Commun. Math. Phys. 344, 889–914 (2016)
Digital Object Identifier (DOI) 10.1007/s00220-016-2640-9

Communications in
**Mathematical
Physics**

Gauss–Manin Connection in Disguise: Calabi–Yau Threefolds

Murad Alim[1,2], Hossein Movasati[3], Emanuel Scheidegger[4], Shing-Tung Yau[1]

[1] Mathematics Department, Harvard University, 1 Oxford Street, Cambridge, MA 02138, USA
[2] Jefferson Physical Laboratory, 17 Oxford Street, Cambridge, MA, 02138, USA
[3] Instituto de Matemática Pura e Aplicada, IMPA, Estrada Dona Castorina, 110, 22460-320, Rio de Janeiro, RJ, Brazil. E-mail: hossein@impa.br
[4] Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Eckerstrasse 1, D-79104 Freiburg, Germany

Received: 22 April 2015 / Accepted: 7 March 2016
Published online: 14 May 2016 –

Abstract: We describe a Lie Algebra on the moduli space of non-rigid compact Calabi–Yau threefolds enhanced with differential forms and its relation to the Bershadsky–Cecotti–Ooguri–Vafa holomorphic anomaly equation. In particular, we describe algebraic topological string partition functions $\mathsf{F}_g^{\mathrm{alg}}$, $g \geq 1$, which encode the polynomial structure of holomorphic and non-holomorphic topological string partition functions. Our approach is based on Grothendieck's algebraic de Rham cohomology and on the algebraic Gauss–Manin connection. In this way, we recover a result of Yamaguchi–Yau and Alim–Länge in an algebraic context. Our proofs use the fact that the special polynomial generators defined using the special geometry of deformation spaces of Calabi–Yau threefolds correspond to coordinates on such a moduli space. We discuss the mirror quintic as an example.

Contents

1. Introduction

Mirror symmetry identifies deformation families of Calabi–Yau (CY) threefolds. It originates from two dimensional sigma models into CY target spaces $\check{X}$ and X and two equivalent twists, which give the A- and the B-model, and which probe the symplectic and complex geometry of $\check{X}$ and X respectively [Wit88, Wit91].

Mirror symmetry is a rich source of far-reaching predictions, especially regarding the enumerative geometry of maps from genus g Riemann surfaces into a CY threefold $\check{X}$. The predictions are made by performing computations on the B-model side which sees the deformations of complex structure of the mirror CY X. The non-trivial step, which is guided by physics, is to identify the equivalent structures on the A-model side and match the two.

The first enumerative predictions of mirror symmetry at genus zero were made by Candelas, de la Ossa, Green and Parkes in Ref. [CDLOGP91]; higher genus predictions were put forward by Bershadsky, Cecotti, Ooguri and Vafa (BCOV) in Refs. [BCOV93, BCOV94]. To prove these predictions and formulate them rigorously is a great mathematical challenge.

The formulation of the moduli spaces of stable maps by Kontsevich [Kon95] provided a mathematical formulation of the A-model and a check of many results confirming the predictions of mirror symmetry. The computations of Ref. [CDLOGP91] for genus zero Gromov–Witten invariants were put in a Hodge theoretical context by Morrison in Ref. [Mor92]. Genus zero mirror symmetry can now be understood as matching two different variations of Hodge structure associated to $\check{X}$ and X, see Refs. [Mor97, CK99, Voi99].

Mirror symmetry at higher genus remains challenging both computationally and conceptually. A fruitful way to think about higher genus mirror symmetry is through geometric quantization as proposed by Witten in Ref. [Wit93]. A mathematical reformulation of BCOV for the B-model was put forward by Costello and Li in Ref. [CL12].

In the present work we will follow a different approach and put forward a new algebraic framework to formulate higher genus mirror symmetry where we can work over an arbitrary field of characteristic zero. Our approach is based on Grothendieck's algebraic de Rham cohomology and Katz–Oda's algebraic Gauss–Manin connection. It further builds on results of Yamaguchi–Yau [YY04] and Alim–Länge [AL07], who uncovered a polynomial structure of higher genus topological string partition functions. This polynomial structure is based on the variation of Hodge structure at genus zero and puts forward variants of BCOV equations which can be understood in a purely algebraic context.

In the algebraic context, surprisingly, no reference to periods or variation of Hodge structures is needed, as all these are hidden in the so called *Gauss–Manin connection in disguise*.[1] As a consequence, we do not need to look for an algebraic definition of

[1] The terminology arose from a private letter of Pierre Deligne to the second author [Del09].

the propagators appearing in the BCOV equations. We will see that the propagators can be expressed as rational functions of the entries of the period matrix. But this is just an indirect algebraic formulation of the propagators that cannot be used as a definition. The advantage of the analytic definition is of course that it is inherent that they transform as sections of bundles on the moduli spaces. Here, we try to combine the best of the frameworks.

The new way of looking at the Gauss–Manin connection was studied by the second author, see Ref. [Mov12c] for elliptic curve case, Ref. [Mov12a] for mirror quintic case and Ref. [Mov13] for a general framework. The richness of this point of view is due to its base space, which is the moduli space of varieties of a fixed topological type and enhanced with differential forms. Computations on such moduli spaces were already implicitly in use by Yamaguchi–Yau [YY04] and Alim–Länge [AL07] without referring to the moduli space itself; however, its introduction and existence in algebraic geometry for special cases go back to the works of the second author. Such moduli spaces give a natural framework for dealing with both automorphic forms and topological string partition functions. In the case of elliptic curves [Mov12c], the theory of modular and quasi-modular forms is recovered. In the case of compact Calabi–Yau threefolds we obtain new types of functions which transcend the world of automorphic forms. In the present text we develop the algebraic structure for any CY threefold. As an example, we study the mirror quintic in detail.

In the following, we recall the basic setting of Refs. [Mov13, Mov12a]. For a background in Hodge theory and algebraic de Rham cohomology we refer to Grothendieck's original article [Gro66] or Deligne's lecture notes in [DMOS82]. Let k be a field of characteristic zero small enough so that it can be embedded in $\mathbb{C}$. For a non-rigid proejctive Calabi–Yau threefold X defined over k let $H^3_{\mathrm{dR}}(X)$ be the third algebraic de Rham cohomology of X and

$$0 = F^4 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H^3_{\mathrm{dR}}(X),$$

be the corresponding Hodge filtration. The intersection form in $H^3_{\mathrm{dR}}(X)$ is defined to be

$$H^3_{\mathrm{dR}}(X) \times H^3_{\mathrm{dR}}(X) \to \mathsf{k}, \quad \langle \omega_1, \omega_2 \rangle = \mathrm{Tr}(\omega_1 \cup \omega_2) := \frac{1}{(2\pi i)^3} \int_X \omega_1 \cup \omega_2. \tag{1}$$

All the structure above, that is, the de Rham cohomology, its Hodge filtration and intersection form, is also defined over k; that is, they do not depend on the embedding $\mathsf{k} \hookrightarrow \mathbb{C}$, see for instance Deligne's lecture notes [DMOS82]. Let $\mathsf{h} = \dim(F^2) - 1$ and let Φ be the following constant matrix:

$$\Phi := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & \mathbb{1}_{\mathsf{h}\times\mathsf{h}} & 0 \\ 0 & -\mathbb{1}_{\mathsf{h}\times\mathsf{h}} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

Here, we use $(2\mathsf{h} + 2) \times (2\mathsf{h} + 2)$ block matrices according to the decomposition $2\mathsf{h} + 2 = 1 + \mathsf{h} + \mathsf{h} + 1$ and $\mathbb{1}_{\mathsf{h}\times\mathsf{h}}$ denotes the $\mathsf{h} \times \mathsf{h}$ identity matrix. The following definition is taken from Ref. [Mov13]. An enhanced Calabi–Yau threefold is a pair $(X, [\omega_1, \omega_2, \ldots, \omega_{2\mathsf{h}+2}])$, where X is as before and $\omega_1, \omega_2, \ldots, \omega_{2\mathsf{h}+2}$ is a basis of $H^3_{\mathrm{dR}}(X)$. We choose the basis such that

1. It is compatible with the Hodge filtration, that is, $\omega_1 \in F^3$, $\omega_1, \omega_2, \ldots, \omega_{\mathsf{h}+1} \in F^2$, $\omega_1, \omega_2, \ldots, \omega_{2\mathsf{h}+1} \in F^1$ and $\omega_1, \omega_2, \ldots, \omega_{2\mathsf{h}+2} \in F^0$.

2. The intersection form in this basis is the constant matrix Φ:

$$[\langle \omega_i, \omega_j \rangle] = \Phi. \tag{3}$$

Let T be the moduli of enhanced Calabi–Yau threefolds of a fixed topological type. The algebraic group

$$\mathsf{G} := \left\{ \mathsf{g} \in \mathrm{GL}(2\mathsf{h}+2, \mathsf{k}) \mid \mathsf{g} \text{ is block upper triangular and } \mathsf{g}^{\mathsf{tr}} \Phi \mathsf{g} = \Phi \right\} \tag{4}$$

acts from the right on T and its Lie algebra plays an important role in our main theorem.

$$\mathrm{Lie}(\mathsf{G}) = \left\{ \mathfrak{g} \in \mathrm{Mat}(2\mathsf{h}+2, \mathsf{k}) \mid \mathfrak{g} \text{ is block upper triangular and } \mathfrak{g}^{\mathsf{tr}} \Phi + \Phi \mathfrak{g} = 0 \right\}. \tag{5}$$

Here, by block triangular we mean triangular with respect to the partition $2\mathsf{h}+2 = 1+\mathsf{h}+\mathsf{h}+1$. We have

$$\dim(\mathsf{G}) = \frac{3\mathsf{h}^2+5\mathsf{h}+4}{2}, \quad \dim(\mathsf{T}) = \mathsf{h} + \dim(\mathsf{G}).$$

Special geometry and period manipulations suggest that T has a canonical structure of an affine variety over $\bar{\mathbb{Q}}$ and the action of G on T is algebraic. We have a universal family X/T of Calabi–Yau threefolds and by our construction we have elements $\tilde{\omega}_i \in H^3(\mathsf{X}/\mathsf{T})$ such that $\tilde{\omega}_i$ restricted to the fiber X_t is the chosen $\omega_i \in H^3_{\mathrm{dR}}(\mathsf{X}_\mathsf{t})$. For simplicity we write $\tilde{\omega}_i = \omega_i$. Here, $H^3(\mathsf{X}/\mathsf{T})$ denotes the set of global sections of the relative third de Rham cohomology of X over T. Furthermore, there is an affine variety $\tilde{\mathsf{T}}$ such that T is a Zarski open subset of $\tilde{\mathsf{T}}$, the action of G on T extends to $\tilde{\mathsf{T}}$ and the quotient $\tilde{\mathsf{T}}/\mathsf{G}$ is a projective variety (and hence compact). All the above statements can be verified for instance for mirror quintic Calabi–Yau threefold, see Sect. 5. Since we have now a good understanding of the classical moduli of Calabi–Yau varieties both in complex and algebraic context (see respectively Refs. [Vie95] and [Tod03]), verifying the above statements is not out of reach. For the purpose of the present text either assume that the universal family X/T over $\bar{\mathbb{Q}}$ exists or assume that T is the total space of the choices of the basis ω_i over a local patch of the moduli space of X. By Bogomolov–Tian–Todorov Theorem such a moduli space is smooth. We further assume that in a local patch of moduli space, the universal family of Calabi–Yau threefolds X exists and no Calabi–Yau threefold X in such a local patch has an isomorphism that acts non-identically on $H^3_{\mathrm{dR}}(X)$. In the last case one has to replace all the algebraic notations below by their holomorphic counterpart. Let

$$\nabla : H^3_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to \Omega^1_\mathsf{T} \otimes_{\mathcal{O}_\mathsf{T}} H^3_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$$

be the algebraic Gauss–Manin connection of the family X/T due to Katz–Oda [KO68], where $\mathcal{O}_\mathsf{T}$ is the $\bar{\mathbb{Q}}$-algebra of regular functions on T and Ω^1_T is the $\mathcal{O}_\mathsf{T}$-module of differential 1-forms in T. For any vector field R in T, let ∇_R be the Gauss–Manin connection composed with the vector field R. We write

$$\nabla_\mathsf{R} \omega = \mathsf{A}_\mathsf{R} \omega,$$

where A_R is a $(2\mathsf{h}+2) \times (2\mathsf{h}+2)$ matrix with entries in $\mathcal{O}_\mathsf{T}$ and $\omega := [\omega_1, \ldots, \omega_{2\mathsf{h}+2}]^{\mathsf{tr}}$.

To state our main theorem we will introduce some physics notation that will be useful in the rest of the paper. We split the notation for the basis ω in the following way

$[\omega_1, \omega_2, \ldots, \omega_{2\mathsf{h}+2}] = [\alpha_0, \alpha_i, \beta^i, \beta^0]$, $i = 1, 2, \ldots, \mathsf{h}$. The distinction between upper and lower indices here does not yet carry particular meaning. They are chosen such that they are compatible with the physics convention of summing over repeated upper and lower indices. We will write out matrices in terms of their components, denoting by an index i the rows and an index j the columns. We further introduce δ_i^j, which is 1 when $i = j$ and zero otherwise.

