2}$$

is the Schwarzian derivative.

The Lefschetz $(1,1)$-theorem implies that the subloci of $\mathbb{P}^1 \times \mathbb{P}^1$ where the polarization is not surjective are given by pairs of isogenous elliptic curves. Let $X_0(d) \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the modular curve of isogenous elliptic curves $f : E_1 \to E_2$ with $\deg(f) = d$. The stronger version of Corollary 2.6 in this case is

Proposition 3.1. *Let S be an algebraic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ and let $(j_1(u), j_2(u)) \in S$ be a local holomorphic parametrization of S. Then*

$$\Box(j_1) = \Box(j_2)$$

if and only if S is a modular curve $X_0(d)$ of degree d isogenous elliptic curves, for some $d \in \mathbb{N}$.

See § 5 and [CDLW09] for a discussion of the corresponding special loci in the moduli of M-polarized K3 surfaces.

Proof. We know that $\det(R^{uu,uuu}) = 0$ is equivalent to $\Box(j_1) = \Box(j_2)$. In turn, this is equivalent to saying that the Picard-Fuchs equation of ω with respect to the parameter u is of order 3. In this case we have $H_2(E_1 \times E_2, \mathbb{Z})_i = H_1(E_1, \mathbb{Z}) \otimes H_1(E_2, \mathbb{Z})$ and so there are constants $a_{ij} \in \mathbb{C},\ i, j = 1, 2$ such that

$$\delta = \sum a_{ij} \delta_{1,i} \otimes \delta_{2,i},\ a_{ij} \in \mathbb{C}$$

and

$$\int_{\sum a_{ij}\delta_{1,i}\otimes\delta_{2,j}} \alpha_1 \otimes \alpha_2 = \sum a_{ij} \int_{\delta_{1,i}} \alpha_1 \int_{\delta_{2,j}} \alpha_2 = 0.$$

where $\{\delta_{i,1}, \delta_{i,2}\},\quad i = 1, 2$ is a basis of $H_1(E_i, \mathbb{Z})$ with $\langle \delta_{i,1}, \delta_{i,2} \rangle = 1$. Therefore, we need only prove that the universal family of pairs of two elliptic curves is perfect in the sense of Definition 2.4.

Let $\tau_k = \frac{\int_{\delta_2} \omega_k}{\int_{\delta_1} \omega_k},\ k = 1, 2$. The above equality becomes

$$\tau_2 = A(\tau_1),\ A \in \mathrm{GL}(2, \mathbb{C}), \tag{6}$$

where $A(\tau_1)$ is the Möbius transformation of τ_1. Now, let assume that the locus described by Equation 6 is algebraic in $\mathbb{P}^1 \times \mathbb{P}^1$. From this we only use the following: For any fixed $j_1 \in \mathbb{P}^1$ there are a finitely many $j_2 \in \mathbb{P}^1$ such that $(j_1, j_2) \in X$. This property using periods is:

$$\#\{\ \mathrm{SL}(2, \mathbb{Z})AB \ \mid\ B \in \mathrm{SL}(2, \mathbb{Z})\} < \infty.$$

This implies that A up to multiplication by a constant has rational coefficients.

□

Remark 3.2. Let C be a curve such that in the decomposition of its Jacobian into simple abelian varieties, there appear two elliptic curves $E_k,\ k = 1, 2$. We have the canonical map $C \to E_1 \times E_2$. It follows from the above arguments that if E_1 and E_2 are not isogenous, then the homology class $[C] \in H_2(E_1 \times E_2, \mathbb{Z})$ of the image of C satisfies $[C] = a_1[E_1] + a_2[E_2]$. That is, no contribution comes from $H_1(E_1, \mathbb{Z}) \otimes H_1(E_2, \mathbb{Z})$.

Remark 3.3. Let P be a reduced polynomial in j_1, j_2. Suppose $P = 0$ is tangent to Equation 5. This property can be written in a purely algebraic way. Consider a solution of the Hamiltonian differential equation

$$\begin{cases} j_1' = \frac{\partial P}{\partial j_2} \\ j_2' = -\frac{\partial P}{\partial j_1} \end{cases} \tag{7}$$

passing through a point of $P = 0$. The solution is entirely contained in $P = 0$. Now, further derivations of j_i's are polynomials in j_1, j_2 and we can substitute all these in $\Box(j_1) = \Box(j_2)$. We obtain an operator $\hat{\Box}$ from $\mathbb{C}[j_1, j_2]$ to itself. $P = 0$ is tangent to the Box equation if and only if

$$P \text{ divides } \hat{\Box}(P).$$

Remark 3.4. All the curves $X_0(d)$ cross $(\infty, \infty) \in \mathbb{P}^1 \times \mathbb{P}^1$, and they are uniquely determined by the box equation. One may conjecture that this data is enough to calculate their equations. Around (∞, ∞) we have the local coordinates (q_1, q_2), where

$$q_i = e^{2\pi i \tau_i}.$$

The curve $X_0(d)$ near (∞, ∞) is reducible and its irreducible components are given by

$$q_1^{d_1} = q_2^{d_2}, \ d_1 d_2 = d.$$

One may calculate an explicit equation of $X_0(d)$ using the q-expansion of the j-function: $j(q) = \frac{1}{q} + 744 + \cdots$. The equation of $X_0(d)$ is a polynomial $P_d(j_1, j_2)$ in two variables such $P_d(j(q), j(q^d)) = 0$.

3.3 The box equation and the Ramanujan differential equation

One may also derive Equation 5 using the following Ramanujan ordinary differential equation:

$$\mathrm{R} : \begin{cases} \dot{t}_1 = t_1^2 - \frac{1}{12} t_2 \\ \dot{t}_2 = 4t_1 t_2 - 6t_3 \\ \dot{t}_3 = 6t_1 t_3 - \frac{1}{3} t_2^2 \end{cases} \quad \dot{t}_k = \frac{\partial t_k}{\partial \tau} \tag{8}$$

which is satisfied by the Eisenstein series:

$$t_k = a_k E_k(q) := a_k \left(1 + b_k \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2k-1} \right) q^n \right), \quad k = 1, 2, 3, \ q = e^{2\pi i \tau} \tag{9}$$

and

$$(b_1, b_2, b_3) = (-24, 240, -504), \quad (a_1, a_2, a_3) = (1, 12, 8).$$

Let

$$j = \frac{t_2^3}{t_2^3 - 27t_3^2}$$

be the j-function. From (8) we can calculate j', j'', j''' as rational functions in t_1, t_2, t_3. Thus, there is a polynomial in four variables which annihilate (j, j', j'', j'''). After calculating this we obtain the Schwarzian differential equation:

$$\Box(j(\tau)) = S(j) + Q(j)(j')^2 = 0,$$

where $Q(j) = \frac{36j^2-41j+32}{72(j-1)^2j^2}$ and $S(j)$ is the Schwarzian derivative of j with respect to τ. The Schwarzian derivative satisfies the properties

$$S(f \circ g) = (S(f) \circ g) \cdot (g')^2 + S(g).$$

$$S(\frac{a\tau + b}{c\tau + d}) = 0.$$

Therefore if g is a Möbius transformation then $S(f \circ g) = (S(f) \circ g) \cdot (g')^2$ and if f is a Möbius transformation then $S(f \circ g) = S(g)$.

Now, if τ is a function of another parameter t then

$$\square(j \circ \tau) = S(\tau)$$

and so for $A \in \mathrm{SL}(2, \mathbb{C})$:

$$\square(j \circ A \circ \tau) = S(A \circ \tau) = S(\tau)$$

4 Lattice-polarized K3 surfaces

In this section we consider three families of lattice-polarized K3 surfaces described by singular hypersurfaces in projective space $\mathbb{P}^3$.

4.1 M-polarized K3 surfaces

In [CD07] and [CDLW09], the authors studied the surfaces described by the following family of polynomials in $\mathbb{P}^3$:

$$q_{a,b,d} = y^2zw - 4x^3z + 3axzw^2 + bzw^3 - \frac{1}{2}(dz^2w^2 + w^4) \quad d \neq 0 \tag{10}$$

After resolving singularities, we obtain a family of M-polarized K3 surfaces $X(a, b, d)$, where M is the rank 18 lattice $H \oplus E_8 \oplus E_8$. In fact, two such $K3$-surfaces are isomorphic if and only if the corresponding parameters are in the same orbit of the $\mathbb{C}^*$-action:

$$k, (a, b, d) \mapsto (k^2a, k^3b, k^6d), \quad k \in \mathbb{C}^*.$$

The coarse moduli space of such $K3$ surfaces is the subset of the weighted projective space $\mathbb{P}^{(2,3,6)}$ where $d \neq 0$. In $\mathbb{P}^{(2,3,6)}$ the loci of parameters such that the polarization $\mathrm{M} \to \mathrm{NS}(X_t)$ is not surjective is given by the curves C_n, $n \in \mathbb{N}$, which parametrize $K3$-surfaces with polarization $M_d := H \oplus E_8 \oplus E_8 \oplus \langle -2d \rangle$. There is a Hodge-theoretic correspondence between pairs of elliptic curves and M-polarized surfaces. Under this correspondence, C_d corresponds to the modular curve $X_0(d)$ (see § 5 and [CDLW09]).

4.2 N-polarized family of $K3$-surfaces

The next step is the N-polarized family of $K3$-surfaces $X(a, b, c, d)$, where $\mathrm{N} = H \oplus E_8 \oplus E_7$, which is studied in [CD12]. These surfaces are realized as the resolution of singularities of the hypersurfaces in $\mathbb{P}^3$ described by the following polynomials:

$$q_{a,b,c,d} = y^2zw - 4x^3z + 3axzw^2 + bzw^3 + cxz^2w - \frac{1}{2}(dz^2w^2 + w^4). \tag{11}$$

$$c \neq 0 \text{ or } d \neq 0.$$

Two N-polarized $K3$-surfaces are isomorphic if and only if the corresponding parameters are in the same orbit of the $\mathbb{C}^*$-action

$$k,(a,b,c,d) \mapsto (k^2 a, k^3 b, k^5 c, k^6 d),\ k \in \mathbb{C}^*.$$

The space $\mathbb{P}^{(2,3,5,6)} \backslash \{c=d=0\}$ is the coarse moduli space of N-polarized K3 surfaces (see [CD12]).

Let D and $\Gamma_{\mathbb{Z}}$ be as in (2.2) associated to N-polarized K3 surfaces and $\mathbb{H}_2$ be the Siegel upper half plane of genus 2:

$$\mathbb{H}_2 = \left\{ z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \mid \mathrm{Im}(z_1)\mathrm{Im}(z_3) > \mathrm{Im}(z_2)^2,\ \ \mathrm{Im}(z_1) > 0 \right\} \tag{12}$$

The rank-five lattice $V_{\mathbb{Z}}$ is naturally isomorphic to the orthogonal direct sum $\mathrm{H} \oplus \mathrm{H} \oplus (2)$. We select an integral basis for $V_{\mathbb{Z}}$ such that the intersection form $\psi_{\mathbb{Z}}$ in this basis is

$$\Psi_0 := [\psi_{\mathbb{Z}}(\delta_i, \delta_j)] = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have an isomorphism of groups

$$\wedge^2 : \mathrm{Sp}(4,\mathbb{Z})/\pm \mathrm{id} \to \Gamma_{\mathbb{Z}}/\pm \mathrm{id},$$

The image of generators of $\mathrm{Sp}(4,\mathbb{Z})$ under this isomorphism is

$$\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \mapsto S := \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} I_2 & \tilde{B} \\ 0 & I_2 \end{pmatrix} \mapsto B := \begin{pmatrix} 1 & -b_1 & 2b_2 & -b_3 & b_2^2 - b_1 b_3 \\ 0 & 1 & 0 & 0 & b_3 \\ 0 & 0 & 1 & 0 & b_2 \\ 0 & 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ where } \tilde{B} = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}. \tag{13}$$

and

$$\begin{pmatrix} \tilde{U}^{-\mathsf{t}} & 0 \\ 0 & \tilde{U} \end{pmatrix} \mapsto U := \det(\tilde{U}) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a^2 & -2ab & b^2 & 0 \\ 0 & -ac & ad+bc & -bd & 0 \\ 0 & c^2 & -2cd & d^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\tilde{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have a biholomorphism

$$\mathbb{H}_2 \to D,\ \ \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \mapsto x = [z_2^2 - z_1 z_3; z_3; z_2; z_1; 1].$$

and so we get the isomorphism

$$\mathrm{Sp}(4,\mathbb{Z})\backslash\mathbb{H}_2 \cong \Gamma_{\mathbb{Z}}\backslash D \tag{14}$$

between the period domain of principally polarized abelian surfaces and the period domain of K3 surfaces as above. For all these see [GrNi]. Since in both sides the period map is a biholomorphism, this determine a bijection between the corresponding coarse moduli spaces. In the left hand side of (16) we have Humbert surfaces which are given by

$$c_1(z_2^2 - z_1z_2) + c_2z_1 + c_3z_2 + c_4z_3 + c_5 = 0, c_i \in \mathbb{Z} \tag{15}$$

The Humbert surfaces parametrize abelian surfaces where the endomorphism ring $\mathrm{End}(A)$ is isomorphic to an order in a real quadratic field. Under the correspondence described in Equation 16, Humbert surfaces parametrize K3 surfaces with a polarization $N \to \mathrm{NS}(X)$of rank 18 and with $H \oplus E_8 \oplus E_7 \subset N$. In this case showing Condition 2.5 holds is equivalent to showing that the hypersurface given by Equation 15 with $c_i \in \mathbb{C}$ induces a hypersurface in $\mathrm{Sp}(4,\mathbb{Z})\backslash\mathbb{H}_2$ if and only if, up to multiplication by a constant, $c_i \in \mathbb{Z}$.

Let $\Gamma_{\mathbb{Z}}\backslash D$ be the period domain associated with N-polarized K3 surfaces, as described in § 2.2, and let $\mathbb{H}_2$ be the Siegel upper half plane of genus 2. We have a bijection

$$\mathrm{Sp}(4,\mathbb{Z})\backslash\mathbb{H}_2 \to \Gamma_{\mathbb{Z}}\backslash D \tag{16}$$

$$\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \mapsto x = [z_1; z_2; z_3; z_2^2 - z_1z_3; 1].$$

between the period domain of principally polarized abelian surfaces and the period domain of N-polarized K3 surfaces. Since on both sides the period map is a biholomorphism, this determines a bijection between the coarse moduli space of principally polarized abelian surfaces and the coarse moduli space of N-polarized K3 surfaces.

4.3 The differential rank-jump property for N-polarized K3 surfaces

To show that N-polarized K3 surfaces have the differential rank-jump property, it is enough to prove that the moduli of N-polarized K3 surfaces of rank 17 have a perfect moduli space, as characterized in Condition 2.5. Let us assume that $cx = 0$ induces an analytic subvariety of $\Gamma_{\mathbb{Z}}\backslash D$. Using the isomorphism (16) we consider the Satake compactification

$$\overline{\Gamma_{\mathbb{Z}}\backslash D} \cong \overline{\mathrm{Sp}(4,\mathbb{Z})\backslash\mathbb{H}_2} = (\mathrm{Sp}(4,\mathbb{Z})\backslash\mathbb{H}_2) \cup D_\infty, \qquad D_\infty := (\mathrm{SL}(2,\mathbb{Z}))\backslash\mathbb{H}) \cup \{\infty\}$$

see for instance [Fr]. Using this topology, if we set $z_2 = 0$ and let either $\mathrm{Im}(z_1)$ or $\mathrm{Im}(z_2)$ go to $+\infty$ then the point converges to a point in D_∞ inside $\overline{\mathrm{Sp}(4,\mathbb{Z})\backslash\mathbb{H}_2}$. By our hypothesis, for all $A \in \Gamma_{\mathbb{Z}}$ the set $cAx = 0$ induces an analytic subvariety of $\mathrm{Sp}(4,\mathbb{Z})\backslash\mathbb{H}_2$. Since the codimension of D_∞ inside $\overline{\mathrm{Sp}(4,\mathbb{Z})\backslash\mathbb{H}_2}$ is bigger than one we conclude that it induces an analytic suvariety H_A in the compactification $\overline{\Gamma_{\mathbb{Z}}\backslash D}$ (Hartog's extension theorem, [Gu]). Now, we set $z_2 = 0$ and send $\mathrm{Im}(z_1)$ to $+\infty$. We conclude that

$$H_A \cap D_\infty = \{\frac{(cA)_2}{(cA)_1} \mid A \in \Gamma_{\mathbb{Z}}\}$$

where for a vector v, v_i is its i-the coordinate. Now this set has no accumulation points in $\mathbb{H}$ and intersects $\mathbb{R}$ only in rational numbers. The same set with matrices $A = C(B^n)D$, $n \in \mathbb{N}$, where $C, D \in \Gamma_{\mathbb{Z}}$ are arbitrary elements and B is given by (13), has no accumulation point in $\mathbb{H}$. We assume that $b_2^2 = b_1 b_2$, let n go to infinity and compute the accumulation set and conclude that

$$(17) \qquad \{\frac{(cA_2}{(cA_1} \mid \quad A = CB_\infty D, \quad C, D \in \Gamma_\mathbb{Z}\} \subset \mathbb{Q}$$

where

$$B_\infty := \begin{pmatrix} 0 & -b_1 & 2b_2 & -b_3 & 0 \\ 0 & 0 & 0 & 0 & b_3 \\ 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 & b_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

For a generic choice of seven matrices A as in the set (17), the $\mathbb{Q}$-vector space generated by seven vectors $A_2 - rA_1$, where $r := \frac{cA_2}{cA_1}$ is of dimension 4, and so the vector c which is orthogonal to it, has rational coordinates (up to multiplication by a constant). For instance take B_∞ with $b_1 = b_2 = b_3 = 1$, a B matrix with $b_1 = b_2 = 1, b_3 = 0$ and a U matrix with $a = d = 0,\ b = 1,\ c = -1$. The seven matrix

$$A_i A_{i-1} \cdots A_1 B_\infty S, \quad i = 1, 2, \ldots, 7$$

satisfy the desired conditions, where the sequence $A_1, \cdots, A_7$ is given by B, U, S, B, S, B, U. For the computer code of this computations see [Mo15].