Theorem 1. *We have the following*

1. *There are unique vector fields* R_k, $k = 1, 2, \ldots, \mathsf{h}$ *in* T *and unique* $\mathsf{C}^{\mathrm{alg}}_{ijk} \in \mathcal{O}_{\mathsf{T}}$, $i, j, k = 1, 2, \ldots, \mathsf{h}$ *symmetric in* i, j, k *such that*

$$\mathsf{A}_{\mathsf{R}_k} = \begin{pmatrix} 0 & \delta_k^j & 0 & 0 \\ 0 & 0 & \mathsf{C}^{\mathrm{alg}}_{kij} & 0 \\ 0 & 0 & 0 & \delta_k^i \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6}$$

Further

$$\mathsf{R}_{i_1}\mathsf{C}^{\mathrm{alg}}_{i_2 i_3 i_4} = \mathsf{R}_{i_2}\mathsf{C}^{\mathrm{alg}}_{i_1 i_3 i_4}. \tag{7}$$

2. *For any* $\mathfrak{g} \in \mathrm{Lie}(\mathsf{G})$ *there is also a unique vector field* $\mathsf{R}_{\mathfrak{g}}$ *in* T *such that*

$$\mathsf{A}_{\mathsf{R}_{\mathfrak{g}}} = \mathfrak{g}^{\mathsf{tr}}. \tag{8}$$

Our proof of Theorem 1 is based on techniques from special geometry which deal with periods of Calabi–Yau varieties, see for instance Refs. [CdlO91, CDLOGP91, Str90, CDF+97, Ali13] and Sect. 2.1. For particular examples, such as the mirror quintic, one can give an algebraic proof which is merely computational, see Sect. 5. Further partial results in this direction are obtained in Ref. [Nik14]. The vector fields $\mathsf{R}_{\mathfrak{g}}$ can be derived from the action of G on T and (8) can be proved in a purely algebraic context. This will be discussed in subsequent works.

The $\mathcal{O}_{\mathsf{T}}$-module $\mathfrak{G}$ generated by the vector fields

$$\mathsf{R}_i, \quad \mathsf{R}_{\mathfrak{g}}, \quad i = 1, 2, \ldots, \mathsf{h}, \quad \mathfrak{g} \in \mathrm{Lie}(\mathsf{G}), \tag{9}$$

form a $\dim(\mathsf{T})$-dimensional Lie algebra with the usual bracket of vector fields. In the case of enhanced moduli of elliptic curves, one gets the classical Lie algebra $\mathfrak{sl}_2$, see for instance Refs. [Mov12b, Gui07]. Our main motivation for introducing such vector fields is that they are basic ingredients for an algebraic version of the Bershadsky–Cecotti–Ooguri–Vafa holomorphic anomaly equations [BCOV93, BCOV94]. First, we choose a basis of $\mathrm{Lie}(\mathsf{G})$:

$$\mathfrak{t}_{ab} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(\delta_a^i\delta_b^j + \delta_b^i\delta_a^j) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mathsf{tr}}, \quad \mathfrak{t}_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\delta_a^i & 0 & 0 & 0 \\ 0 & \delta_a^j & 0 & 0 \end{pmatrix}^{\mathsf{tr}}, \quad \mathfrak{t} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}^{\mathsf{tr}},$$

$$\mathfrak{k}_a := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta_i^a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_j^a & 0 \end{pmatrix}^{\mathsf{tr}}, \quad \mathfrak{g}_b^a := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\delta_i^a\delta_b^j & 0 & 0 \\ 0 & 0 & \delta_b^i\delta_j^a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mathsf{tr}}, \quad \mathfrak{g}_0 := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mathsf{tr}} \tag{10}$$

and we call it the canonical basis. The Lie algebra structure of $\mathfrak{G}$ is given by the following tables.

	$\mathsf{R}_{\mathfrak{g}_0}$	$\mathsf{R}_{\mathfrak{g}^d_c}$	$\mathsf{R}_{\mathsf{t}_{cd}}$	$\mathsf{R}_{\mathsf{t}_c}$
$\mathsf{R}_{\mathfrak{g}_0}$	0	0	0	$-\mathsf{R}_{\mathsf{t}_c}$
$\mathsf{R}_{\mathfrak{g}^a_b}$	0	0	$-\delta^a_c\mathsf{R}_{\mathsf{t}_{bd}}-\delta^a_d\mathsf{R}_{\mathsf{t}_{bc}}$	$-\delta^a_c\mathsf{R}_{\mathsf{t}_b}$
$\mathsf{R}_{\mathsf{t}_{ab}}$	0	$\delta^d_a\mathsf{R}_{\mathsf{t}_{bc}}+\delta^d_b\mathsf{R}_{\mathsf{t}_{ac}}$	0	0
$\mathsf{R}_{\mathsf{t}_a}$	$\mathsf{R}_{\mathsf{t}_a}$	$\delta^d_a\mathsf{R}_{\mathsf{t}_c}$	0	0
R_{t}	$2\mathsf{R}_{\mathsf{t}}$	0	0	0
$\mathsf{R}_{\mathfrak{k}_a}$	$\mathsf{R}_{\mathfrak{k}_a}$	$-\delta^a_c\mathsf{R}_{\mathfrak{k}_d}$	$-\frac{1}{2}(\delta^a_c\mathsf{R}_{\mathsf{t}_d}+\delta^a_d\mathsf{R}_{\mathsf{t}_c})$	$-2\delta^a_c\mathsf{R}_{\mathsf{t}}$
R_a	$-\mathsf{R}_a$	$\delta^d_a\mathsf{R}_c$	$\frac{1}{2}(\mathsf{C}^{\mathrm{alg}}_{ade}\mathsf{R}_{\mathfrak{g}^e_c}+\mathsf{C}^{\mathrm{alg}}_{ace}\mathsf{R}_{\mathfrak{g}^e_d})$	$-2\mathsf{R}_{\mathsf{t}_{ac}}+\mathsf{C}^{\mathrm{alg}}_{ace}\mathsf{R}_{\mathfrak{k}_e}$

(11)

	R_{t}	$\mathsf{R}_{\mathfrak{k}_c}$	R_c
$\mathsf{R}_{\mathfrak{g}_0}$	$-2\mathsf{R}_{\mathsf{t}}$	$-\mathsf{R}_{\mathfrak{k}_c}$	R_c
$\mathsf{R}_{\mathfrak{g}^a_b}$	0	$\delta^c_b\mathsf{R}_{\mathfrak{k}_a}$	$-\delta^a_c\mathsf{R}_b$
$\mathsf{R}_{\mathsf{t}_{ab}}$	0	$\frac{1}{2}(\delta^c_a\mathsf{R}_{\mathsf{t}_b}+\delta^c_b\mathsf{R}_{\mathsf{t}_a})$	$-\frac{1}{2}(\mathsf{C}^{\mathrm{alg}}_{cbd}\mathsf{R}_{\mathfrak{g}^d_a}+\mathsf{C}^{\mathrm{alg}}_{acd}\mathsf{R}_{\mathfrak{g}^d_b})$
$\mathsf{R}_{\mathsf{t}_a}$	0	$2\delta^c_a\mathsf{R}_{\mathsf{t}}$	$2\mathsf{R}_{\mathsf{t}_{ac}}-\mathsf{C}^{\mathrm{alg}}_{acd}\mathsf{R}_{\mathfrak{k}_d}$
R_{t}	0	0	$\mathsf{R}_{\mathsf{t}_c}$
$\mathsf{R}_{\mathfrak{k}_a}$	0	0	$-\delta^a_c\mathsf{R}_{\mathfrak{g}_0}+\mathsf{R}_{\mathfrak{g}^a_c}$
R_a	$-\mathsf{R}_{\mathsf{t}_a}$	$\delta^c_a\mathsf{R}_{\mathfrak{g}_0}-\mathsf{R}_{\mathfrak{g}^c_a}$	0

(12)

The genus one topological string partition function $\mathsf{F}^{\mathrm{alg}}_1$ belongs to $\log(\mathcal{O}^*_{\mathsf{T}})$, where $\mathcal{O}^*_{\mathsf{T}}$ is the set of invertible regular functions in T, and it satisfies the following equations:

$$\mathsf{R}_{\mathfrak{g}_0}\mathsf{F}^{\mathrm{alg}}_1=-\frac{1}{2}(2+\mathsf{h}+\frac{\chi}{12}), \tag{13}$$

$$\mathsf{R}_{\mathfrak{g}^a_b}\mathsf{F}^{\mathrm{alg}}_1=-\frac{1}{2}\delta^b_a, \tag{14}$$

$$\mathsf{R}_{\mathfrak{g}}\mathsf{F}^{\mathrm{alg}}_1=0,\quad \text{all other } \mathfrak{g} \text{ of the canonical basis of } \mathrm{Lie}(\mathsf{G}). \tag{15}$$

Here, χ is the Euler number of the Calabi–Yau variety X. The genus g topological string partition function $\mathsf{F}^{\mathrm{alg}}_g\in\mathcal{O}_{\mathsf{T}}$ turns out to be a regular function in T. The holomorphic anomaly equations in the polynomial formulation of Refs. [YY04, AL07] can be written in terms of vector fields:

$$\begin{aligned}
\mathsf{R}_{\mathfrak{t}_{ab}}\mathsf{F}^{\mathrm{alg}}_g&=\frac{1}{2}\sum_{h=1}^{g-1}\mathsf{R}_a\mathsf{F}^{\mathrm{alg}}_h\,\mathsf{R}_b\mathsf{F}^{\mathrm{alg}}_{g-h}+\frac{1}{2}\mathsf{R}_a\mathsf{R}_b\mathsf{F}^{\mathrm{alg}}_{g-1},\\
\mathsf{R}_{\mathfrak{k}_a}\mathsf{F}^{\mathrm{alg}}_g&=0,\\
\mathsf{R}_{\mathfrak{g}_0}\mathsf{F}^{\mathrm{alg}}_g&=(2g-2)\mathsf{F}^{\mathrm{alg}}_g,\\
\mathsf{R}_{\mathfrak{g}^b_a}\mathsf{F}^{\mathrm{alg}}_g&=0.
\end{aligned} \tag{16}$$

The functions $\mathsf{F}^{\mathrm{alg}}_g$ are not defined uniquely by the algebraic holomorphic anomaly equation as above. Let S be the moduli of (X,ω_1), where X is a Calabi–Yau threefold as above and ω_1 is a holomorphic differential 3-form on X. We have a canonical projection $\mathsf{T}\to\mathsf{S}$ which is obtained by neglecting all ω_i's except ω_1. It is characterized by the fact that $f\in\mathcal{O}_{\mathsf{S}}$ does not depend on the choice of $\omega_2,\ldots,\omega_{2\mathsf{h}+2}$. We also expect that S

has a canonical structure of an affine variety over $\bar{\mathbb{Q}}$ such that $\mathsf{T} \to \mathsf{S}$ is a morphism of affine varieties. We get a sub-algebra $\mathcal{O}_{\mathsf{S}}$ of $\mathcal{O}_{\mathsf{T}}$ which is characterized by the following:

Theorem 2. *We have*

$$\bigcap_{\mathfrak{g} \in \text{ canonical basis } \mathfrak{g} \neq \mathfrak{g}_0} \ker(\mathsf{R}_{\mathfrak{g}}) = \mathcal{O}_{\mathsf{S}}, \tag{17}$$

where we regard a vector field in T *as a derivation in* $\mathcal{O}_{\mathsf{T}}$.

This means that $\mathsf{F}_g^{\mathrm{alg}},\ \ g \geq 2$ (resp. $\mathsf{F}_1^{\mathrm{alg}}$) is defined up to addition of an element of $\mathcal{O}_{\mathsf{S}}$ (resp. $\log(\mathcal{O}_{\mathsf{S}}^*)$), which is called the ambiguity of $\mathsf{F}_g^{\mathrm{alg}}$. The algebra $\mathcal{O}_{\mathsf{T}}$ can be considered as a generalization of the classical algebra of quasi-modular forms. For a discussion of the q-expansion of its elements see Refs. [YY04,AL07,Mov11]. Once we compute the q-expansion of a set of generators of the algebra $\mathcal{O}_{\mathsf{T}}$ we substitute them in $\mathsf{F}_g^{\mathrm{alg}}$ and compute the Gromov–Witten invariants of the A-model Calabi–Yau threefold $\check{X}$, see Sect. 5 for the case of mirror quintic. The results are based on a considerable amount of machine and hand computations, which are suppressed in this paper to enhance readability.

The text is organized in the following way. In Sect. 2 we review basic facts about special geometry, the original BCOV holomorphic anomaly equation, the polynomial structure of topological string partition functions. New manipulations of periods inspired by our geometric approach are explained in Sects. 3.2 and 3.4. In these sections we work in the analytic setting, i.e. with complex differential geometry. The main aspect of Sect. 3, however, is that the results admit an algebraic formulation. This fact is essential for Sect. 4 which is dedicated to the proofs of our main theorems. We start with a brief discussion of how the analytic and algebraic settings are related in general, and we explain the various descriptions of the topological string partition functions and their interrelations between the analytic and algebraic formulations. Then, in Sect. 4.2 we first recall the definition of a generalized period domain for Calabi–Yau threefolds. Via the generalized period maps, we interpret the polynomial generators and topological string partition functions as functions on the moduli space T. The algebraic content of the period manipulations of special geometry are explained in Sect. 4.6. Explicit computations of the vector fields (9) and the construction of the moduli space T in the case of mirror quintic is explained in Sect. 5. Finally, in Sect. 6 we review some works for the future and possible applications of our algebraic anomaly equation.