4.4 A lattice-polarized family of K3 surfaces of rank 12

In this section we consider a nine-parameter family of $K3$ surfaces

$$X(t_i, i = 4, 6, 7, 10, 12, 15, 16, 18, 24)$$

polarized by a lattice of rank 12. The family is obtained by resolving the singularities of the hypersurfaces described by the following polynomials in $\mathbb{P}^3$:

$$(18) \qquad q_{t_i} = -4x^3 + y^2 w - \frac{1}{2} w^4 + t_4 z x w^2 + \\ t_6 z^2 y^2 + t_7 z^2 y x + t_{10} z^2 x w + t_{12} z^2 w^2 + t_{15} z^3 y + t_{16} z^3 x + t_{18} z^3 w + t_{24} z^4$$

The parameters t_i satisfy the condition that two such K3 surfaces are isomorphic if and only if the corresponding parameters are in the same orbit of the $\mathbb{C}^*$-action

$$k, (t_i,\ i \in I) \mapsto (k^{\frac{i}{2}} t_i,\ i \in I), \quad I := \{4, 6, 7, 10, 12, 15, 16, 18, 24\}.$$

See also §6.

For a generic choice of the parameters, the surface given by $q_{t_i} = 0$ has a unique singularity

$$P_1 = [0 : 1 : 0 : 0]$$

which is a rational double point singularity of type A_5. Let E be the intersection of X with $w = 0$:

$$-4x^3 + t_6 z^2 y^2 + t_7 z^2 y x + t_{15} z^3 y + t_{16} z^3 x + t_{24} z^4 = 0.$$

This is an elliptic curve. After a blow-up in $\mathbb{P}_1$ we obtain the following configuration of lines and curves in X:

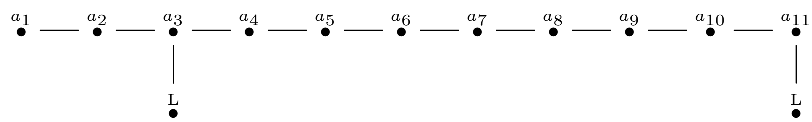

Figure 1: A rank 12 lattice

5 The Griffiths-Dwork technique

5.1 The Griffiths-Dwork technique for smooth hypersurfaces

We want to compute the Picard-Fuchs equations satisfied by the period $\int \omega$ of the holomorphic $(2,0)$-form on our K3 surfaces. Let us recall the Griffiths-Dwork method for a family of smooth hypersurfaces $Y(t)$ in $\mathbb{P}^m$ given by a family of polynomials $q(t)$. (For a more detailed exposition, see [CK99] or [DGJ08].)

First, we observe that we may write forms in $H^{m-1}(Y(t))$ as residues of meromorphic forms in $H^m(\mathbb{P}^m - Y(t))$:

$$\operatorname{Res}\left(\frac{p\Omega}{q^k}\right) \in H^{m-k,k-1}(Y).$$

Here Ω is the usual holomorphic form on $\mathbb{P}^m$ and p is a homogeneous polynomial satisfying

$$\deg p = k \deg q - (m+1).$$

In particular, $\omega = \operatorname{Res}\left(\frac{\Omega}{q}\right)$. The image of the residue map is the *primitive cohomology*, which consists of the classes in $H^{m-1}(Y(t))$ orthogonal to the hyperplane class.

Next, we differentiate the period with respect to our parameter. Note that we may move the derivative under the integral sign. For any class $\alpha = \operatorname{Res}\left(\frac{p\Omega}{q^k}\right)$, we have

$$\begin{aligned} \frac{d}{dt}\int \alpha &= \frac{d}{dt}\int \operatorname{Res}\left(\frac{p\Omega}{q^k}\right) \\ &= \int \operatorname{Res}\left(\Omega \frac{d}{dt}\frac{p}{q^k}\right). \end{aligned} \tag{19}$$

We then observe that because $H^{m-1}(Y(t))$ is finite-dimensional, if we take enough derivatives, we will obtain a $\mathbb{C}(t)$-linear relationship between $\int \operatorname{Res}\left(\frac{\Omega}{q}\right)$ and its derivatives. In order to compute this relationship, we use a *reduction of pole order* formula to compare classes of the form $\operatorname{Res}\left(\frac{r\Omega}{f^{q+1}}\right)$ to classes of the form $\operatorname{Res}\left(\frac{p\Omega}{q^k}\right)$. Suppose $r = \sum_i A_i \frac{\partial q}{\partial x_i}$, where the A_i are polynomials of the appropriate degree. Then,

$$\frac{\Omega}{q^{k+1}} \sum_i A_i \frac{\partial q}{\partial x_i} = \frac{1}{k}\frac{\Omega}{q^k}\sum_i \frac{\partial A_i}{\partial x_i} + \text{exact terms.} \tag{20}$$

We may generalize the Griffiths-Dwork technique to a multi-parameter family $Y(t_1, \ldots, t_j)$ by taking partial derivatives with respect to each parameter.

If our hypersurfaces $Y(t)$ are K3 surfaces, then $H^2(Y(t), \mathbb{C})$ is a 22-dimensional vector space, and the primitive cohomology $PH(Y)$ is 21-dimensional. Thus, we are guaranteed a

21st order ordinary differential equation, since there must be a linear relationship between $\int \omega$ and its first 21 derivatives. If $Y(t)$ has high Picard rank, however we expect a much lower order differential equation: the holomorphic $(2,0)$ form ω and its derivatives are orthogonal to $\mathrm{Pic}(Y(t))$, so we should obtain an ordinary differential equation of order $22-\rho$, where ρ is the generic Picard rank of $Y(t)$.

To implement the Griffiths-Dwork method, we shift our attention from the cohomology ring to a related ring. Let $S = \mathbb{C}(t)[x_0, \ldots, x_m]$, and let $J(q) = \langle \frac{\partial f}{\partial x_i} \rangle$ be the *Jacobian ideal* of f. Using Equation 20, we see that if $r \in J(q)$ and $\deg r = (k+1)\deg q - (m+1)$, we may reduce the pole order of $\mathrm{Res}(\frac{r\Omega}{f^{k+1}})$. Let $R(q) = S/J(q)$ be the *Jacobian ring.* The grading on S induces a grading on $R(q)$, and we have injective maps

$$R(q)_{k \deg q-(m+1)} \hookrightarrow H^{m-k,k-1}(Y) \tag{21}$$

for $k = 1, 2, \ldots$. The image of these *induced residue maps* is the primitive cohomology $PH(Y(t))$. Thus, we may implement the Griffiths-Dwork method by using a computer algebra system such as [BCP97] to work with graded pieces of $R(q)$.

5.2 The Griffiths-Dwork technique for M- and N-polarized surfaces

If the K3 surfaces under examination have ADE singularities, the residue map is still well-defined, since we can extend ω uniquely to the resolution of singularities. The authors of [CDLW09] applied the Griffiths-Dwork technique to the singular K3 surfaces $X(a,b,d)$ described in Equation 10. The computation yields the following characterization of one-parameter loci in moduli where the map $\mathrm{M} \to NS(X_t)$ fails to be surjective (compare Proposition 3.1):

Theorem 5.1 ([CDLW09])**.** *A one-parameter family of* M*-polarized K3 surfaces X_t generically has Picard-Fuchs equation of rank* 4*. The following are equivalent:*

- *Each surface X_t is polarized by the enhanced lattice* $\mathrm{M}_n = H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle$
- *The Picard-Fuchs equation drops to rank* 3
- *The corresponding pairs of elliptic curves* $\mathrm{E}_1(t)$ *and* $\mathrm{E}_2(t)$ *are n-isogenous*
- *The j-invariants of* $\mathrm{E}_1(t)$ *and* $\mathrm{E}_2(t)$ *satisfy* $\Box(j_1(t)) = \Box(j_2(t))$.