2. Holomorphic Anomaly Equations

In this section we review some basic formulas used in special geometry of Calabi–Yau threefolds. We use physics conventions of writing out the components of geometric objects and for handling indices. In general (lower) upper indices will denote components of (co-) tangent space. Identical lower and upper indices are summed over, i. e. $x^i y_i := \sum_i x^i y_i$. For derivatives w.r.t. coordinates x^i we will write $\partial_i := \frac{\partial}{\partial x^i}$ and $\partial_{\bar{i}} := \frac{\partial}{\partial \bar{x}^{\bar{i}}}$. The inverse of a matrix $[M_{ij}]$ is denoted by $[M^{ij}]$. We define δ^i_j to be 1 if $i = j$ and 0 otherwise.

2.1. Special geometry. By Bogomolov–Tian–Todorov the moduli space $\mathcal{M}$ of projective Calabi–Yau threefolds is smooth and hence we can take local coordinates $z=(z^1,z^2,\dots,z^{\mathsf{h}})\in(\mathbb{C}^{\mathsf{h}},0)$ for an open set U in such a moduli space. In our context, $\mathcal{M}$ is just the quotient of T by the action of the algebraic group G. We denote by $\Omega=\Omega_z$ a holomorphic family of 3-forms on the Calabi–Yau threefold X_z. The geometry of $\mathcal{M}$ can be best described using the third cohomology bundle $\mathcal{H}\to\mathcal{M}$, where the fiber of $\mathcal{H}$ at a point $z\in\mathcal{M}$ is $\mathcal{H}_z=H^3(X_z,\mathbb{C})$. This bundle can be decomposed into sub-bundles in the following way:

$$\mathcal{H}=\mathcal{L}\oplus(\mathcal{L}\otimes T\mathcal{M})\oplus\overline{(\mathcal{L}\otimes T\mathcal{M})}\oplus\overline{\mathcal{L}}, \tag{18}$$

where $\mathcal{L}$ is the line bundle of holomorphic $(3,0)$ forms in X_z, $T\mathcal{M}$ denotes the holomorphic tangent bundle and the overline denotes complex conjugation. This structure gives the Hodge decomposition of the variation of Hodge structure arising from the Calabi–Yau threefolds X_z. It carries the intersection form in cohomology

$$\langle\cdot,\cdot\rangle:\mathcal{H}\times\mathcal{H}\to\mathbb{C},$$

which is given by the same formula as in (1). Let

$$\partial_i=\partial/\partial z^i,\ \partial_{\bar{\jmath}}=\partial/\partial\bar{z}^J$$

These are sections of $T\mathcal{M}$ and Ω is a section of $\mathcal{L}$. We get in canonical way global sections of the (symmetric) tensor product of $\mathcal{L}$, $T\mathcal{M}$ and $T^*\mathcal{M}$ which form a basis at each fiber. Any other section can be written as a linear combination of such a basis and we treat such coefficients as if they are sections themselves. Let

$$K:=-\log\langle\Omega,\overline{\Omega}\rangle,$$

be the Kähler potential. It provides a Kähler form for a Kähler metric on $\mathcal{M}$, whose components and Levi–Civita connection are given by:

$$G_{i\bar{\jmath}}:=\partial_i\partial_{\bar{\jmath}}K,\quad \Gamma^k_{ij}=G^{k\bar{k}}\partial_i G_{j\bar{k}}. \tag{19}$$

The description of the change of the decomposition of $\mathcal{H}$ into sub-bundles is captured by the holomorphic Yukawa couplings or threepoint functions

$$C_{ijk}:=-\langle\Omega,\partial_i\partial_j\partial_k\Omega\rangle\in\Gamma\left(\mathcal{L}^2\otimes\mathrm{Sym}^3T^*\mathcal{M}\right), \tag{20}$$

which satisfy

$$\partial_{\bar{\imath}}C_{ijk}=0,\quad D_iC_{jkl}=D_jC_{ikl}, \tag{21}$$

the curvature is then expressed as [BCOV94]:

$$[\bar{\partial}_{\bar{\imath}},D_i]^l{}_j=\bar{\partial}_{\bar{\imath}}\Gamma^l_{ij}=\delta^l_iG_{j\bar{\imath}}+\delta^l_jG_{i\bar{\imath}}-C_{ijk}\overline{C}^{kl}_{\bar{\imath}}, \tag{22}$$

where

$$\overline{C}^{jk}_{\bar{\imath}}:=e^{2K}G^{k\bar{k}}G^{j\bar{\jmath}}\overline{C}_{\bar{\imath}\bar{k}\bar{\jmath}}, \tag{23}$$

and where D_i is the covariant derivative defined using the connections Γ^k_{ij} and K_i, for example for A^k_j a section of $\mathcal{L}^n\otimes T^*\mathcal{M}\otimes T\mathcal{M}$ we have

$$D_iA^k_j:=\partial_i\,A^k_j-\Gamma^m_{ij}A^k_m+\Gamma^k_{im}A^m_j+n\,K_iA^k_j.$$

We further introduce the objects S^{ij}, S^i, S, which are sections of $\mathcal{L}^{-2}\otimes\mathrm{Sym}^mT\mathcal{M}$ with $m=2,1,0$, respectively, and give local potentials for the non-holomorphic Yukawa couplings:

$$\partial_{\bar{\imath}}S^{ij}=\overline{C}^{ij}_{\bar{\imath}},\qquad \partial_{\bar{\imath}}S^j=G_{i\bar{\imath}}S^{ij},\qquad \partial_{\bar{\imath}}S=G_{i\bar{\imath}}S^i. \tag{24}$$

2.2. Special coordinates. We pick a symplectic basis $\{A^I, B_J\}$, $I, J = 0, \ldots, \mathsf{h}$ of $H_3(X_z, \mathbb{Z})$, satisfying

$$A^I \cdot B_J = \delta_I^J, \quad A^I \cdot A^J = 0, \quad B_I \cdot B_J = 0, \tag{25}$$

We write the periods of the holomorphic (3, 0) form over this basis:

$$X^I(z) := \int_{A^I} \Omega_z, \quad \mathcal{F}_J(z) := \int_{B_J} \Omega_z. \tag{26}$$

The periods $X^I(z), \mathcal{F}_J(z)$ satisfy the Picard–Fuchs equations of the Calabi–Yau family X_z. The periods X^I can be identified with projective coordinates on $\mathcal{M}$ and $\mathcal{F}_J$ with derivatives of a homogeneous function $\mathcal{F}(X^I)$ of weight 2 such that $\mathcal{F}_J = \frac{\partial \mathcal{F}(X^I)}{\partial X^J}$. In a patch where $X^0(z) \neq 0$ a set of special coordinates can be defined

$$t^a = \frac{X^a}{X^0}, \quad a = 1, \ldots, \mathsf{h}. \tag{27}$$

The normalized holomorphic (3, 0) form $\tilde{\Omega}_t := (X^0)^{-1}\Omega_z$ has the periods:

$$\int_{A^0, A^a, B_b, B_0} \tilde{\Omega}_t = \left(1, t^a, F_b(t), 2F_0(t) - t^c F_c(t)\right), \tag{28}$$

where

$$F_0(t) = (X^0)^{-2}\mathcal{F} \quad \text{and} \quad F_a(t) := \partial_a F_0(t) = \frac{\partial F_0(t)}{\partial t^a}.$$

$F_0(t)$ is the called the prepotential and

$$C_{abc} = \partial_a \partial_b \partial_c F_0(t). \tag{29}$$

are the Yukawa coupling in the special coordinates t^a. See Refs. [CDF+97, Ali13] for more details.

2.3. Holomorphic anomaly equations. The genus g topological string amplitude $\mathsf{F}_g^{\text{non}}$ are defined in Ref. [BCOV93] for $g = 1$ and Ref. [BCOV94] for $g \geq 2$. It is a section of the line bundle $\mathcal{L}^{2-2g}$ over $\mathcal{M}$. They are related recursively in g by the holomorphic anomaly equations [BCOV93]

$$\bar{\partial}_{\bar{i}} \partial_j \mathsf{F}_1^{\text{non}} = \frac{1}{2} C_{jkl} \overline{C}_{\bar{i}}^{kl} + (1 + \frac{\chi}{24}) G_{j\bar{i}}, \tag{30}$$

where χ is the Euler character of B-model CY threefold, and [BCOV94]

$$\bar{\partial}_{\bar{i}} \mathsf{F}_g^{\text{non}} = \frac{1}{2} \overline{C}_{\bar{i}}^{jk} \left(\sum_{r=1}^{g-1} D_j \mathsf{F}_r^{\text{non}} D_k \mathsf{F}_{(g-r)}^{\text{non}} + D_j D_k \mathsf{F}_{g-1}^{\text{non}} \right). \tag{31}$$

Note that $D_i \mathsf{F}_g^{\text{non}}$ is a section of $\mathcal{L}^{2-2g} \otimes T^*\mathcal{M}$.

2.4. Polynomial structure. In Ref. [YY04] it was shown that the topological string amplitudes for the mirror quintic can be expressed as polynomials in finitely many generators of differential ring of multi-derivatives of the connections of special geometry. This construction was generalized in Ref. [AL07] for any CY threefold. It was shown there that $\mathsf{F}_g^{\text{non}}$ is a polynomial of degree $3g-3$ in the generators S^{ij}, S^i, S, K_i, where degrees 1, 2, 3, 1 were assigned to these generators respectively. The proof was given inductively and relies on the closure of these generators under the holomorphic derivative [AL07]. The purely holomorphic part of the construction as well as the coefficients of the monomials would be rational functions in the algebraic moduli, this was further discussed in Refs. [ALM10,Hos08].

It was further shown in Ref. [AL07], following Ref. [YY04], that the L.H.S. of Eq. (31) could be written in terms of the generators using the chain rule:

$$\partial_{\bar{\imath}} \mathsf{F}_g^{\text{non}} = \overline{C}_{\bar{\imath}}^{jk} \frac{\partial \mathsf{F}_g^{\text{non}}}{\partial S^{jk}} + G_{i\bar{\imath}} \left(S^{ij} \frac{\partial \mathsf{F}_g^{\text{non}}}{\partial S^j} + S^i \frac{\partial \mathsf{F}_g^{\text{non}}}{\partial S} + \frac{\partial \mathsf{F}_g^{\text{non}}}{\partial K_i} \right), \tag{32}$$

assuming the linear independence of $\overline{C}_{\bar{\imath}}^{jk}$ and $G_{i\bar{\imath}}$ over the field generated by the generators in Definition 1, the holomorphic anomaly equations (31) could then be written as two different sets of equations [AL07]:

$$\frac{\partial \mathsf{F}_g^{\text{non}}}{\partial S^{jk}} = \frac{1}{2} \sum_{r=1}^{g-1} D_j \mathsf{F}_r^{\text{non}} D_k \mathsf{F}_{(g-r)}^{\text{non}} + \frac{1}{2} D_j D_k \mathsf{F}_{(g-1)}^{\text{non}}, \tag{33}$$

$$S^{ij} \frac{\partial \mathsf{F}_g^{\text{non}}}{\partial S^j} + S^i \frac{\partial \mathsf{F}_g^{\text{non}}}{\partial S} + \frac{\partial \mathsf{F}_g^{\text{non}}}{\partial K_i} = 0. \tag{34}$$

This linear independence assumption is not at all a trivial statement. Its proof in the one parameter case can be done using a differential Galois theory argument as in the proof of Theorem 2 in [Mov11] and one might try to generalize such an argument to multi parameter case. However, for the purpose of the present text we do not need to prove it. The reason is that (33) and (34) always have a solution which gives a solution to (32).

3. Algebraic Structure of Topological String Theory

In this section we develop the new ingredients and tools which will allow us to phrase the algebraic structure of topological string theory. We start by enhancing the differential polynomial ring of Ref. [AL07] with further generators which parameterize a choice of section of the line bundle $\mathcal{L}$ and a choice of coordinates as was done in Ref. [ASYZ13] for one dimensional moduli spaces. We will then show that these new generators parameterize different choices of forms compatible with the Hodge filtration and having constant symplectic pairing.

3.1. Special polynomial rings. We first fix the notion of holomorphic limit discussed in Ref. [BCOV94]. For our purposes we think of the limit as an assignment:

$$e^{-K}|_{\text{hol}} = \mathsf{h}_0 X^0, \quad G_{i\bar{j}}|_{\text{hol}} = \mathsf{h}_{a\bar{j}} \frac{\partial t^a}{\partial z^i} \tag{35}$$

for a given choice of section X^0 of $\mathcal{L}$ and a choice of special coordinates t^a where h_0 is a constant and $\mathsf{h}_{a\bar{\imath}}$ denote the components of a constant matrix.

Definition 1. The generators of the special polynomial differential ring are defined by

$$g_0 := \mathsf{h}_0^{-1} e^{-K}, \tag{36}$$
$$g_i^a := e^{-K} G_{i\bar{j}} \mathsf{h}^{\bar{j}a}, \tag{37}$$
$$T^{ab} := g_i^a\, g_j^b S^{ij}, \tag{38}$$
$$T^a := g_0\, g_i^a (S^i - S^{ij} K_j), \tag{39}$$
$$T := g_0^2 (S - S^i K_i + \frac{1}{2} S^{ij} K_i K_j), \tag{40}$$
$$L_a = g_0 (g^{-1})_a^i \partial_i K. \tag{41}$$

We will use the same notation for these generators and for their holomorphic limit.

Proposition 1. *The generators of the special polynomial ring satisfy the following differential equations, called the differential ring*:

$$\partial_a g_0 = -L_a\, g_0, \tag{42}$$
$$\partial_a g_i^b = g_i^c \left(\delta_a^b\, L_c - C_{cad} T^{bd} + g_0\, s_{ca}^b\right), \tag{43}$$
$$\partial_a T^{bc} = \delta_a^b (T^c + T^{cd} L_d) + \delta_a^c (T^b + T^{bd} L_d) - C_{ade} T^{bd} T^{ce} + g_0\, h_a^{bc}, \tag{44}$$
$$\partial_a T^b = 2\delta_a^b (T + T^c L_c) - T^b L_a - k_{ac} T^{bc} + g_0^2\, h_a^b, \tag{45}$$
$$\partial_a T = \frac{1}{2} C_{abc} T^b T^c - 2 L_a T - k_{ab} T^b + g_0^3\, h_a, \tag{46}$$
$$\partial_a L_b = -L_a L_b - C_{abc} T^c + g_0^{-2}\, k_{ab}. \tag{47}$$

Proof. The first two equations follow from Definition 1 and the special geometry discussed in Sect. 2.1, the other equations follow from the definitions and the equations which were proved in Ref. [AL07]. □

The generators g_0, g_i^a are chosen such that their holomorphic limits become:

$$g_0|_{\text{hol}} = X^0, \quad g_i^a|_{\text{hol}} = X^0 \frac{\partial t^a}{\partial z^i}, \tag{48}$$

In these equations the functions s_{ij}^k, h_i^{jk}, h_i^j, h_i and k_{ij} are fixed once a choice of generators has been made and we transformed the indices from arbitrary local coordinates to the special coordinates using g_i^a and its inverse.

The freedom in choosing the generators S^{ij}, S^i, S was discussed in Refs. [ALM10, Hos08] and translates here to a freedom of adding holomorphic sections $\mathcal{E}^{ij}$, $\mathcal{E}^i$, $\mathcal{E}$ of $\mathcal{L}^{-2} \otimes \text{Sym}^m T\mathcal{M}$ with $m = 2, 1, 0$, respectively to the generators as follows:

$$T^{ab} \to T^{ab} + g_i^a\, g_j^b\, \mathcal{E}^{ij}, \tag{49}$$
$$T^a \to T^a + g_0\, g_i^a\, \mathcal{E}^i, \tag{50}$$
$$T \to T + g_0^2 \mathcal{E}. \tag{51}$$

It can be seen from the equations that there is additional freedom in defining the generators g_0, g_i^a and L_a given by:

$$L_a \to L_a + g_0 (g^{-1})_a^i \mathcal{E}_i, \tag{52}$$

$$g_0 \to \mathcal{C} g_0 \,, \tag{53}$$
$$g_i^a \to \mathcal{C}^i_j g_i^a \,, \tag{54}$$

where $\mathcal{C}$ denotes a holomorphic function, $\mathcal{C}^i_j$ a holomorphic section of $T\mathcal{M} \otimes T^*\mathcal{M}$ and $\mathcal{E}_i$ a holomorphic section of $T^*\mathcal{M}$.

The number of special polynomial generators matches $\dim(\mathsf{G})$, where G is the algebraic group in the Introduction.

Definition 2. We introduce:

$$\tilde{\mathsf{F}}_g^{\text{non}} := g_0^{2g-2} \mathsf{F}_g^{\text{non}} \,, \tag{55}$$

which defines a section of $\overline{\mathcal{L}}^{2g-2}$. After taking the holomorphic limit discussed earlier we get $\mathsf{F}_g^{\text{hol}}$ which will be a holomorphic function (and no longer a section) on the moduli space $\mathcal{M}$.

Proposition 2. $\tilde{\mathsf{F}}_g^{\text{non}}$*'s satisfy the following equations:*

$$\left(g_0 \frac{\partial}{\partial g_0} + L_a \frac{\partial}{\partial L_a} + T^a \frac{\partial}{\partial T^a} + 2T \frac{\partial}{\partial T} \right) \tilde{\mathsf{F}}_g^{\text{non}} = (2g-2) \tilde{\mathsf{F}}_g^{\text{non}} \tag{56}$$

$$\left(g_m^a \frac{\partial}{\partial g_m^b} + 2T^{ac} \frac{\partial}{\partial T^{bc}} + T^a \frac{\partial}{\partial T^b} - L_b \frac{\partial}{\partial L_a} \right) \tilde{\mathsf{F}}_g^{\text{non}} = 0, \tag{57}$$

$$\begin{aligned} &\left(\frac{\partial}{\partial T^{ab}} - \frac{1}{2}(L_b \frac{\partial}{\partial T^a} + L_a \frac{\partial}{\partial T^b}) + \frac{1}{2} L_a L_b \frac{\partial}{\partial T} \right) \tilde{\mathsf{F}}_g^{\text{non}} \\ &= \frac{1}{2} \sum_{r=1}^{g-1} \partial_a \tilde{\mathsf{F}}_r^{\text{non}} \, \partial_b \tilde{\mathsf{F}}_{g-r}^{\text{non}} + \frac{1}{2} \partial_a \partial_b \tilde{\mathsf{F}}_{g-1}^{\text{non}}, \end{aligned} \tag{58}$$

$$\frac{\partial \tilde{\mathsf{F}}_g^{\text{non}}}{\partial L_a} = 0. \tag{59}$$

$\mathsf{F}_g^{\text{hol}}$*'s satisfy the same equations in the holomorphic limit.*

Proof. The first two equations follow from the definition of $\tilde{\mathsf{F}}_g^{\text{non}}$ and the proof of Ref. [AL07], bearing in mind that the dependence on the generators g_0, g_i^a is introduced through the definition of the special polynomial generators and the factor g_0^{2g-2} in $\tilde{\mathsf{F}}_g^{\text{non}}$. The third and fourth equation are a re-writing of Eq. (33) using the special polynomial generators defined earlier. □

3.2. Different choices of Hodge filtrations. In order to parameterize the moduli space of a Calabi–Yau threefold enhanced with a choice of forms compatible with the Hodge filtration and having constant intersection, we seek to parameterize the relation between different choices of Hodge filtrations. We start with a non-holomorphic choice of a Hodge filtration $\vec{\omega}_z$ defined by arbitrary local coordinates on the moduli space and relate this to a choice of filtration in special coordinates $\vec{\omega}_t$. The choices are given by

$$\vec{\omega}_z = \begin{pmatrix} \alpha_{z,0} \\ \alpha_{z,i} \\ \beta_z^i \\ \beta_z^0 \end{pmatrix} = \begin{pmatrix} \Omega \\ \partial_i \Omega \\ (C_\sharp^{-1})^{ik} \partial_\sharp \partial_k \Omega \\ \partial_\sharp (C_\sharp^{-1})^{\sharp k} \partial_* \partial_k \Omega \end{pmatrix}, \tag{60}$$

where C_{ijk} are given by (20). Here, $A_\sharp = (g^{-1})^i_* A_i$, where $*$ denotes a fixed choice of special coordinate. One can write the holomorphic anonaly equation of [BCOV94] using the non-holomorphic basis $\vec{\omega}_z$ which is different from the standard holomorphic basis. This is the following after taking holomorphic limit

$$\vec{\omega}_t = \begin{pmatrix} \alpha_{t,0} \\ \alpha_{t,a} \\ \beta_t^a \\ \beta_t^0 \end{pmatrix} = \begin{pmatrix} \tilde{\Omega} \\ \partial_a \tilde{\Omega} \\ (C_*^{-1})^{ae} \partial_* \partial_e \tilde{\Omega} \\ \partial_* (C_*^{-1})^{*e} \partial_* \partial_e \tilde{\Omega} \end{pmatrix}, \tag{61}$$

where $\tilde{\Omega} = \tilde{\Omega}_t$ is given by (28) and $*$ denotes a fixed choice of special coordinate.

Proposition 3. *The period matrix of $\vec{\omega}_t$ over the symplectic basis of $H_3(X_z, \mathbb{Z})$ given in Eq.* (25) *has the following special format*:

$$[\int_{A^0, A^c, B_c, B_0} \begin{pmatrix} \alpha_{t,0} \\ \alpha_{t,a} \\ \beta_t^a \\ \beta_t^0 \end{pmatrix}] = \begin{pmatrix} 1 & t^c & F_c & 2F_0 - t^d F_d \\ 0 & \delta_a^c & F_{ac} & F_a - t^d F_{ad} \\ 0 & 0 & \delta_c^a & -t^a \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{62}$$

with $F_a := \partial_a F_0, \quad \partial_a = \frac{\partial}{\partial t^a}$.

Proof. This follows from the defitions of Sect. 2.2. □

Proposition 4. *The symplectic form for both bases $\vec{\omega}_z$ and $\vec{\omega}_t$ is the matrix Φ in* (2).

Proof. The computation of the symplectic form for $\vec{\omega}_t$ follows from the Proposition 3. The symplectic form of $\vec{\omega}_z$ follows from the definition of C_{ijk} in (20) and from Griffiths transversality, for instance,

$$\int_{X_z} \partial_i \Omega \wedge (C_\sharp^{-1})^{jk} \partial_\sharp \partial_k \Omega = -(C_\sharp^{-1})^{jk} \int_{X_z} \Omega \wedge \partial_i \partial_\sharp \partial_k \Omega = (C_\sharp^{-1})^{jk} C_{\sharp ik} = \delta_i^j. \tag{63}$$

□

Proposition 5. *The flat choice $\vec{\omega}_t$ satisfies the following equation:*

$$\partial_b \begin{pmatrix} \tilde{\Omega} \\ \partial_a \tilde{\Omega} \\ (C_*^{-1})^{ae} \partial_* \partial_e \tilde{\Omega} \\ \partial_* (C_*^{-1})^{*e} \partial_* \partial_e \tilde{\Omega} \end{pmatrix} = \begin{pmatrix} 0 & \delta_b^c & 0 & 0 \\ 0 & 0 & C_{abc} & 0 \\ 0 & 0 & 0 & \delta_b^a \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\Omega} \\ \partial_c \tilde{\Omega} \\ (C_*^{-1})^{ce} \partial_* \partial_e \tilde{\Omega} \\ \partial_* (C_*^{-1})^{*e} \partial_* \partial_e \tilde{\Omega} \end{pmatrix}. \tag{64}$$

Proof. This follows from Eq. (62). □

We now want to find the matrix relating:

$$\vec{\omega}_t = \mathsf{B} \cdot \vec{\omega}_z, \tag{65}$$

and express its entries in terms of the polynomial generators. The matrix B is given by:

$$\mathsf{B} = \begin{pmatrix} g_0^{-1} & 0 & 0 & 0 \\ g_0^{-1} L_a & (g^{-1})^i_a & 0 & 0 \\ -g_0^{-1} \widehat{T}^a & (g^{-1})^i_d \widehat{T}^{ad} & g_i^a & 0 \\ -g_0^{-1} \left(2T + \widehat{T}^d L_d\right) + g_0 \mathcal{H} & (g^{-1})^i_d (T^d + \widehat{T}^{de} L_e) + g_0 \mathcal{H}^i & g_i^e L_e & g_0 \end{pmatrix}, \tag{66}$$

where a is an index for the rows and i for the columns and where:

$$\begin{aligned}
\widehat{T}^{a} &= T^{a} - g_0\, g_i^d \mathcal{E}^m, &(67)\\
\widehat{T}^{ab} &= T^{ab} - g_m^a\, g_n^b\, \mathcal{E}^{mn}, &(68)\\
\mathcal{H} &= g_j^*(g^{-1})_*^i(\partial_i \mathcal{E}^j + C_{imn}\mathcal{E}^{lj}\mathcal{E}^m - h_i^j) &(69)\\
\mathcal{H}^i &= g_m^*(g^{-1})_*^n(-\partial_n \mathcal{E}^{im} - C_{nlk}\mathcal{E}^{il}\mathcal{E}^{km} + \delta_n^i \mathcal{E}^m + h_n^{im}) &(70)\\
\mathcal{E}^{ik} &= (C_\sharp^{-1})^{ij} s_{\sharp j}^k, &(71)\\
\mathcal{E}^{i} &= (C_\sharp^{-1})^{ij} k_{\sharp j}. &(72)
\end{aligned}$$

The computation of G is done using the chain rule and the equalities in Proposition 1. >From Proposition 4 and the definition of the algebraic group G it follows that $\mathsf{B}^{\mathsf{tr}} \in \mathsf{G}$. This can be also verified using the explicit expression of B in (66). Note that we have written the matrix B after inserting the freedom in choosing the generators S^{ij}, S^i, S in (49) till (54). In this way the ambiguities $\mathcal{E}$, h_i^j e.t.c. appear in the expression of B and we get the most general base change $\mathsf{B}\cdot\vec{\omega}_z$ such that $\mathsf{B} \in \mathsf{G}$.