One may attempt to compute the Picard-Fuchs equations for N-polarized K3 surfaces by the Griffiths-Dwork technique, using the realization as singular hypersurfaces given in Equation 11. In practice, the computation is sensitive to the choice of parametrization on the moduli space. Recall that an N-polarized K3 surface is given by a point $(a,b,c,d) \in \mathbb{P}^{(2,3,5,6)}$, where c and d are not simultaneously 0. The locus where $c = 0$ yields the M-polarized K3 surfaces. We may work in the affine chart on $\mathbb{P}^{(2,3,5,6)}$ where $c \neq 0$ by applying the Griffiths-Dwork technique to the polynomials $q_{a,b,1,d}$. If we do so, we find that the elements of $R(q_{a,b,1,d})$ corresponding to $\mathrm{Res}\left(\frac{\Omega}{q}\right)$ and its first and second derivatives with respect to the parameters a, b, and d lie in the $\mathbb{C}(a,b,d)$-vector space spanned by the 5 elements

$$1, w^4, w^3x, w^3z, w^2z^2.$$

Thus, we will obtain $10 - 5 = 5$ second-order Picard-Fuchs equations, and a generic one-parameter subfamily specified by choosing $a(t)$, $b(t)$, and $d(t)$ will satisfy a fifth-order ordinary differential equation, as expected.

If we choose a chart on $\mathbb{P}^{(2,3,5,6)}$ where c is not constant, such as the chart where $d = 1$, we obtain a different number of equations. The elements of $R(q_{a,b,c,1})$ corresponding to $\mathrm{Res}\left(\frac{\Omega}{q}\right)$ and its first and second derivatives with respect to the parameters a and b lie in the $\mathbb{C}(a, b, c)$-vector space spanned by the same 5 basis elements, $1, w^4, w^3x, w^3z$, and w^2z^2. The first derivative $\frac{\partial}{\partial c}\mathrm{Res}\left(\frac{\Omega}{q}\right)$ and the mixed second derivatives involving c also lie in this vector space. However, the element of $R(q_{a,b,c,1})$ corresponding to $\frac{\partial^2}{\partial c^2}\mathrm{Res}\left(\frac{\Omega}{q}\right)$ contains a term in a sixth basis element, wz^3. Thus, in this chart we find only 4 second-order Picard-Fuchs equations, and an arbitrary one-parameter subfamily specified by equations $a(t)$, $b(t)$, and $c(t)$ yields a sixth-order ordinary differential equation. A similar result holds if we work with the full polynomials $q_{a,b,c,d}$: the element of $R(q_{a,b,c,d})$ corresponding to $\frac{\partial^2}{\partial c^2}\mathrm{Res}\left(\frac{\Omega}{q}\right)$ is independent of the ring elements corresponding to $\mathrm{Res}\left(\frac{\Omega}{q}\right)$ and the other first and second derivatives.

By using the Griffiths-Dwork technique applied to the polynomials $q_{a,b,c,d}$ or the methods of § 6, one can show that $\int \omega$ and its first derivatives are linearly dependent:

$$4a\frac{\partial}{\partial a}\int \omega + 6b\frac{\partial}{\partial b}\int \omega + 10c\frac{\partial}{\partial c}\int \omega + 12d\frac{\partial}{\partial d}\int \omega + \int \omega = 0. \tag{22}$$

Thus, $\frac{\partial^2}{\partial c^2}\mathrm{Res}\left(\frac{\Omega}{q}\right)$ cannot be linearly independent of the other first and second derivatives of $\mathrm{Res}\left(\frac{\Omega}{q}\right)$. The discrepancy demonstrates that the induced residue maps are not injective; this problem arises because our representative polynomials are not smooth.

5.3 Griffiths-Dwork for weighted projective hypersurfaces

In order to fix the problem that we encountered at the end of Section 5.2, we will pass from the expression for N-polarized K3 surfaces as singular hypersurfaces in projective space to an expression as generic hypersurfaces in a weighted projective space.

According to Table 1.1 of [B97], any generic hypersurface in $\mathbb{WP}^3(3, 4, 10, 13)$ has an embedding of the lattice N into its Néron-Severi lattice. Conversely, if X is an N-polarized K3 surface it can be written in the form of Equation 11, then X is birational to an anticanonical hypersurface in $\mathbb{WP}^3(3, 4, 10, 13)$ of the form

$$x_0^{10} + bx_0^6x_1^3 + \frac{d}{4}x_0^2x_1^6 + 3ax_0^4x_1^2x_2 - \frac{c}{2}x_1^5x_2 + x_0^2x_1x_2^2 + 2x_0x_1x_2x_3 - 4x_2^3 + x_1x_3^2 = 0 \tag{23}$$

with parameters $(a, b, c, d) \in \mathbb{WP}^3(2, 3, 5, 6)$, and where the variables x_0, x_1, x_2 and x_3 have weights $3, 4, 10$ and 13 respectively. To see how this birational transformation comes about, one restricts the weighted projective family of K3 surfaces to a copy of $(\mathbb{C}^\times)^3 \subseteq \mathbb{WP}^3(3, 4, 10, 13)$, exhibits an elliptic fibration over $\mathbb{CP}^1$ with singular fibers of types II^* and III^*, then matches parameters with the natural fibration of this form on the projective hypersurfaces in Equation 11.

In Section 4 of [Do82] it is shown that one may apply version of Griffiths residues to compute the orbifold cohomology of the hypersurfaces in Equation 23, since a generic member of the family in Equation 23 is quasismooth (in other words, its only singularities are inherited from the weighted projective space in which it lives). Since the primitive orbifold Hodge structure contains the transcendental Hodge structure of the minimal resolution of an orbifold K3 surface as a direct summand, the Griffiths-Dwork method will succeed in producing differential relations between the periods of the family of K3 surfaces in Equation 11. See Section 5.3.2 of [CK99] for details of how this technique differs from the Griffiths-Dwork technique for hypersurfaces in projective space as described in Section 5.1.

Therefore, if we attempt to compute the Picard-Fuchs equation of the N-polarized family on the chart $d = 1$, then this technique must produce the correct results and the problem encountered at the end of Section 5.2 vanishes. However if one does this computation, the result is a family of differential equations with complicated polynomial coefficients. For the sake of obtaining a differential equation that we can write down in a few pages, we will choose the family of K3 surfaces over $\mathbb{C}^3$ obtained by setting $a = 1$ in Equation 23. The resulting family of K3 surfaces is a family over $\mathbb{C}^3$ with coordinates b, c and d and according to [SY89] to the differential ideal annihilating periods of the three parameter family in Equation 23 is generated by a system of 5 equations, and each equation is expressed in the form

$$A\frac{\partial^2}{\partial a_i \partial a_j}\int \omega = A_{d,d}^{a_i,a_j}\frac{\partial^2}{\partial d^2}\int\omega + A_b^{a_i,a_j}\frac{\partial}{\partial b}\int\omega + A_c^{a_i,a_j}\frac{\partial}{\partial c}\int\omega + A_d^{a_i,a_j}\frac{\partial}{\partial d}\int\omega + A_0^{a_i,a_j}\int\omega$$

for (a_i, a_j) one of the pairs

$$(b,d),(b,c),(b,b),(c,c) \text{ and } (c,d).$$

We refer to these linear differential equations as $\mathcal{D}_{(a_i,a_j)}$

The polynomial A does not depend upon our choice of a_i and a_j. In our situation, we find that

$$A = 1296b^4c - 2340b^2cd - 2592b^2c - 4320bc^2 - 875c^3 - 432bd^2 + 900cd^2 - 2412cd + 1296c.$$

The other coefficients of our Picard-Fuchs equations are given as follows.

5.3.1 The equation $\mathcal{D}_{(b,d)}$

$$\begin{aligned}
A_{d,d}^{b,d} =& \frac{2}{3}\cdot(-648b^3cd - 1296b^2c^2 - 2700bc^3 - 625c^4 + 648b^2d^2 + 1080bcd^2 + 648bcd \\
&- 810c^2d + 648d^3 + 1296c^2 - 648d^2) \\
A_b^{b,d} =& -\frac{1}{6}\cdot c\cdot(1296b^2 - 8100bc - 3125c^2 + 180d - 1296) \\
A_c^{b,d} =& -\frac{1}{2}\cdot(-3060b^2c - 625bc^2 - 432bd + 1050cd - 1260c) \\
A_d^{b,d} =& -648b^3c + 432b^2d + 2730bcd + 625c^2d + 648bc + 450c^2 + 648d^2 - 432d \\
A_0^{b,d} =& \frac{5}{12}\cdot(360bc + 125c^2 + 36d)
\end{aligned}$$

5.3.2 The equation $\mathcal{D}_{(b,c)}$

$$
\begin{aligned}
A^{b,c}_{d,d} =& -4\cdot(216b^3c^2+150b^2c^3-108b^3d^2+108b^2cd-45bc^2d-125c^3d+108bd^3\\
&-216bc^2-60c^3+108bd^2+216cd^2-108cd)\\
A^{b,c}_{b} =& -\frac{1}{2}\cdot(-900b^3c-432b^2d+750bcd-3420bc-1225c^2)\\
A^{b,c}_{c} =& -1\cdot c\cdot(216b^3-750b^2c-330bd+625cd-216b-960c)\\
A^{b,c}_{d} =& -6\cdot(-72b^3d-150b^2cd+108b^2c+105bc^2+48bd^2+125cd^2+72bd\\
&+31cd-108c)\\
A^{b,c}_{0} =& -\frac{5}{2}\cdot(-30b^2c-6bd+25cd-30c)
\end{aligned}
$$

5.3.3 The equation $\mathcal{D}_{(b,b)}$

$$
\begin{aligned}
A^{b,b}_{d,d} =& -4\cdot(-720b^2c^3-250bc^4-36b^2cd^2-1296bc^2d-475c^3d+432bd^3\\
&+180cd^3-144c^3+36cd^2)\\
A^{b,b}_{b} =& 6\cdot c\cdot(-432b^3-125b^2c+60bd+432b+245c)\\
A^{b,b}_{c} =& -2\cdot c\cdot(720b^2c+625bc^2-432bd+750cd-1152c)\\
A^{b,b}_{d} =& -6\cdot(264b^2cd+250bc^2d-432bc^2-175c^3+288bd^2+480cd^2-408cd)\\
A^{b,b}_{0} =& -5\cdot c\cdot(36b^2+25bc+30d-36)
\end{aligned}
$$

5.3.4 The equation $\mathcal{D}_{(c,c)}$

$$
\begin{aligned}
A^{c,c}_{d,d} =& 4\cdot c^{-1}\cdot(90b^3c^3+324b^4d^2-9b^2c^2d-150bc^3d-648b^2d^3+126bc^3+25c^4\\
&-648b^2d^2-1404bcd^2-495c^2d^2+324d^4+117c^2d-648d^3+324d^2)\\
A^{c,c}_{b} =& -18\cdot c^{-1}\cdot(15b^4c-36b^3d-25b^2cd+6b^2c+10bc^2+6bd^2\\
&+36bd+42cd-21c)\\
A^{c,c}_{c} =& -3\cdot c^{-1}\cdot(432b^4c+150b^3c^2-936b^2cd-250bc^2d-864b^2c\\
&-1662bc^2-425c^3-72bd^2+180cd^2-648cd+432c)\\
A^{c,c}_{d} =& 18\cdot c^{-1}\cdot(72b^4d-30b^3cd+21b^2c^2-120b^2d^2+50bcd^2\\
&-144b^2d-234bcd-55c^2d+108d^3+15c^2-180d^2+72d)\\
A^{c,c}_{0} =& \frac{5}{4}\cdot c^{-1}\cdot(-36b^3c+36b^2d+60bcd+36bc+25c^2+36d^2-36d)
\end{aligned}
$$

5.3.5 The equation $\mathcal{D}_{(c,d)}$

$$\begin{aligned}
A^{c,d}_{d,d} =& -2 \cdot (648b^4d - 360b^2c^2 - 125bc^3 - 1188b^2d^2 - 1296b^2d - 2808bcd \\
& - 675c^2d + 540d^3 - 72c^2 - 1188d^2 + 648d) \\
A^{c,d}_{b} =& \frac{3}{2} \cdot (-432b^3 - 125b^2c + 60bd + 432b + 245c) \\
A^{c,d}_{c} =& -\frac{1}{2} \cdot (720b^2c + 625bc^2 - 432bd + 750cd - 1152c) \\
A^{c,d}_{d} =& -\frac{1}{2} \cdot (2592b^4 - 3888b^2d + 750bcd - 5184b^2 - 9936bc - 2275c^2 \\
& + 3240d^2 - 6048d + 2592) \\
A^{c,d}_{0} =& -\frac{5}{4} \cdot (36b^2 + 25bc + 30d - 36)
\end{aligned}$$

5.3.6 Comments

First note that in order to produce the Picard-Fuchs equations in Section 5.3 we have chosen a family of K3 surfaces whose period map is dominant onto the moduli space of N-polarized K3 surfaces with degree 2. Other choices of families that we have tried produced differential equations which are too complicated to be written concisely.

Secondly, the construction that we have described is effective for any chart on the moduli space of K3 surfaces, but we have not been able to use this to produce global results on the moduli space of N-polarized K3 surfaces. In Section 6, we will use another technique based upon the method of tame polynomials in order to produce an expression for the Gauss-Manin connection on the moduli space of N-polarized K3 surfaces which encompasses all of the data that we might be able to determine from the Griffiths-Dwork method applied to various charts on the moduli space of N-polarized K3 surfaces. The only disadvantage of the technique in Section 6 is that it produces very large equations.

Finally, we would like to point out that, in theory, the technique described above is valid for the moduli space of L-polarized K3 surfaces for any lattice L which appears as the Picard lattice of a generic anticanonical K3 surface in Reid's list of 95 weighted projective threefolds as listed in Table 1.1 of [B97].

5.4 The Elkies-Kumar parametrizations and Picard-Fuchs equations

Before moving on, we will describe how one may apply the technique of Griffiths-Dwork for weighted projective hypersurfaces along with Equation 23 and the work of Elkies and Kumar [EK14] to produce Picard-Fuchs equations for any family of K3 surfaces living over a Humbert surface in the moduli space of N-polarized K3 surfaces.

In Theorem 11 of [K12], Kumar determines an expression for the Shioda-Inose partner for the Jacobian of a given curve of genus 2. We recall that there are invariants I_2, I_4, I_6 and I_{10} which determine a curve of genus 2 up to isomorphism, called Igusa-Clebsch invariants. Kumar writes down a family of elliptically fibered K3 surfaces varying with parameters the Igusa-Clebsch invariants so that the Shioda-Inose partner of such a K3 surface is the Jacobian of the genus 2 curve determined by the Igusa-Clebsch invariants appearing in the equation for the K3 surface. This family is written as

$$y^2 = x^3 - t^3\left(\frac{I_4}{12}t + 1\right)x + t^5\left(\frac{I_{10}}{4}t^2 + \frac{(I_2I_4 - 3I_6)}{108}t + \frac{I_2}{24}\right).$$

In Sections 6 to 35 of [EK14], Elkies and Kumar determine explicit parametrizations for all rational Humbert surfaces of discriminant less than 100. One can use these parametrizations to provide Picard-Fuchs equations for the corresponding Humbert surfaces in the moduli space of N-polarized K3 surfaces. First, it is understood that the family written by Kumar is exactly the same as the family of K3 surfaces written in [CD12], since both are Shioda-Inose partners of Jacobians of genus 2 curves, and thus are N-polarized K3 surfaces. The exact relationship between sets of parameters is given by the map

$$(a,b,c,d) = \left(\left(\frac{1}{36}\right) I_4, \left(-\frac{1}{216}\right) I_2 I_4 + \left(\frac{1}{72}\right) I_6, \left(\frac{1}{4}\right) I_{10}, \left(\frac{1}{96}\right) I_2 I_{10}\right).$$

For any of the Humbert surfaces whose parametrization is determined by Elkies and Kumar, it is then possible to compute the corresponding Picard-Fuchs equation by straightforward application of the Griffiths-Dwork method in Section 5.3, since an N-polarized K3 surface of Picard rank 18 expressed as in Equation 23 is quasi-smooth if its transcendental lattice has discriminant greater than 4.

As an example, the Humbert surface $\mathcal{H}_5$ is parametrized by

$$u, v \mapsto (a(u,v), b(u,v), c(u,v), d(u,v)) = \left(\frac{u^2}{4}, \frac{u^3}{8} + \frac{v}{2}, v^2, uv^2 + \frac{v^2}{4}\right).$$

Applying the Griffiths-Dwork method to the family of weighted projective hypersurfaces obtained from this parametrization and Equation 23 produces the Picard-Fuchs operators

$$\begin{aligned} &-4u^2(270u^3 + 99u^2 + 9u + 125v)\frac{\partial^2}{\partial u^2} + 18u^2 v(10u+3)(15u+2)\frac{\partial^2}{\partial u \partial v} \\ &+ 2u(-810u^3 - 207u^2 - 9u + 250v)\frac{\partial}{\partial u} + 54uv(5u-1)(5u+1)\frac{\partial}{\partial v} - 375v \end{aligned}$$

and

$$\begin{aligned} &2uv(270u^3 + 99u^2 + 9u + 125v)\frac{\partial^2}{\partial v^2} + 2u(108u^4 + 36u^3 + 3u^2 + 100uv + 10v)\frac{\partial^2}{\partial u \partial v} \\ &+ 10u(10u+1)\frac{\partial}{\partial u} + 2(270u^4 + 99u^3 + 9u^2 + 150uv - 5v)\frac{\partial}{\partial v} - 5(6u+1) \end{aligned}$$

which generate the differential ideal which annihilates the periods of the family of K3 surfaces over the Humbert surface $\mathcal{H}_5$. These Picard-Fuchs equation should agree, after change of variables, with the Picard-Fuchs equations given in Section 3 of [N10].