3.3. Lie Algebra description. In the following, we regard all the generators as independent variables and compute the following matrices:

$$\mathsf{M}_g = \frac{\partial}{\partial g}\mathsf{B}\cdot\mathsf{B}^{-1}, \tag{73}$$

where g denotes a generator and $\partial_g := \frac{\partial}{\partial g}$. We find

$$\begin{aligned}
\mathsf{M}_{T^{ab}} &= \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\delta_a^i L_b & \frac{1}{2}(\delta_a^i\delta_b^j + \delta_b^i\delta_a^j) & 0 & 0\\ -L_a L_b & \delta_b^j L_a & 0 & 0\end{pmatrix},\\
\mathsf{M}_{T^{a}} &= \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\delta_a^i & 0 & 0 & 0\\ -2L_a & \delta_a^j & 0 & 0\end{pmatrix},\\
\mathsf{M}_{T} &= \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -2 & 0 & 0 & 0\end{pmatrix}, &(74)\\
\mathsf{M}_{L_a} &= \begin{pmatrix} 0 & 0 & 0 & 0\\ \delta_i^a & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \delta_j^a & 0\end{pmatrix},\\
\mathsf{M}_{g_0} &= \begin{pmatrix} -g_0^{-1} & 0 & 0 & 0\\ -g_0^{-1}L_i & 0 & 0 & 0\\ g_0^{-1}T^i & 0 & 0 & 0\\ 2g_0^{-1}(2T_0 + T^d L_d) & -g_0^{-1}T^j & -g_0^{-1}L_j & g_0^{-1}\end{pmatrix}.
\end{aligned}$$

$$
\mathsf{M}_{g^a_m} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ g^m_i L_a & -\delta^j_a g^m_i & 0 & 0 \\ T^{id} g^m_d L_a + \delta^i_a g^m_d (T^d + T^{de} L_e) & -\delta^i_a T^{jd} g^m_d - \delta^j_a T^{id} g^m_d & \delta^i_a (g^{-1})^m_j & 0 \\ 2 L_a g^m_d (T^d + T^{de} L_e) & -\delta^j_a g^m_d (T^d + T^{de} L_e) - g^m_d T^{dj} L_a & (g^{-1})^m_j L_a & 0 \end{pmatrix}, \tag{75}
$$

We now look for combinations of the vector fields which give constant vector fields. We find the following:

$$
\begin{aligned}
\mathfrak{t}_{ab} &= \mathsf{M}_{T^{ab}} - \frac{1}{2}(L_a \mathsf{M}_{T^b} + L_b \mathsf{M}_{T^a}) + \frac{1}{2} L_a L_b \mathsf{M}_T, \\
\mathfrak{t}_a &= \mathsf{M}_{T^a} - L_a \mathsf{M}_T, \\
\mathfrak{t} &= \frac{1}{2} \mathsf{M}_T, \\
\mathfrak{k}^a &= \mathsf{M}_{L_a}, \\
\mathfrak{g}^a_b &= g^a_m \mathsf{M}_{\partial g^b_m} - L_b \mathsf{M}_{L_a} + 2 T^{ad} \mathsf{M}_{T^{db}} + T^a \mathsf{M}_{T^b}, \\
\mathfrak{g}_0 &= g_0 \mathsf{M}_{g_0} + L_a \mathsf{M}_{L_a} + T^a \mathsf{M}_{T^a} + 2 \mathsf{M}_T.
\end{aligned} \tag{76}
$$

Therefore, we get all the elements of the Lie algebra $\mathrm{Lie}(\mathsf{G})$ and for each $\mathfrak{g} \in \mathrm{Lie}(\mathsf{G})$ a derivation $\mathsf{R}_{\mathfrak{g}}$ in the generators.

3.4. Algebraic master anomaly equation. In genus zero we have

$$
C_{abc} = C_{ijk} (g^{-1})^i_a (g^{-1})^j_b (g^{-1})^k_c g_0, \tag{77}
$$

Its holomorphic limit is (29) and for simplicity we have used the same notation. The anomaly equations read:

$$
\mathsf{R}_{\mathfrak{g}_0} C_{abc} = C_{abc}, \tag{78}
$$

$$
\mathsf{R}_{\mathfrak{g}^b_a} C_{cde} = -\delta^b_c C_{ade} - \delta^b_d C_{cae} - \delta^b_e C_{cda}, \tag{79}
$$

and in genus one we obtain the anomaly equations (13) and (16) in the holomorphic context. We combine all $\mathsf{F}^{\mathrm{hol}}_g$ into the generating function:

$$
Z = \exp \sum_{g=1}^{\infty} \lambda^{2g-2} \mathsf{F}^{\mathrm{hol}}_g. \tag{80}
$$

The master anomaly equations become:

$$
\mathsf{R}_{\mathfrak{g}_0} Z = \left(-\frac{2 + \mathsf{h}}{2} - \frac{\chi}{24} + \theta_\lambda \right) Z, \tag{81}
$$

$$
\mathsf{R}_{\mathfrak{g}^b_a} Z = -\frac{1}{2} \delta^b_a Z, \tag{82}
$$

$$
\mathsf{R}_{\mathfrak{t}_{ab}} Z = \frac{\lambda^2}{2} \mathsf{R}_a \mathsf{R}_b Z, \tag{83}
$$

$$
\mathsf{R}_{\mathfrak{k}_a} Z = 0, \tag{84}
$$

$$\mathsf{R}_{\mathsf{t}_a} Z = \lambda^2 \left(-\frac{\chi}{24} + \theta_\lambda \right) \mathsf{R}_a Z, \tag{85}$$

$$\mathsf{R}_{\mathsf{t}} Z = \frac{\lambda^2}{2} \left(-\frac{\chi}{24} + \theta_\lambda \right) \left(-\frac{\chi}{24} - 1 + \theta_\lambda \right) Z. \tag{86}$$

4. Proofs

So far, we have used the language of special geometry in order to find several derivations in the special polynomial ring generators. This is essentially the main mathematical content of period manipulations in the B-model of mirror symmetry. In this section we interpret such derivations as vector fields in the moduli space T introduced in the Introduction and hence prove Theorems 1 and 2. We denote an element of T by t and hopefully it will not be confused with the special coordinates t of Sect. 2.2.

4.1. Holomorphic versus algebraic description. In order to give this interpretation we first need to relate the derivations found in the framework of special geometry, i.e., complex differential geometry, to the framework of algebraic geometry used in the Introduction.

The main observation is the following. Since polynomials over $\mathbb{C}$ can be viewed as holomorphic functions, an affine variety $X \subset \mathbb{A}^n_{\mathbb{C}}$ is also closed in the analytic topology on $\mathbb{C}^n$ and hence defines a complex analytic space. If X is smooth, it is even a complex submanifold. This assignment behaves well under gluing, and hence can be extended to the setting of projective varieties. More precisely, there is a functor, called analytification, from the category of schemes of finite type over $\mathbb{C}$ to the category of analytic spaces that associates to a scheme X its analytification X^{an}. Furthermore, there is a natural morphism of locally ringed spaces $\alpha : (X^{\mathrm{an}}, \mathcal{O}^{\mathrm{hol}}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$ such that $^{\mathrm{an}}$ is universal with this property, and is the identity on points. Here, $\mathcal{O}^{\mathrm{hol}}_{X^{\mathrm{an}}}$ is the sheaf of holomorphic functions on X^{an}, while $\mathcal{O}_X$ is the structure sheaf of the variety X. If X is smooth, then X^{an} is a complex manifold.

In the analytic setting, the holomorphic de Rham cohomology of the complex manifold X^{an} is defined as the hypercohomology $H^i_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}}) = H^i(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}})$ of the holomorphic de Rham complex. The latter is built from the sheaf of holomorphic differential p-forms on X^{an}. In a similar manner in the algebraic setting, the algebraic de Rham cohomology of the smooth variety X over $\mathbb{C}$ is defined as the hypercohomology $H^i_{\mathrm{dR}}(X) = H^i(X, \Omega^\bullet_X)$ of the algebraic de Rham complex. The latter is built from the p-th exterior product of the sheaf of Kähler differentials over $\mathbb{C}$ on X.

The observation that any algebraic differential is holomorphic yields a natural morphism of complexes $\alpha^{-1}\Omega^\bullet_X \to \Omega^\bullet_{X^{\mathrm{an}}}$ which induces a map $\alpha^* : H^i_{\mathrm{dR}}(X) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}})$ on cohomology. Serre has shown that if X is smooth and projective, then α^* is an isomorphism. This allows us to go back and forth from the analytic to the algebraic description at the level of cohomology. The analytifcation functor is also compatible with inverse and direct images. This property can be used to show that the algebraic and analytic Gauss–Manin connections are compatible.

This change of framework is important from the arithmetic point of view. It is well known that the Fourier coefficients of the topological string partition functions F_g satisfy integrality properties. The algebraic framework allows us to change the base field from $\mathbb{C}$ to, say, the field $\overline{\mathbb{Q}}$ of algebraic numbers, in which such arithmetic questions are more natural to study.

The results of Sect. 3 are originally derived for differential forms in complex geometry. The manipulations though were purely algebraic, hence as far as cohomological quantities are concerned they are also valid in the algebraic setting. In particular, the change of trivialization of the Hodge bundle in (65) also holds in the algebraic de Rham cohomology.

As far as the topological string partition functions F_g in the analytic and the algebraic setting are concerned, they cannot directly be compared due to the fact that the global map of the moduli space $\mathcal{M}$ into the classical period domain is known to be nonalgebraic in the case of Calabi–Yau threefolds. The embedding itself is not known (which manifests itself in the absence of a global Torelli theorem for Calabi–Yau threefolds). For these reasons, there are various versions of the topological string partition function. In Sect. 2.3 we introduced the nonholomorphic topological string partition function $\mathsf{F}_g^{\mathrm{non}}$ which is a nonholomorphic section of a line bundle over $\mathcal{M}$. In Sect. 3.1 we defined the notion of the holomorphic limit $f \mapsto f|_{\mathrm{hol}}$ which yields holomorphic sections $\mathsf{F}_g^{\mathrm{hol}} = \mathsf{F}_g^{\mathrm{non}}|_{\mathrm{hol}}$, still over $\mathcal{M}$. In the algebraic setting of the Introduction, we considered regular functions $\mathsf{F}_g^{\mathrm{alg}} \in \mathcal{O}_{\mathsf{T}}$, which are living on the moduli space T of enhanced Calabi–Yau threefolds, a much bigger space than $\mathcal{M}$. In order to relate $\mathsf{F}_g^{\mathrm{alg}}$ to $\mathsf{F}_g^{\mathrm{hol}}$ we would need a regular embedding $\mathcal{M} \to T$. As in the classical case, such an embedding is not known and also not expected to be regular. A workaround to this problem was first suggested in [Mov12a]. It consists of restricting to appropriate open subsets $U \subset \mathcal{M}$ and $\mathbb{H} \subset \mathsf{T}$, respectively. U can be thought of as a neighborhood around the point of maximal unipotent monodromy of the family $\pi : \mathcal{X} \to \mathcal{M}$, while $\mathbb{H}$ is chosen such that the generalized period map of the family $\mathsf{X} \to \mathsf{T}$ takes the standard form in special geometry, see Definition 3 below. With these choices, the relation between the analytic and the algebraic versions of the topological string partition function is

$$\mathsf{F}_g^{\mathrm{hol}}(X^{\mathrm{an}})|_U = \mathsf{F}_g^{\mathrm{alg}}(X)|_{\mathbb{H}},$$

and similarly for the Yukawa couplings

$$C_{ijk}|_U = \mathsf{C}_{ijk}^{\mathrm{alg}}|_{\mathbb{H}}$$

as formal power series in variables $q_1, \dots, q_{\mathsf{h}}$. These formal power series are expected to be Fourier series of generalized automorphic forms on the generalized period domain T. The left hand side has been worked out in [BCOV94] while the right hand side is the main focus of the present work and will be discussed below in more detail when proving the statements from the Introduction. This is new in the sense that we provide a purely algebraic framework for the topological string partition functions which does not rely anymore on analytic, i.e. Kähler geometry. Instead they are solutions to a set of algebraic differential equations, the algebraic version of the holomorphic anomaly equations. The advantage is that it is not necessary to look for an algebraic version of the propagators, even though it is implicitly provided.

4.2. Generalized period domain. The generalized period domain Π introduced in Ref. [Mov13], is the set of all $(2\mathsf{h}+2) \times (2\mathsf{h}+2)$-matrices P with complex entries which satisfies $\mathsf{P}^{\mathrm{tr}}\Psi\mathsf{P} = \Phi$ and a positivity condition which can be derived from the description below. Here, Ψ is the standard symplectic matrix and Φ is given in (3). The symplectic group $\mathrm{Sp}(2\mathsf{h}+2, \mathbb{Z})$ acts on Π from the left and the quotient

$$\mathsf{U} := \mathrm{Sp}(2\mathsf{h}+2, \mathbb{Z})\backslash\Pi,$$

parameterizes the set of all lattices L inside $H^3_{\mathrm{dR}}(X_0)$ such that the data $(L, H^3_{\mathrm{dR}}(X_0), F^*_0, \langle\cdot,\cdot\rangle)$ form a polarized Hodge structure. Here, X_0 is a fixed Calabi–Yau threefold as in the Introduction and $H^3_{\mathrm{dR}}(X_0)$, F^*_0, $\langle\cdot,\cdot\rangle$ are its de Rham cohomology, Hodge filtration and intersection form, respectively. It is defined in such a way that we have the period map

$$\mathsf{P} : \mathsf{T} \to \mathsf{U}, \ \ \mathsf{t} \mapsto [\int_{\delta_i} \omega_j], \tag{87}$$

where t now represent the pair (X, ω), δ_i is a symplectic basis of $H_3(X, \mathbb{Z})$. In order to have a modular form theory attached to the above picture, one has to find a special locus $\mathbb{H}$ in T. In the case of Calabi–Yau threefolds special geometry in Sect. 2.1 gives us the following candidate.