Another approach to computing Picard-Fuchs equations for families of K3 surfaces over Humbert surfaces will be presented in Section 6.11.

6 Calculating the Gauss-Manin connection

In this section we calculate the Gauss-Manin connection for M- and N-polarized K3 surfaces using *tame polynomials*; the relevant techniques were first introduced in [Mo11]. The method of tame polynomials differs from our naive application of the Griffiths-Dwork technique in two ways: we work with affine hypersurfaces rather than hypersurfaces in projective space, and we use a one-parameter deformation to remove computational problems caused by singularities in the representative hypersurfaces. [1]

[1] See `http://w3.impa.br/∼hossein/k3surfaces` for explicit equations for all calculations in this section.

6.1 Tame polynomial

We follow the notation of [Mo11, Chapter 4]. Let $\mathsf{R} = \mathbb{Q}[a_1, \ldots, a_n]$, where we view the a_i as arbitrary parameters. Let us consider the homogeneous polynomial of degree 24

$$g := -4x^3 + y^2 w - \frac{1}{2} w^4$$

in the weighted ring

$$\mathsf{R}[x, y, z],\ \deg(x) = 8,\ \deg(y) = 9,\ \deg(w) = 6 \tag{24}$$

The polynomial has an isolated singularity at the origin $0 \in \mathbb{C}^3$. A *tame polynomial in* R *with the last homogeneous polynomial* g is a polynomial of the form $f := g + f_1$, where $f_1 \in \mathsf{R}[x, y, w]$ is of degree strictly less than 24.

Our main example is the case $\mathsf{R} = \mathbb{Q}[a, b, c, d]$ and

$$f = y^2 w - 4x^3 + 3axw^2 + bw^3 + cxw - \frac{1}{2}(dw^2 + w^4). \tag{25}$$

We may obtain f by starting with the family of polynomials $q_{a,b,c,d}$ which describe N-polarized K3 surfaces given in Equation 11, and converting to the affine chart where $z = 1$. Note, however, that we have changed the weights on w, x, and y.

Remark 6.1. $\{g = 0\}$ induces a rational curve L_2 in $\mathbb{P}^{(8,9,6)}$. After a resolution of singularities, L_2 is the same L_2 as in [CD12]. We consider the weights

$$\deg(a) = 4,\ deg(b) = 6,\ \deg(c) = 10,\ \deg(d) = 12$$

and in this way f becomes a homogeneous polynomial of degree 24 in 7 variables x, y, w, a, b, c, d. These weights are compatible with the weights of Eisenstein series computed in [CD12], Theorem 1.7.

Remark 6.2. The monomials of degree less than 24 in (24) are:

$$y^2, y, yw, yw^2, yx, yxw, 1, w, w^2, w^3, x, xw, xw^2, x^2, x^2 w.$$

Thus, the most general tame polynomial f that we can write is g plus a linear combination of the above monomials with coefficients in R. If we are interested in such a tame polynomial up to linear transformations $x \mapsto x + * + *w,\ y \mapsto * + *x + *w,\ w \mapsto w + *$, all $*$ in R, then we may discard the monomials $x^2, x^2 w, yw, yxw, yw^2, w^3$ in the definition of f. In this way we obtain an affine version of the equation for the family of rank 12 K3 surfaces introduced in § 4.4:

$$f = -4x^3 + y^2 w - \frac{1}{2} w^4 + t_4 xw^2 + t_6 y^2 + t_7 yx + t_{10} xw + t_{12} w^2 + t_{15} y + t_{16} x + t_{18} w + t_{24} \tag{26}$$

6.2 Algebraic De Rham cohomology

The R-module $V_g := \mathsf{R}[x, y, w]/\mathrm{Jacob}(g)$ is free of rank 10. In fact, the set of monomials

$$I := \{xw^3, xw^2, w^3, xy, xw, w^2, y, x, w, 1\} \tag{27}$$

form a basis for both V_g and $V_f := \mathsf{R}[x, y, w]/\mathrm{Jacob}(f)$ (see [Mo11, Proposition 4.6]). Let

$$\mathbb{U}_0 := \mathrm{Spec}(\mathsf{R}),\ \mathbb{U}_1 = \mathrm{Spec}(\mathsf{R}[x, y, w]).$$

The Brieskorn module or the relative de Rham cohomology of $\mathbb{U}_1/\mathbb{U}_0$ is by definition:

$$\mathsf{H} = H^2_{\mathrm{dR}}(\mathbb{U}_1/\mathbb{U}_0) := \frac{\Omega^3_{\mathbb{U}_1/\mathbb{U}_0}}{f\Omega^3_{\mathbb{U}_1/\mathbb{U}_0} + df \wedge d\Omega^1_{\mathbb{U}_1/\mathbb{U}_0}} \tag{28}$$

where $\Omega^i_{\mathbb{U}_1/\mathbb{U}_0}$ is the set of differential i-forms of $\mathsf{R}[x, y, w]$ over R (the differential of the elements of R is zero). It is an R-module in a canonical way. It can be shown that H is a free R-module generated by

$$\alpha dx \wedge dy \wedge dw, \alpha \in I. \tag{29}$$

(See [Mo11, Theorem 4.1 and Corollary 4.1].)

6.3 Discriminant

Let

$$A_f : V_f \to V_f, \quad A(P) = P \cdot f.$$

We use the monomial basis of the free R-module V_f defined in Equation 27 in order to write A_f as a matrix. Let $\Delta(s)$ be the minimal polynomial of A_f. It is a factor of the characteristic polynomial $\det(A_f - s \cdot I) \in \mathsf{R}[s]$. By definition, the discriminant of f is $\Delta := \Delta(0)$.

6.4 Gauss-Manin connection

In order to introduce the Gauss-Manin connection on H it is more convenient to use the R-module

$$\mathsf{M} \ := \ \frac{\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}[\frac{1}{f}]}{\Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0} + d(\Omega^{n}_{\mathbb{U}_1/\mathbb{U}_0}[\frac{1}{f}])}$$

which we call the Gauss-Manin system of f. Here $\Omega^i_{\mathbb{U}_1/\mathbb{U}_0}[\frac{1}{f}]$ is the set of polynomials in $\frac{1}{f}$ with coefficients in $\Omega^i_{\mathbb{U}_1/\mathbb{U}_0}$. The Gauss-Manin system has a natural filtration given by the pole order along $\{f = 0\}$, namely

$$\mathsf{M}_i := \{[\frac{\omega}{f^i}] \in \mathsf{M} \mid \omega \in \Omega^{n+1}_{\mathbb{U}_1/\mathbb{U}_0}\},$$

$$\mathsf{M}_1 \subset \mathsf{M}_2 \subset \cdots \subset \mathsf{M}_i \subset \cdots \subset \mathsf{M}_\infty := \mathsf{M}.$$

We have a canonical map $\mathsf{H} \to \mathsf{M}$, $\omega \mapsto \frac{\Omega}{f}$. If the discriminant $\Delta \in \mathsf{R}$ of f is not zero then this map is an inclusion and an isomorphism of R-modules $\mathsf{H} \cong \mathsf{M}_1$. The Gauss-Manin connection on M is the map

$$\nabla : \mathsf{M} \to \Omega^1_{\mathbb{U}_0} \otimes_{\mathsf{R}} \mathsf{M}$$

which is obtained by derivation with respect to the elements of R (the derivation of x, y and w is zero). More precisely

$$\nabla(\frac{P\omega}{f^i}) = \frac{d_{\mathsf{R}}P \cdot f - iP \cdot d_{\mathsf{R}}f}{f^{i+1}}\omega, \ P \in \mathsf{R}[x, y, w], \tag{30}$$

where $\omega = dx \wedge dy \wedge dw$ and $d_{\mathsf{R}} : \mathsf{R}[x] \to \mathsf{R}[x]$ is the differential with respect to elements in R. (Note that Equation 30 is the algebraic, affine counterpart of Equation 19.)

6.5 Gauss-Manin connection, $\Delta \neq 0$

If the discriminant of f is not zero then the Gauss-Manin connection on M induces a connection on all $\mathsf{M}_i,\ i \in \mathbb{N}$:

$$\nabla : \mathsf{M}_i \to \frac{1}{\Delta}\Omega^1_{\mathbb{U}_0} \otimes_{\mathsf{R}} \mathsf{M}_i.$$

The R-module $\mathsf{H} = \mathsf{M}_1$ is freely generated by (29) and so theoretically we can calculate a 10×10 matrix $\tilde{A}$ with entries in $\Omega_{\mathbb{U}_0}$ such that

$$\nabla\Omega = \frac{1}{\Delta}\tilde{A} \otimes \Omega$$

where Ω is a 10×1 matrix formed by (29). However, in practice performing this calculation for the tame polynomial (26) using the algorithms in [Mo11] was beyond the power of our computers.

The discriminant of the tame polynomial (25) with $a, b, c, d \in \mathsf{R}$ is zero. In order to obtain a tame polynomial with non-zero discriminant, we introduce a new parameter s and work with the ring $\tilde{\mathsf{R}} := \mathsf{R}[s]$ and the tame polynomial $\tilde{f} = f - s$. The polynomial $\Delta(s)$ turns out to be the discriminant of the tame polynomial $f - s$ with coefficients in $\mathsf{R}[s]$. We use the algorithms in [Mo11] to calculate the Gauss-Manin connection of $\tilde{f}$. This means that we take the 10×1 matrix Ω formed by (29) and calculate the 10×10 matrices $A(s), B(s), C(s), D(s)$ and $S(s)$ with entries in $\mathsf{R}[s]$ satisfying the equality

$$\nabla\Omega = \frac{1}{\Delta(s)}\tilde{A}(s) \otimes \Omega,$$

$$\tilde{A}(s) = A(s)da + B(s)db + C(s)dc + D(s)dd + S(s)ds^2$$

Here d stands for both differential and a parameter. Therefore, dd means the differential of $d \in \mathsf{R}$. For our example (25) we have $\Delta(s) = s(\Delta+\Delta_1 s+\cdots),\ \Delta_i \in \mathsf{R}$. It turns out that $A(0) = B(0) = C(0) = D(0) = 0$. Therefore, we get the claculation of the Gauss-Manin connection for f.

$$\nabla\Omega = \frac{1}{\Delta}\tilde{A} \otimes \Omega,$$

$$\tilde{A} = A \cdot da + B \cdot db + C \cdot dc + D \cdot dd$$

where A, B, C and D are 10×10 matrices with entries in R. Using these calculations we can check that:

Proposition 6.3. *We have*

1. *The free* R*-module generated by*

   $$\alpha dx \wedge dy \wedge dw, \alpha = xw^3, xw^2, w^3, xw, w^2, \tag{31}$$

   in invariant under $\Delta \cdot \nabla$.

2. $\omega = \alpha dx \wedge dy \wedge dw, \alpha = xy, y$ *is a flat section, that is,* $\nabla(\omega) = 0$.

[2]The data of $\Delta(s), A(s), \cdots$ as a text file is around 500KB.

6.6 Gauss-Manin connection, $\Delta = 0$

Let f be a tame polynomial with zero discriminant. For $P \in \ker(A_f)$ we have $fPdx = df \wedge \omega_P$ for some $\omega_P \in \Omega^n_{\mathbb{U}_1/\mathbb{U}_0}$ and so

$$\begin{aligned}\frac{Pdx}{f^i} &= \frac{fPdx}{f^{i+1}} = \frac{df \wedge \omega_P}{f^{i+1}} \\ &= \frac{1}{i}\left(\frac{d\omega_P}{f^i} - d(\frac{\omega_P}{f^i})\right) \\ &= \frac{1}{i}\frac{d\omega_P}{f^i} \quad \text{in } \mathsf{M}\end{aligned}$$

We conclude that

$$\frac{Pdx - \frac{1}{i}d\omega_P}{f^i} = 0, \quad \text{in } \mathsf{M}$$

For $i = 1$ we conclude that there are many R-linear relations between (29) in M. We get

Proposition 6.4. *For arbitrary a, b, c, d, we have the following equalities in M:*

$$\alpha dx \wedge dy \wedge dw = 0 \ \text{in} \quad \mathsf{M} \tag{32}$$

where α is one of the polynomials:

$$\begin{aligned}&xy,\\ &y,\\ &300dxw^3 - 432bdxw^2 - (216a^2d - 17c^2)w^3 - (48ac^2 - 132d^2)xw - (102acd + 24bc^2)w^2 - 6c^3x - 3c^2dw,\\ &150xw^3 - 216bxw^2 - 108a^2w^3 + 66dxw - 51acw^2 - 5c^2w,\\ &-(1350ac^2 - 1800d^2)xw^3 + (1944abc^2 - 2592bd^2 + 114c^3)xw^2 + (972a^3c^2 - 1296a^2d^2 + 102c^2d)w^3 - (882ac^2d + 144bc^3 - 792d^3)xw+\\ &(387a^2c^3 - 612acd^2 - 144bc^2d)w^2 - 6c^3dx + (18ac^4 - 18c^2d^2)w - c^5\end{aligned}$$

and for $c = 0$ we have the equalities (32), where α is one of the polynomials:

$$\begin{aligned}&25xw^3 - 36bxw^2 - 18a^2w^3 + 11dxw\\ &xy\\ &y\\ &-75bxw^3 + (108b^2 - 19d)xw^2 + 54a^2bw^3 - 9bdxw + 12a^2dw^2 - 5d^2x\\ &-150axw^3 + 216abxw^2 + (108a^3 - 17d)w^3 - 18adxw + 24bdw^2 - 7d^2w\\ &-900abxw^3 + (1296ab^2 - 114ad)xw^2 + (648a^3b - 51bd)w^3 - 108abdxw + (72a^3d + 72b^2d - 11d^2)w^2 - 6ad^2x - 9bd^2w - d^3\end{aligned}$$

6.7 M-polarized $K3$ surfaces

In this section we compute the Gauss-Manin connection for M-polarized $K3$ surfaces by setting $c = 0$ in Equation 25. In Proposition 6.4 we found 6 relations between the differential forms (29). Let $\mathsf{R} = \mathbb{Q}[a, b, d]$. We consider the R-module generated by four elements

$$\alpha dx \wedge dy \wedge dw, \quad \alpha = 1, w, w^2, w^3,$$

of H and write the Gauss-Manin connection on this module. This means that we consider the 4×1 matrix Ω with the above entries and calculate:

$$\nabla\Omega = \frac{1}{\Delta}(A \cdot da + B \cdot db + D \cdot dd) \otimes \Omega$$

where Δ and all entries of A, B, D are explicit polynomials in a, b, d with coefficients in $\mathbb{Q}$ (see [Mo15]). For instance,

$$\Delta := a(a^6d^6 - 2a^3b^2d^6 - 2a^3d^7 + b^4d^6 - 2b^2d^7 + d^8)$$

We can use this data and calculate the differential equations (22) and (23) of [CDLW09]. We can also calculate the Picard-Fuchs equations of $\omega = dx \wedge dy \wedge dw$ when a, b and d depend on a parameter t. For instance, for

$$a = (16 + t)(256 + t),\ b = (-512 + t)(-8 + t)(64 + t),\ c = 2985984t^3$$

ω satisfy the Picard-Fuchs equation:

$$y + (26t + 512)y' + (36t^2 + 1536t)y'' + (8t^3 + 512t^2)y''' = 0,$$

with $' = \frac{\partial}{\partial t}$. In a similar way for

$$a = (t^2+246t+729)(t+27)^2,\ b = (t^2-486t-19683)(t^2+18t-27)(t+27)^2,\ d = 2^{12}3^6t^4(t+27)^4$$

it satisfies:

$$(t + 15)y(7t^2 + 192t + 729)y' + (6t^3 + 243t^2 + 2187t)y'' + (t^4 + 54t^3 + 729t^2)y''' = 0.$$

6.8 N-polarized K3 surfaces

In this section we analyze the Gauss-Manin connection for N-polarized K3 surfaces using the full family given in Equation 25. By Proposition 6.4, we have 5 relations between the differential forms (29). Let $\mathsf{R} = \mathbb{Q}[a, b, c, d]$. We consider the R-module generated by

$$\alpha dx \wedge dy \wedge dw, \quad \alpha = 1, w, w^2, w^3, xw \tag{33}$$

and calculate the Gauss-Manin connection on this module:

$$\nabla(\omega) = A\omega, \quad A = \frac{1}{\Delta}(Ada + Bdb + Cdc + Ddd).^3 \tag{34}$$

Here Δ and all the entries of the 5×5 matrices A, B, C and D are in R. For instance,

$$\Delta := c(34992a^7c^3d+23328a^6bc^4-11664a^6cd^3+3888a^5c^5-69984a^4b^2c^3d-71928a^4c^3d^2-46656a^3b^3c^4+$$
$$23328a^3b^2cd^3 - 184680a^3bc^4d + 23328a^3cd^4 - 97200a^2b^2c^5 + 46656a^2bc^2d^3 - 37125a^2c^5d+$$
$$34992ab^4c^3d-68040ab^2c^3d^2-33750abc^6+48600ac^3d^3+23328b^5c^4-11664b^4cd^3-48600b^3c^4d+$$
$$23328b^2cd^4 + 27000bc^4d^2 - 3125c^7 - 11664cd^5)$$

Using this data, we can calculate the Picard-Fuchs of $dx\wedge dy\wedge dw$ restricted to the decagon curve:

$$\begin{aligned}
a &= 625(-3 + t)^2,\\
b &= -(625/2)(-1134 + 1458t - 504t^2 + 23t^3),\\
c &= -(759375/4)(-2 + t)^2(2 + t)^4,\\
d &= 18984375/4(-2 + t)^2(2 + t)^4(9 + 2t).
\end{aligned}$$

It is the fourth order differential equation:

$$504y+(9000t)y'+(15500t^2-22000)y''+(6250t^3-25000t)y'''+(625t^4-5000t^2+10000)y'''' = 0$$

[3] The data of $A, B, \cdots$ as a text file is around 50KB.