Definition 3. We define $\mathbb{H}$ to be the set of all elements $\mathsf{t} \in \mathsf{T}$ such that $\mathsf{P}(\mathsf{t})^{\mathsf{tr}}$ is of the form (62).

In the case of elliptic curves the set $\mathbb{H}$ is biregular to a punctured disc of radius one, see Ref. [Mov12c]. In our context we do not have a good understanding of the global behavior of $\mathbb{H}$. This is related to the analytic continuation of periods of Calabi–Yau threefolds. The set $\mathbb{H}$ is neither an algebraic nor an analytic subvariety of T. One can introduce a holomorphic foliation in T with $\mathbb{H}$ as its leaf and study its dynamics, see for instance [Mov08] for such a study in the case of elliptic curves. For the purpose of q-expansions, one only needs to know that a local patch of $\mathbb{H}$ is biregular to a complement of a normal crossing divisor in a small neighborhood of 0 in $\mathbb{C}^{\mathsf{h}}$.

In this abstract context of periods, we think of t^i as h independent variables and of F_0 as a function in the t^i's. The restriction of the ring of regular functions in T to $\mathbb{H}$ gives a $\mathbb{C}$-algebra which can be considered as a generalization of quasi-modular forms. We can do q-expansion of such functions around a degeneracy point of Calabi–Yau threefolds, see Ref. [Mov13] for more details.

4.3. Proof of Theorem 1. First, we assume that $\mathsf{k} = \mathbb{C}$ and work in a local patch U of the moduli space of Calabi–Yau threefolds $X_{z(t)},\ t \in U$, where $z(t)$ is the inverse map to (27). Therefore, we have assumed that over U the universal family $\mathcal{X}$ of Calabi–Yau threefolds exists. In (61) we have defined $\vec{\omega}_t$ and in Proposition 3 we have proved that $(X_{z(t)}, \vec{\omega}_t) \in \mathbb{H}$. Therefore by the discussion in Sect. 4.1, we have the following map

$$f : U \to \mathbb{H},\ t \mapsto (X_{z(t)}, \vec{\omega}_t). \tag{88}$$

Here, we implicitly use the fact that these objects both have an analytic and an algebraic description. In particular, the entries of the matrix B in (66) are rational functions in $z_i(t),\ i = 1, 2, \ldots, \mathsf{h}$ and the generators in Definition 1. The Gauss–Manin connection matrix restricted to the image of the map (88) and computed in the flat coordinates t is just $\sum_{i=1}^{\mathsf{h}} \tilde{\mathsf{A}}_i dt^i$, where $\tilde{\mathsf{A}}_i$ is the matrix computed in (64). The analytic functions $C_{ijk}|_U$ appearing there, however, are now replaced by their algebraic counterparts, $\mathsf{C}^{\mathrm{alg}}_{ijk}|_{\mathbb{H}}$. From this we get the Gauss–Manin connection matrix in the basis $\mathsf{B}\vec{\omega}_t$:

$$d\mathsf{B}\cdot\mathsf{B}^{-1} + \sum_{i=1}^{\mathsf{h}} \tilde{\mathsf{A}}_i dt^i. \tag{89}$$

We consider the generators in Definition 1 as independent variables

$$x_1, x_2, \ldots, x_a, \quad a := \dim(\mathsf{G})$$

independent of z_i's, and so, we write $\mathsf{B} = \mathsf{B}_{z,x}$ and $\tilde{\mathsf{A}}_i = \tilde{\mathsf{A}}_{i,z,x}$. We get an open subset V of $U \times \mathbb{C}^a$ such that for $(t,x) \in V$ we have $\mathsf{B}_{z(t),x} \in \mathsf{G}$. In this way

$$\tilde{U} := \{(t, \mathsf{B}_{z(t),x}\vec{\omega}_t) \mid (t,x) \in V\},$$

is an open subset of T. Using the action of G on T, the holomorphic limit of polynomial generators can be regarded as holomorphic functions in $\tilde{U}$. Now the Gauss–Manin connection of the enhanced family X/T in the open set $\tilde{U}$ is just (89) replacing B and $\tilde{\mathsf{A}}_i$ with $\mathsf{B}_{z,x}$ and $\tilde{\mathsf{A}}_{i,z,x}$, respectively. The existence of the vector fields in Theorem 1 follows from the same computations as in Sect. 3.3. Note that from (76) we get

$$\begin{aligned}
\mathsf{R}_{\mathfrak{t}_{ab}} &= \frac{\partial}{\partial T^{ab}} - \frac{1}{2}(L_a\frac{\partial}{\partial T^b} + L_b\frac{\partial}{\partial T^a}) + \frac{1}{2}L_aL_b\frac{\partial}{\partial T},\\
\mathsf{R}_{\mathfrak{t}_a} &= \frac{\partial}{\partial T^a} - L_a\frac{\partial}{\partial T},\\
\mathsf{R}_{\mathfrak{t}} &= \frac{1}{2}\frac{\partial}{\partial T},\\
\mathsf{R}_{\mathfrak{k}^a} &= \frac{\partial}{\partial L_a},\\
\mathsf{R}_{\mathfrak{g}^a_b} &= g^a_m\frac{\partial}{\partial g^b_m} - L_b\frac{\partial}{\partial L_a} + 2T^{ad}\frac{\partial}{\partial T^{db}} + T^a\frac{\partial}{\partial T^b},\\
\mathsf{R}_{\mathfrak{g}_0} &= g_0\frac{\partial}{\partial g_0} + L_a\frac{\partial}{\partial L_a} + T^a\frac{\partial}{\partial T^a} + 2\frac{\partial}{\partial T}
\end{aligned} \tag{90}$$

and R_a's are given by the vector fields computed in Proposition 1.

We now prove the uniqueness. Let us assume that there are two vector fields $\mathsf{R}', \mathsf{R}''$ such that $\mathsf{A}_{\mathsf{R}'} = \mathsf{A}_{\mathsf{R}''}$ is in one of the special matrix formats in Theorem 1. Let us assume that $\mathsf{R} = \mathsf{R}' - \mathsf{R}''$ is not zero and so it has a non-zero solution $\gamma(y)$ which is a regular map from a neighborhood of 0 in $\mathbb{C}$ to T and satisfies $\partial_y\gamma = \mathsf{R}(\gamma)$. We have $\nabla_{\mathsf{R}}\omega_1 = 0$ and so ω_1 restricted to the image of γ is a flat section of the Gauss–Manin connection. Since $z \to t$ is a coordinate change, we conclude that z as a function of y is constant, and so if $\gamma(y) := (X_y, \{\omega_{1,y}, \ldots, \omega_{2\mathsf{h}+2,y}\}) \in \mathsf{T}$ then X_y and $\omega_{1,y}$ do not depend on y. For the case of the special matrix formats in (10) we have also $\nabla_{\mathsf{R}}\omega_{i,y} = 0$ and so $\gamma(y)$ is a constant map. In the case of the special matrix format in (6), in a similar way all $\omega_{1,y}, \omega_{\mathsf{h}+2,y}, \omega_{\mathsf{h}+3,y}, \ldots, \omega_{2\mathsf{h}+2,y}$ do not depend on y. For others we argue as follows. Since $X = X_z$ does not depend on y the differential forms $\omega_{2,y}, \omega_{3,y}, \ldots, \omega_{\mathsf{h}+1,y}$ are linear combinations of elements $F^2H^3_{\mathrm{dR}}(X)$ (which do not depend on y) with coefficients which depend on y. This implies that the action of ∇_{R} on them is still in $F^2H^3_{\mathrm{dR}}(X)$. Using (6) we conclude that they are also independent of y.

Now let us consider the algebraic case, where the universal family X/T as in the Introduction exists. We further assume that over a local moduli U no Calabi–Yau threefold $X_z,\ z \in U$ has an isomorphism which acts non-identically on $H^3_{\mathrm{dR}}(X_z)$. In this way, the total space of choices of ω_i's over U gives us an open subset $\tilde{\mathsf{T}}$ of the moduli space T. The existence of the algebraic vector fields R in Theorem 1 is equivalent to the existence of the same analytic vector fields in some small open subset of T. This is because to find such vector fields we have to solve a set of linear equations with coefficients in $\mathcal{O}_{\mathsf{T}}$. That is why our argument in the algebraic context cannot guarantee that the vector fields in Theorem 1 are holomorphic everywhere in T.

4.4. Proof of Theorem 2. We use the equalities (76) and (74) and conclude that the two sets of derivations

$$\mathsf{R}_{\mathfrak{g}_a^b}, \mathsf{R}_{\mathfrak{t}_{ab}}, \mathsf{R}_{\mathfrak{k}_a}, \mathsf{R}_{\mathfrak{t}_a}, \mathsf{R}_{\mathfrak{t}}$$

and

$$\frac{\partial}{\partial T^{ab}}, \frac{\partial}{\partial T^{a}}, \frac{\partial}{\partial T}, \frac{\partial}{\partial L_a}, \frac{\partial}{\partial g_m^a}$$

are linear combinations of each other. Therefore, if $f \in \mathcal{O}_{\mathsf{T}}$ is in the left hand side of (17) then its derivation with respect to all variables $T^{ab}, T^a, T, L_a, g_m^a$ is zero and so it depends only on z_i's and g_0. In a similar way we can derive the fact that $\bigcap_{\mathfrak{g}\in \mathrm{Lie}(\mathsf{G})} \ker(\mathsf{R}_{\mathfrak{g}})$ is the set of G invariant functions in T.

4.5. The Lie Algebra $\mathfrak{G}$. In this section we describe the computation of Lie bracket structure of (9) resulting in the Table 11. Let $\mathsf{R}_1, \mathsf{R}_2$ be two vector fields in T and let $\mathsf{A}_i := \mathsf{A}_{\mathsf{R}_i}$. We have

$$\nabla_{[\mathsf{R}_1,\mathsf{R}_2]}\omega = ([\mathsf{A}_2, \mathsf{A}_1] + \mathsf{R}_1(\mathsf{A}_2) - \mathsf{R}_2(\mathsf{A}_1))\omega$$

In particular, for $\mathfrak{g}_i \in \mathrm{Lie}(\mathsf{G})$ we get

$$[\mathsf{R}_{\mathfrak{g}_1}, \mathsf{R}_{\mathfrak{g}_2}] = [\mathfrak{g}_1, \mathfrak{g}_2]^{\mathsf{tr}}.$$

We have also

$$[\mathsf{R}_i, \mathsf{R}_j] = 0, \quad i = 1, 2, \ldots, \mathsf{h}.$$

because $\mathsf{R}_i(\mathsf{A}_{\mathsf{R}_j}) = \mathsf{R}_j(\mathsf{A}_{\mathsf{R}_i})$ and $[\mathsf{A}_{\mathsf{R}_i}, \mathsf{A}_{\mathsf{R}_j}] = 0, i, j = 1, \ldots, \mathsf{h}$. These equalities, in turn, follow from the fact that $\mathsf{C}_{ijk}^{\mathrm{alg}}$ are symmetric in i, j, k. It remains to compute

$$[\mathsf{R}_{\mathfrak{g}}, \mathsf{R}_i], \quad i = 1, 2, \ldots, \mathsf{h}, \quad \mathsf{g} \in \mathsf{G}.$$

which is done for each element of the canonical basis of $\mathrm{Lie}(\mathsf{G})$.

4.6. Two fundamental equalities of the special geometry. The Gauss–Manin connection matrix A satisfies the following equalities:

$$d\mathsf{A} = -\mathsf{A} \wedge \mathsf{A} \tag{91}$$

$$0 = \mathsf{A}\Phi + \Phi\mathsf{A}^{\mathsf{tr}}. \tag{92}$$

The first one follows from the integrability of the Gauss–Manin connection and the second equality follows after taking the differential of the equality (3). Note that Φ is constant and so $d\Phi = 0$. Note also that the base space of our Gauss–Manin connection matrix is T. For the mirror quintic this is of dimension 7, and so the integrability is a non-trivial statement, whereas the integrability over the classical moduli space of mirror quintics (which is of dimension one) is a trivial identity. The Lie algebra $\mathrm{Lie}(\mathsf{G})$ is already hidden in (92) and it is consistent with the fact that after composing $\mathsf{R}_{\mathfrak{g}}$ with A we get $\mathfrak{g}^{\mathrm{tr}}$. Assuming the existence of R_i's and $\mathsf{C}^{\mathrm{alg}}_{ijk}$ in Theorem 1, the equalities (91) and (92) composed with the vector fields $\mathsf{R}_i, \mathsf{R}_j$ imply that $\mathsf{C}^{\mathrm{alg}}_{ijk}$ are symmetric in i, j, k and the equality (7). We want to argue that most of the ingredients of the special geometry can be derived from (91) and (92). Special geometry, in algebraic terms, aims to find a h-dimensional sub-locus $(\mathbb{C}^{\mathsf{h}}, 0) \cong M \subset \mathsf{T}$ such that A restricted to M is of the form

$$\tilde{\mathsf{A}} := \begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ 0 & 0 & \omega_2 & 0 \\ 0 & 0 & 0 & \omega_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

where the entries of ω_i's are differential 1-forms in M. In Sect. 4.2 the union of such loci is denoted by $\mathbb{H}$. The equality (92) implies that $\omega_3 = \omega_1^{\mathrm{tr}}$ and $\omega_2 = \omega_2^{\mathrm{tr}}$ and the equality (91) implies that $\omega_1 \wedge \omega_2 = 0$ and all the entries of ω_i's are closed, and since $M \cong (\mathbb{C}^{\mathsf{h}}, 0)$, they are exact. Let us write $\omega_1 = dt,\ \omega_2 = dP$ for some matrices t, P with entries which are regular functions on M and so we have

$$dt \wedge dP = 0. \tag{93}$$

Special geometry takes the entries of t as coordinates on M and the equation (93) gives us the existence of a holomorphic function F on M such that $P = [\frac{\partial F}{\partial t_i \partial t_j}]$. This is exactly the prepotential discussed in Sect. 2.2. One can compute the special period matrix P in (62), starting from the initial data

$$\mathsf{P}^{\mathrm{tr}} = \begin{pmatrix} 1 & * & * & * \\ 0 & \delta^c_a & * & * \\ 0 & 0 & \delta^a_c & * \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{94}$$

and the equality $d\mathsf{P}^{\mathrm{tr}} = \tilde{\mathsf{A}}\mathsf{P}^{\mathrm{tr}}$, where $\tilde{\mathsf{A}}$ is the Gauss–Manin connection restricted to M.