6.9 A canonical basis

The choice of the differential forms (33) is not canonical. Moreover it is not compatible with the Hodge filtration. In this section we discuss a choice of another basis. For any function I in a, b, c, d let us define:

$$\partial_a I = I_a := -4a\frac{\partial I}{\partial a},\ \partial_b I = I_b := -6b\frac{\partial I}{\partial b},\ \partial_c I = I_c := -10c\frac{\partial I}{\partial c},\ \partial_d I = I_d := -12d\frac{\partial I}{\partial d}$$

Note that our notations for I_a and ω_a is different from those in the Introduction. Let k be the fractional field of R. The k-vector V space generated by $\omega := dx \wedge dy \wedge dw$ and all its derivatives is at most 5 dimensional. We have

$$\omega_a + \omega_b + \omega_c + \omega_d - \omega = 0$$

and so we cannot take $\omega, \omega_a, \cdots$ as a basis of V. Our calculations show that we can take

$$\omega, \omega_b, \omega_c, \omega_d, \omega_{dd}$$

as a basis of V. We calculate the Gauss-Manin connection in this basis. In other words we calculate all 5×5 matrices $M^x,\ x = a, b, c, d$ in the equalities:

$$\begin{pmatrix} \omega_x \\ \omega_{bx} \\ \omega_{cx} \\ \omega_{dx} \\ \omega_{ddx} \end{pmatrix} = M^x \begin{pmatrix} \omega, \\ \omega_b \\ \omega_c \\ \omega_d \\ \omega_{dd} \end{pmatrix}, \quad x = a, b, c, d, \tag{35}$$

Some identities above are trivial. For instance, the first row of M^a is $[1, -1, -1, -1, 0]$, the first two of M^b is $[0, 1, 0, 0, 0]$ and so on.

6.10 The differential algebra annihilating the holomorphic 2-form

Let I be the left ideal of $\mathbb{Q}(a, b, c, d)[\partial_a, \partial_b, \partial_c, \partial_d]$ containing all differential operators which annihilate $\omega := dx\wedge dy\wedge dw$. The ideal I is generated by the differential operators obtained by 20 equalities in (35). From 20 equalities, 4 of them are trivial equalities such as $\omega_b = \omega_b$ and so on. There are 3 pair of equations which are repetition. For instance we have two equation of the type $\omega_{bc} = \cdots,\ \omega_{cb} = \cdots$. From the 13 non-trivial differential operators, one is $P\omega = 0$, where P is

$$\partial_a + \partial_b + \partial_c + \partial_d - 1 \tag{36}$$

Among them, we have 8 second order differential equations with the differential operator:

$$p_1 + p_2\partial_b + p_3\partial_c + p_4\partial_d + p_5\partial_d\partial_d - X \tag{37}$$

$$X = \partial_b\partial_b,\ \partial_c\partial_c,\ \partial_b\partial_c,\ \partial_c\partial_d,\ \partial_b\partial_d,\ \partial_b\partial_a,\ \partial_c\partial_a,\ \partial_d\partial_a$$

where $p_i \in \mathsf{k}$ and they do depend on the choice of X. Four of them are third order differential equations with the differential operators (37), where X is

$$X = \partial_d\partial_d\partial_a,\ \partial_d\partial_d\partial_b,\ \partial_d\partial_d\partial_c,\ \partial_d\partial_d\partial_d.$$

All these $13 = 1 + 8 + 4$ differential operator generate the differential module which annihilates the holomorphic differential form ω.

6.11 Differential equations for Humbert surfaces

Let us consider the system (34). We consider a, b, c, d as functions of three other parameters u, v and w. In this way $da = \frac{\partial a}{\partial u}du + \frac{\partial a}{\partial v}dv + \frac{\partial a}{\partial w}dw$ and so on. More generally, we consider three vector field $\partial_u,\ \partial_v,\ \partial_w$ on the (a, b, c, d)-space. Therefore, we may have $\partial_u\partial_v \neq \partial_v\partial_u$. What we first calculate is the 3×5 matrix R in the equality:

$$\begin{pmatrix} \omega_u \\ \omega_{uv} \\ \omega_{uvw} \end{pmatrix} = R \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{pmatrix}$$

where $\omega_i,\ i = 1, 2, \ldots, 5$ are the differential forms (33). For instance, the first row of R is the first row of the matrix

$$\frac{1}{\Delta}(Aa_u + Bb_u + Cc_u + Dd_u).$$

The entries of the second row are in $\mathbb{Q}(a, b, c, d)[a_u, b_u, c_u, d_u, a_v, b_v, c_v, d_v, a_{uv}, b_{uv}, c_{uv}, d_{uv}]$ and the entries of the third row are in

$$\mathbb{Q}(a, , b, c, d)[a_u, b_u, c_u, d_u, a_v, b_v, c_v, d_v, a_{uv}, b_{uv}, c_{uv}, d_{uv},$$

$$a_w, b_w, c_w, d_w, a_{uw}, b_{uw}, c_{uw}, d_{uw}, a_{vw}, b_{vw}, c_{vw}, d_{vw}, a_{uvw}, b_{uvw}, c_{uvw}, d_{uvw}]$$

[4] We then calculate

(38) $$\det(R^x) = 0, \text{ where } \begin{pmatrix} \omega \\ \omega_u \\ \omega_v \\ \omega_{uv} \\ \omega_x \end{pmatrix} = R^x \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{pmatrix}$$

where

$$x = uu,\ vu,\ uvu,\ vv,\ uvv$$

if $\partial_u\partial_v = \partial_v\partial_u$ then we have essentially two differential equations

(39) $$\det(R^{uu}) = 0, \quad \det(R^{uvv}) = 0,$$

The first one is second order and the second one is third order. [5]

6.12 Checking well-known Humbert surfaces

Let us be given a hypersurface

$$Z(H) := \{H(a, b, c, d) = 0\}$$

We want to check that whether $Z(H)$ is tangent to the partial differential equations (38). Let us take one of a, b, c or d as a constant. If the compactification of $Z(H)$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ then we could parametrize $Z(H)$ with algebraic coordinates (u, v) and we could

[4]The size of the matrix R stored in a computer is 5.9 MB.

[5]The data of (39) as a text file is respectively 385.8 KB and 4.0 MB.

check if such a parametrization satisfies the differential equations (38). But in general we have to work with transcendental parametrization which might be difficult to find them in practice. There is another algebraic method which we explain it below:

We consider two linearly independent algebraic vector fields

$$U = \sum_{x=a,b,c,d} U^x \frac{\partial}{\partial x},\ V = \sum_{x=a,b,c,d} V^x \frac{\partial}{\partial x},\ U^x, V^x \in \mathbb{Q}[a,b,c,d].$$

in the parameter space which are tangent to $Z(H)$. For instance take

$$U = \frac{\partial H}{\partial b}\frac{\partial}{\partial a} - \frac{\partial H}{\partial a}\frac{\partial}{\partial b}$$

$$V = \frac{\partial H}{\partial d}\frac{\partial}{\partial c} - \frac{\partial H}{\partial c}\frac{\partial}{\partial d}$$

Let u and v be a (transcendental) coordinate system in the domain of a solution of U and V, respectively. All the parameters a, b, c, d become functions of u and v, but not simultaneously:

$$x_u = U^x, x_v = V^x, \quad x = a, b, c, d.$$

Note that $[\frac{\partial}{\partial u}, \frac{\partial}{\partial u}]$ may not be zero and so we may have $x_{uv} \neq x_{vu}$. Now, we can use the chain rule and calculate all a_{uv} and so on. The rest is just substitution and checking the equalities (38). Using this method we have checked that the differential equations (38) are tangent to the zero set of

$$H := (d - b^2 - a^3)^2 - 4a(c - ab)^2.$$

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