5. Mirror Quintic Case

In Refs. [Mov11,Mov12a] it was proven that the universal family $\mathsf{X} \to \mathsf{T}$ exists in the case of mirror quintic Calabi–Yau threefolds and it is defined over $\mathbb{Q}$. More precisely we have

$$\mathsf{S} = \mathrm{Spec}(\mathbb{Q}[\mathsf{t}_0, \mathsf{t}_4, \frac{1}{(\mathsf{t}_0^5 - \mathsf{t}_4)\mathsf{t}_4}]),$$

where for $(\mathsf{t}_0, \mathsf{t}_4)$ we associate the pair $(\mathsf{X}_{\mathsf{t}_0,\mathsf{t}_4}, \omega_1)$. In the affine coordinates (x_1, x_2, x_3, x_4), that is $x_0 = 1$, $\mathsf{X}_{\mathsf{t}_0,\mathsf{t}_4}$ is given by

$$\begin{aligned} \mathsf{X}_{\mathsf{t}_0,\mathsf{t}_4} &:= \{f(x) = 0\}/G, \\ f(x) &:= -\mathsf{t}_4 - x_1^5 - x_2^5 - x_3^5 - x_4^5 + 5\mathsf{t}_0 x_1 x_2 x_3 x_4, \end{aligned}$$

and

$$\tilde{\omega}_1 := \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{df}.$$

We have also

$$\mathsf{T} = \mathrm{Spec}(\mathbb{Q}[\mathsf{t}_0, \mathsf{t}_1, \ldots, \mathsf{t}_6, \frac{1}{(\mathsf{t}_0^5 - \mathsf{t}_4)\mathsf{t}_4\mathsf{t}_5}]) \tag{95}$$

Here, for $(\mathsf{t}_0, \mathsf{t}_1, \ldots, \mathsf{t}_6)$ we associate the pair $(\mathsf{X}_{\mathsf{t}_0,\mathsf{t}_4}, [\omega_1, \omega_2, \omega_3, \omega_4])$, where $\mathsf{X}_{\mathsf{t}_0,\mathsf{t}_4}$ is as before and ω is given by

$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{pmatrix} := \begin{pmatrix} -1 & 0 & 0 & 0\\ \frac{-5^5\mathsf{t}_0^4-\mathsf{t}_3}{\mathsf{t}_5} & \frac{-5^4(\mathsf{t}_4-\mathsf{t}_0^5)}{\mathsf{t}_5} & 0 & 0\\ \frac{(5^5\mathsf{t}_0^4+\mathsf{t}_3)\mathsf{t}_6-(5^5\mathsf{t}_0^3+\mathsf{t}_2)\mathsf{t}_5}{5^4(\mathsf{t}_4-\mathsf{t}_0^5)} \;\; \frac{5^4(\mathsf{t}_0^5-\mathsf{t}_4)}{\mathsf{t}_5} & \mathsf{t}_6 & \mathsf{t}_5 & 0\\ \mathsf{t}_1 & \mathsf{t}_2 & \mathsf{t}_3 & 625(\mathsf{t}_4 - \mathsf{t}_0^5)\end{pmatrix} \begin{pmatrix}\tilde{\omega}_1\\ \nabla_{\frac{\partial}{\partial \mathsf{t}_0}}\tilde{\omega}_1\\ (\nabla_{\frac{\partial}{\partial \mathsf{t}_0}})^{(2)}\tilde{\omega}_1\\ (\nabla_{\frac{\partial}{\partial \mathsf{t}_0}})^{(3)}\tilde{\omega}_1\end{pmatrix} \tag{96}$$

The above definition is the algebraic counterpart of the equality (65), up to a change of trivialization. Note that in this context t_i's are just parameters, whereas the generators of the special polynomial differential ring are functions in a local patch of the classical moduli space of Calabi–Yau threefolds. The relation between these two sets are explained in Sect. 4.3. The genus one topological string partition function F_1^A is given by

$$\mathsf{F}_1^{\mathrm{alg}} := -5^{-1}\ln(\mathsf{t}_4^{\frac{25}{12}}(\mathsf{t}_4 - \mathsf{t}_0^5)^{\frac{-5}{12}}\mathsf{t}_5^{\frac{1}{2}}) \tag{97}$$

and for $g \geq 2$, we have

$$\mathsf{F}_g^{\mathrm{alg}} = \frac{Q_g}{(\mathsf{t}_4 - \mathsf{t}_0)^{2g-2}\mathsf{t}_5^{3g-3}}, \tag{98}$$

where Q_g is a homogeneous polynomial of degree $69(g-1)$ with weights

$$\deg(\mathsf{t}_i) := 3(i+1),\ i = 0, 1, 2, 3, 4,\quad \deg(\mathsf{t}_5) := 11,\quad \deg(\mathsf{t}_6) := 8. \tag{99}$$

and with rational coefficients, and so $\mathsf{F}_g^{\mathrm{alg}}$ is of degree $6g-6$. Further, any monomial $\mathsf{t}_0^{i_0}\mathsf{t}_1^{i_1}\cdots\mathsf{t}_6^{i_6}$ in $\mathsf{F}_g^{\mathrm{alg}}$ satisfies $i_2 + i_3 + i_4 + i_5 + i_6 \geq 3g - 3$. For the q-expansion of t_i's see Ref. [Mov12a]. We have

$$\mathsf{C}_{111}^{\mathrm{alg}} = \frac{5^8(\mathsf{t}_4 - \mathsf{t}_0)^2}{\mathsf{t}_5^3} \tag{100}$$

and

$$\begin{aligned}\mathsf{R}_1 \;=\; & -\frac{3750\mathsf{t}_0^5 + \mathsf{t}_0\mathsf{t}_3 - 625\mathsf{t}_4}{\mathsf{t}_5}\frac{\partial}{\partial \mathsf{t}_0} + \frac{390625\mathsf{t}_0^6 - 3125\mathsf{t}_0^4\mathsf{t}_1 - 390625\mathsf{t}_0\mathsf{t}_4 - \mathsf{t}_1\mathsf{t}_3}{\mathsf{t}_5}\frac{\partial}{\partial \mathsf{t}_1}\\ & + \frac{5859375\mathsf{t}_0^7 + 625\mathsf{t}_0^5\mathsf{t}_1 - 6250\mathsf{t}_0^4\mathsf{t}_2 - 5859375\mathsf{t}_0^2\mathsf{t}_4 - 625\mathsf{t}_1\mathsf{t}_4 - 2\mathsf{t}_2\mathsf{t}_3}{\mathsf{t}_5}\frac{\partial}{\partial \mathsf{t}_2}\\ & + \frac{9765625\mathsf{t}_0^8 + 625\mathsf{t}_0^5\mathsf{t}_2 - 9375\mathsf{t}_0^4\mathsf{t}_3 - 9765625\mathsf{t}_0^3\mathsf{t}_4 - 625\mathsf{t}_2\mathsf{t}_4 - 3\mathsf{t}_3^2}{\mathsf{t}_5}\frac{\partial}{\partial \mathsf{t}_3}\\ & - \frac{15625\mathsf{t}_0^4\mathsf{t}_4 + 5\mathsf{t}_3\mathsf{t}_4}{\mathsf{t}_5}\frac{\partial}{\partial \mathsf{t}_4} + \frac{625\mathsf{t}_0^5\mathsf{t}_6 - 9375\mathsf{t}_0^4\mathsf{t}_5 - 2\mathsf{t}_3\mathsf{t}_5 - 625\mathsf{t}_4\mathsf{t}_6}{\mathsf{t}_5}\frac{\partial}{\partial \mathsf{t}_5}\\ & - \frac{9375\mathsf{t}_0^4\mathsf{t}_6 - 3125\mathsf{t}_0^3\mathsf{t}_5 - 2\mathsf{t}_2\mathsf{t}_5 + 3\mathsf{t}_3\mathsf{t}_6}{\mathsf{t}_5}\frac{\partial}{\partial \mathsf{t}_6}\end{aligned}$$

$$
\begin{aligned}
\mathsf{R}_{\mathfrak{g}_1^1} &= \mathsf{t}_5 \frac{\partial}{\partial \mathsf{t}_5} + \mathsf{t}_6 \frac{\partial}{\partial \mathsf{t}_6}, \\
\mathsf{R}_{\mathfrak{g}_0} &= \mathsf{t}_0 \frac{\partial}{\partial \mathsf{t}_0} + 2\mathsf{t}_1 \frac{\partial}{\partial \mathsf{t}_1} + 3\mathsf{t}_2 \frac{\partial}{\partial \mathsf{t}_2} + 4\mathsf{t}_3 \frac{\partial}{\partial \mathsf{t}_3} + 5\mathsf{t}_4 \frac{\partial}{\partial \mathsf{t}_4} + 3\mathsf{t}_5 \frac{\partial}{\partial \mathsf{t}_5} + 2\mathsf{t}_6 \frac{\partial}{\partial \mathsf{t}_6}, \\
\mathsf{R}_{\mathfrak{k}_1} &= -\frac{5\mathsf{t}_0^4\mathsf{t}_6 - 5\mathsf{t}_0^3\mathsf{t}_5 - \frac{1}{625}\mathsf{t}_2\mathsf{t}_5 + \frac{1}{625}\mathsf{t}_3\mathsf{t}_6}{\mathsf{t}_0^5 - \mathsf{t}_4} \frac{\partial}{\partial \mathsf{t}_1} + \mathsf{t}_6 \frac{\partial}{\partial \mathsf{t}_2} + \mathsf{t}_5 \frac{\partial}{\partial \mathsf{t}_3}, \\
\mathsf{R}_{\mathfrak{t}_{11}} &= \frac{625\mathsf{t}_0^5 - 625\mathsf{t}_4}{\mathsf{t}_5} \frac{\partial}{\partial \mathsf{t}_6}, \\
\mathsf{R}_{\mathfrak{t}_1} &= \frac{-3125\mathsf{t}_0^4 - \mathsf{t}_3}{\mathsf{t}_5} \frac{\partial}{\partial \mathsf{t}_1} + \frac{625(\mathsf{t}_0^5 - \mathsf{t}_4)}{\mathsf{t}_5} \frac{\partial}{\partial \mathsf{t}_2}, \\
\mathsf{R}_{\mathfrak{t}} &= \frac{\partial}{\partial \mathsf{t}_1}.
\end{aligned} \tag{101}
$$

Using the asymptotic behavior of $\mathsf{F}_g^{\mathrm{alg}}$'s in Ref. [BCOV93], we know that the ambiguities of $\mathsf{F}_g^{\mathrm{alg}}$ arise from the coefficients of

$$
\frac{P_g(\mathsf{t}_0, \mathsf{t}_4)}{(\mathsf{t}_4 - \mathsf{t}_0^5)^{2g-2}}, \quad \deg(P_g) = 36(g-1). \tag{102}
$$

Knowing that we are using the weights (99), we observe that it depends on $[\frac{12(g-1)}{5}] + 1$ coefficients. The monomials in (102) are divided into two groups, those meromorphic in $\mathsf{t}_4 - \mathsf{t}_0^5$ and the rest which is

$$
\mathsf{t}_0^a(\mathsf{t}_4 - \mathsf{t}_0^5)^b, \quad a + 5b = 2g - 2, \ a, b \in \mathbb{N}_0.
$$

The coefficients of the first group can be fixed by the so called gap condition and the asymptotic behavior of F_g^A at the conifold, see for instance Ref. [HKQ09]. One of the coefficients in the second group can be solved using the asymptotic behavior of $\mathsf{F}_g^{\mathrm{alg}}$ at the maximal unipotent monodromy point. In total, we have $[\frac{2g-2}{5}]$ undetermined coefficients. It is not clear whether it is possible to solve them using only the data attached to the mirror quintic Calabi–Yau threefold. Using the generating function role that $\mathsf{F}_g^{\mathrm{alg}}$'s has on the $A-$model side for counting curves in a generic quintic, one may solve all the ambiguities given enough knowledge of enumerative invariants, such computations are usually hard to perform, see for instance Ref. [HKQ09] for a use of boundary data in the B-model and $A-$model counting data which determines the ambiguities up to genus 51.

Finally, we discuss the q-expansion of the $\mathsf{F}_g^{\mathrm{alg}}$. Recall the special subset $\mathbb{H}$ of T chosen in Definition 3 whose inclusion map we denote by i. Furthermore, recall the map $f : U \to \mathbb{H}$ in (88), where $U \subset \mathcal{M}$ is a neighborhood of the point of maximal unipotent monodromy with a local coordinate q such that $q = 0$ corresponds to the point of maximal unipotent monodromy. Pulling back functions $g \in \mathcal{O}_\mathsf{T}$ by $i \circ f$ yields functions $g(q)$ on U. A formal q–expansion of the coordinate functions $\mathsf{t}_i \in \mathcal{O}_\mathsf{T}$ can be obtained using the vector field R_1 in the following way. We write each t_i as a formal power series in q, $\mathsf{t}_i = \sum_{n=0}^{\infty} \mathsf{t}_{i,n} q^n$, and make the ansatz that the pull-back vector field $(i \circ f)^* \mathsf{R}_1$ to U is $5q\frac{\partial}{\partial q}$. We see that this ansatz determines uniquely all the coefficients $\mathsf{t}_{i,n}$ with the initial values:

$$
\mathsf{t}_{0,0} = \frac{1}{5}, \ \mathsf{t}_{0,1} = 24, \ \mathsf{t}_{4,0} = 0 \tag{103}
$$

and assuming that $\mathsf{t}_{5,0} \neq 0$, see [Mov12a] for few coefficients of t_i's. Substituting these q-expansions in $\mathsf{F}_g^{\mathrm{alg}}$ yields the q-expansions of the $\mathsf{F}_g^{\mathrm{alg}}$ pulled back to U, and we recover the generating functions of the genus g Gromov–Witten invariants. For instance, we get the following product formula for the expression inside ln of $\mathsf{F}_1^{\mathrm{alg}}$:

$$t_4^{\frac{25}{12}}(t_4-t_0^5)^{-\frac{5}{12}}t_5^{\frac{1}{2}}=5^{-\frac{5}{12}}q^{\frac{25}{12}}\Big(\prod_{n=1}^{\infty}(1-q^n)^{\sum_{r|n}d_r}\Big)\Big(\prod_{s=1}^{\infty}(1-q^s)^{n_s}\Big)^{\frac{1}{12}} \tag{104}$$

where $n_s=2875, 609250, 317206375, \cdots$ (resp. $d_s=0, 0, 609250, 3721431625, \cdots$) is the virtual number of rational (resp. elliptic) curves of degree s in a generic quintic.

6. Final Remarks

We strongly believe that a mathematical verification of mirror symmetry at higher genus will involve the construction of the Lie algebra $\mathfrak{G}$ in the A-model Calabi–Yau variety $\check{X}$. The genus zero case was established by Givental [Giv96] and Lian et al. [LLY97] for many cases of Calabi–Yau threefolds and in particular the quintic case, that is, the period manipulations of the B-model lead to the virtual number of rational curves in the A-model. The genus one case was proved by Zinger in Ref. [Zin09]. The amount of computations and technical difficulties from genus zero to genus one case is significantly large. For higher genus there has been no progress. The period expressions involved in higher genus, see for instance Ref. [YY04], are usually huge and this is the main reason why the methods used in Refs. [Giv96,LLY97,Zin09] do not generalize. This urges us for an alternative description of the generating function of the number of higher genus curves in the A-model. The original formulation in Ref. [BCOV94] using holomorphic anomaly equation for genus g topological string partition functions is completely absent in the mathematical formulation of A-model using quantum differential equations. Motivated by this, we formulated Theorem 1 and the algebraic holomorphic anomaly equation (16). Our work opens many other new conjectures in the A-model Calabi–Yau varieties. The most significant one is the following. Let C^A_{ijk} and F^A_g, $g\geq 1$ be the generating function of genus zero and genus g Gromov–Witten invariants of the A-model Calabi–Yau threefold $\check{X}$, respectively

Conjecture 1. Let $\check{X}$ be a Calabi–Yau threefold with $\mathsf{h}:=\dim(H^2_{\mathrm{dR}}(X))$ and let M be the sub-field of formal power series generated by $\mathsf{C}^A_{ijk}, \exp(\mathsf{F}^A_1), \mathsf{F}^A_g,\quad g\geq 2$ and their derivations under $q_i\frac{\partial}{\partial q_i},\quad i=1,2,\ldots,\mathsf{h}$. The transcendental degree of M over $\mathbb{C}$ is at most $a_{\mathsf{h}}:=\frac{3\mathsf{h}^2+7\mathsf{h}+4}{2}$, that is, for any $a_{\mathsf{h}}+1$ elements $x_1,x_2,\ldots,x_{a_{\mathsf{h}}+1}$ of M there is a polynomial P in $a_{\mathsf{h}}+1$ variables and with coefficients in $\mathbb{C}$ such that $P(x_1,x_2,\ldots,x_{a_{\mathsf{h}}+1})=0$.

The number a_{h} is the dimension of the moduli space T in the Introduction. Our mathematical knowledge in enumerative algebraic geometry of Calabi–Yau threefolds is still far from any solution to the above conjecture.

Our reformulation of the BCOV anomaly equation opens an arithmetic approach to Topological String partition functions. For many interesting example such as mirror quintic, T can be realized as an affine scheme over $\mathbb{Z}[\frac{1}{N}]$ for some integer N. In this way we can do mod primes of $\mathsf{F}^{\mathrm{alg}}_g$'s which might give some insight into the arithmetic of Fourier expansions of $\mathsf{F}^{\mathrm{alg}}_g$'s.

In the case of mirror quintic we have partial compactifications of T given by $\mathsf{t}_4 = 0$, $\mathsf{t}_4 - \mathsf{t}_0^5 = 0$ and $\mathsf{t}_5 = 0$. The first two correspond to the maximal unipotent and conifold singularities. The degeneracy locus $\mathsf{t}_5 = 0$ corresponds to degeneration of differential forms and not the mirror quintic itself. Our computations show that the vector fields in Theorem 1 are holomorphic everywhere except t_5. These statements cannot be seen for the proof of Theorem 1 and one may conjecture that similar statements in general must be valid. Of course one must first construct the universal family X/T and enlarge it to a bigger family using similar moduli spaces for limit mixed Hodge structures.

Acknowledgments. We would like to thank Jie Zhou for discussions and for collaborations on related projects. H.M. would like to acknowledge the generous support he received from Brazilian Science without Borders program in order to stay for a sabbatical year at Harvard University. E.S. would like to thank the mathematics department at Harvard University for support and hospitality. This work has been supported by NSF grants PHY-1306313 and DMS-0804454.

References

[AL07] Alim, M., Lange, J.D.: Polynomial Structure of the (open) Topological String Partition Function. JHEP **0710**, 045 (2007)

[Ali13] Alim, M.: Lectures on Mirror Symmetry and Topological String Theory. Open Probl. Surv. Contemp Math. **6**, 1 (2013)

[ALM10] Alim, M., Lange, J.D., Mayr, P.: Global Properties of Topological String Amplitudes and Orbifold invariants. JHEP **1003**, 113 (2010)

[ASYZ13] Alim, M., Scheidegger, E., Yau, S.-T., Zhou, J.: Special Polynomial Rings, Quasi Modular Forms and Duality of Topological Strings. Adv. Theor. Math. Phys. **18**(2), 401–467 (2014)

[BCOV93] Bershadsky, M., Cecotti, S., Ooguri, H., Vafa, C.: Holomorphic Anomalies in Topological Field theories. Nucl. Phys. B **405**, 279–304 (1993)

[BCOV94] Bershadsky, M., Cecotti, S., Ooguri, H., Vafa, C.: Kodaira–spencer theory of gravity and exact results for quantum string amplitudes. Commun. Math. Phys. **165**, 311–428 (1994)

[CDF+97] Ceresole, A., D'Auria, R., Ferrara, S., Lerche, W., Louis, J., Regge, T.: Picard–Fuchs equations, special geometry and target space duality. In: Mirror symmetry, II, vol. 1, AMS/IP Stud. Adv. Math., pp. 281–353. American Mathematical Society, Providence (1997)

[CdlO91] Candelas, P., Ossa, X. de la : Moduli space of calabi–yau manifolds. Nucl. Phys. B **355**, 455–481 (1991)

[CDLOGP91] Candelas, P., DeLaOssa, X.C., Green, P.S., Parkes, L.: A pair of calabi–yau manifolds as an exactly soluble superconformal theory. Nucl. Phys. B **359**, 21–74 (1991)

[CK99] Cox, D.A., Katz, S.: Mirror symmetry and algebraic geometry, vol. 68, Mathematical Surveys and Monographs. American Mathematical Society, Providence (1999)

[CL12] Costello, K.J., Li, S.: Quantum BCOV theory on Calabi–Yau manifolds and the higher genus B-model (2012)

[Del09] Deligne, P.: Private letter. http://w3.impa.br/~hossein/myarticles/deligne6-12-2008 (2009)

[DMOS82] Deligne, P., Milne, J.S., Ogus, A., Shih, K.-Y.: Hodge cycles, motives, and Shimura varieties, vol. 900, Lecture Notes in Mathematics. Springer-Verlag, Berlin (1982)

[Giv96] Givental, A.B.: Equivariant Gromov–Witten invariants. Int. Math. Res. Notices (13):613–663 (1996)

[Gro66] Grothendieck, A.: On the de rham cohomology of algebraic varieties. Inst. Hautes Études Sci. Publ. Math. **29**, 95–103 (1966)

[Gui07] Guillot, A.: Sur les équations d'halphen et les actions de $\mathrm{sl}_2(\mathbf{c})$. Publ. Math. Inst. Hautes Études Sci. **105**, 221–294 (2007)

[HKQ09] Huang, M.-X., Klemm, A., Quackenbush, S.: Topological string theory on compact calabi–yau: modularity and boundary conditions. Lect. Notes Phys. **757**, 45–102 (2009)

[Hos08] Hosono, S.: BCOV ring and holomorphic anomaly equation. In: New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008), Advanced Studies in Pure Mathematics, vol. 59, pp. 79–110. A publication of the Mathematical Society of Japan, Tokyo (2010)

[KO68] Katz, N.M., Oda, T.: On the differentiation of de rham cohomology classes with respect to parameters. J. Math. Kyoto Univ. **8**, 199–213 (1968)

[Kon95] Kontsevich, M.: Homological algebra of mirror symmetry. In: Proceedings of the International Congress of Mathematicians, vol. 1, 2 (Zürich, (1994)) pp. 120–139. Birkhäuser, Basel (1995)

[LLY97] Lian, B.H., Liu, K., Yau, S.-T.: Mirror principle. i. Asian J. Math. **1**(4), 729–763 (1997)

[Mor92] Morrison, D.R.: Picard–Fuchs equations and mirror maps for hypersurfaces. In: Essays on mirror manifolds, pp. 241–264. International Press, Hong Kong (1992)

[Mor97] Morrison, D.R.: Mathematical aspects of mirror symmetry. In *Complex algebraic geometry (Park City, UT, 1993)*, vol. 3 of IAS/Park City Math. Ser., pp. 265–327. American Mathematical Society, Providence (1997)

[Mov08] Movasati, H.: On elliptic modular foliations. Indag. Math. (N.S.) **19**(2), 263–286 (2008) arXiv:0806.3926v1 [math.AG]

[Mov11] Movasati, H.: Eisenstein type series for calabi–yau varieties. Nuclear Phys. B **847**(2), 460–484 (2011)

[Mov12a] Movasati, H.: Modular-type functions attached to mirror quintic calabi–yau varieties. Math. Zeit. **281**(3), 907–929 (2015)

[Mov12b] Movasati, H.: On ramanujan relations between eisenstein series. Manuscr. Math. **139**(3–4), 495–514 (2012)

[Mov12c] Movasati, H.: Quasi-modular forms attached to elliptic curves, i. Ann. Math. Blaise Pascal **19**(2), 307–377 (2012)

[Mov13] Movasati, H.: Quasi-modular forms attached to Hodge structures. In: Arithmetic and geometry of K3 surfaces and Calabi–Yau threefolds, vol 67 of Fields Inst. Commun., pp. 567–587. Springer, New York (2013)

[Nik14] Nikdelan, Y.: Darboux–halphen–ramanujan vector field on a moduli of calabi–yau manifolds. Qual. Theory Dyn. Syst. **14**(1), 71100 (2015)

[Str90] Strominger, A.: Special geometry. Commun. Math. Phys. **133**, 163–180 (1990)

[Tod03] Todorov, A.: Local and global theory of the moduli of polarized Calabi–Yau manifolds. In: Proceedings of the International Conference on Algebraic Geometry and Singularities (Spanish) (Sevilla, 2001), vol. 19, pp. 687–730 (2003)

[Vie95] Viehweg, E.: Quasi-projective moduli for polarized manifolds, volume 30 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin (1995)

[Voi99] Voisin, C.: Mirror symmetry, vol. 1, SMF/AMS Texts and Monographs. American Mathematical Society, Providence; Société Mathématique de France, Paris (1999) (Translated from the 1996 French original by Roger Cooke)

[Wit88] Witten, E.: Topological sigma models. Commun. Math. Phys. **118**, 411 (1988)

[Wit91] Witten, E.: Mirror manifolds and topological field theory. In: Essays on mirror manifolds, pp. 120–158. International Press, Hong Kong (1992)

[Wit93] Witten, E.: Quantum background independence in string theory (1993) arXiv:hep-th/9306122

[YY04] Yamaguchi, S., Yau, S.-T.: Topological string partition functions as polynomials. JHEP **07**, 047 (2004)

[Zin09] Zinger, A.: The reduced genus 1 gromov–witten invariants of calabi–yau hypersurfaces. J. Am. Math. Soc. **22**(3), 691–737 (2009)

Communicated by H. T. Yau