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Introduction to Algebraic Curves and Foliations

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Preface

The guiding principal in this book is to give a detailed and historical exposition of the theory of holomorphic foliations in the projective space of dimension two. Its objective is to introduce the reader with a basic knowledge in holomorphic foliations. Our approach is purely algebraic and we avoid many transcendental arguments in the literature. For our purpose we take foliations in the two dimensional plane and given by polynomial vector fields. We would like to rise the need for working with arbitrary fields instead of the field of complex numbers. This makes our text different from the available texts in the literature such as Camacho-Sad's monograph [CS87] which emphasizes local aspects, Brunella's monograph [Bru00] which emphasizes the classification of holomorphic foliations similar to classification of two dimensional surfaces, Lins Neto-Scárdua's book [LNS] and Ilyashenko-Yakovenko's book [IY08] which both emphasize analytic and holomorphic aspects. We have in mind an audience with a basic knowledge of Complex Analysis in one variable and Algebraic Geometry of curves in the two dimensional projective space. The text is mainly written for two primary target audiences: undergraduate students who want to have a flavor of an important class of holomorphic foliations and algebraic geometers who want to learn how the theory of holomorphic foliations can be written in the framework of Algebraic Geometry.

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Chapter 1

Hilbert's sixteen problem

In this chapter we introduce limit cycles of polynomial differential equations in \mathbb{R}^2 and state the well-known Hilbert 16-th problem. Despite the fact that this is not the main problem of the present text, it must be considered the most important unsolved problem on polynomial differential equations. Our aim is not to collect all the developments and theorems in direction of Hilbert 16-th problem (for this see for instance [Ily02]), but to present a way of breaking the problem in many pieces and observing the fact that even such partial problems are extremely difficult to treat. Our point of view is algebraic and we want to point out that both real and complex Algebraic Geometry would be indispensable for a systematic approach to Hilbert 16-th problem.

1.1 Real foliations

What we want to study is the following ordinary differential equation:

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases}, \quad (1.1)$$

where P, Q are two polynomials in x and y with coefficients in \mathbb{R} and $\dot{x} = \frac{dx}{dt}$, where x and y are functions depending on the real parameter t which is sometimes called the time. The differential equation (1.1) is called an autonomous differential equation as its right hand side is independent of t . We may assume that P and Q do not have common factors. The solutions of (1.1) are the trajectories of the vector field:

$$X := P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}$$

We will also write $X = (P, Q)$. The reader may interpret $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ as

$$\frac{\partial}{\partial x} := (1, 0), \quad \frac{\partial}{\partial y} := (0, 1).$$

This is introduced in order to distinguish between points and vectors in \mathbb{R}^2 . Let us first recall the first basic theorem of ordinary differential equations.

Theorem 1.1 *For $a \in \mathbb{R}^2$ if $X(a) \neq 0$ then there is a unique analytic function*

$$\varphi : (\mathbb{R}, 0) \rightarrow \mathbb{R}^2$$

such that

$$\varphi(0) = a, \quad \dot{\varphi} = X(\varphi(t))$$

Proof. Let us write formally

$$\varphi = \sum_{i=0}^{\infty} \varphi_i t^i, \quad \varphi_i \in \mathbb{R}^2, \quad \varphi_0 := a$$

and substitute it in $\dot{\varphi} = X(\varphi)$. It turns out that φ_i can be written in a unique way in terms of φ_j , $j < i$. This guaranties the existence of a unique formal φ . Note that if $X(a) = 0$ then $\varphi_i = 0$ for all $i \geq 1$ and so φ is the constant map $\varphi(t) = a$. It is not at all clear why φ must be convergent. For this we use Picard operator associated with the differential equation 1.1 and the contracting map principle. For more details see [IY08, §1.4, page 4]. Let

$$P : \mathcal{O}_U \rightarrow \mathcal{O}_U, \quad P(f) := a + \int_0^t X(f(t)).$$

where U is a small neighborhood of 0 in \mathbb{R} , let us say $U = (\varepsilon, \varepsilon)$ for some positive small number ε , \mathcal{O}_U is the set of analytic functions f in a neighborhood of \bar{U} with $f(0) = a$. We regard X as a function $(\mathbb{R}^2, a) \rightarrow (\mathbb{R}^2, X(a))$, and we can find suitable a neighborhood of a and a constant C such that $|X(b) - X(c)| < C|b - c|$ for all $b, c \in (\mathbb{R}^2, a)$, where we have used the usual norm of \mathbb{R}^2 . We have

$$|Pf - Pg| < \left| \int_0^t (X(f(t)) - X(g(t))) dt \right| < C|f - g|\varepsilon.$$

For $C\varepsilon < 1$ this implies that for an arbitrary analytic function f , $P^n f$ is a Cauchy sequence of functions and hence it converges to an analytic function φ which is the desired function. For instance, if $f(t) = a$ is the constant function then $Pf = a + t(P(a), Q(a))$ whose image is the line tangent to the solution of X through a . The entries of the second iteration $P^2 f$ are polynomials in t of degree $\leq \max\{\deg(P), \deg(Q)\} + 1$. \square

Exercise 1.1 Rewrite the proof of Theorem 1.2 in many variables $x = (x_1, x_2, \dots, x_n)$ instead of (x, y) and replacing \mathbb{R} with \mathbb{C} .

Exercise 1.2 In the proof of Theorem 1.1 we have introduced two methods to approximate a solution of a vector field X . First, by taking a formal power series φ and

finding its coefficients by recursion given by X . Second, iterating Picard's operator starting from a constant function. Which method converges faster? Justify your answer. For instance, implement both methods in a computer and justify your answer upon this.

In Theorem 1.1 we even claim that φ depends on a analytically, that is, there is a small neighborhood (\mathbb{R}^2, a) of a in \mathbb{R}^2 and an analytic function

$$\Gamma : (\mathbb{R}^2, a) \times (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, a)$$

such that $\Gamma(b, \cdot)$ for all $b \in (\mathbb{R}^2, A)$ is the solution in Theorem 1.1 crossing the point b . In this way we may reformulate the following theorem:

Theorem 1.2 For $a \in \mathbb{R}^2$ if $X(a) \neq 0$ then there is an analytic isomorphism $F : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, a)$ such that the push-forward of $\frac{\partial}{\partial x}$ by F is X .

Proof. The push forward of the vector field $\frac{\partial}{\partial x}$ by F is X . This is equivalent to

$$\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} P(F_1, F_2) \\ Q(F_1, F_2) \end{pmatrix}$$

where $F = (F_1, F_2)$. In a similar way as in Theorem 1.1 we have a unique solution (F_1, F_2) to the above differential equation with

$$(F_1(0, y), F_2(0, y)) = a + (0, y).$$

We have

$$\begin{pmatrix} \frac{\partial F_1}{\partial x}(0, 0) & \frac{\partial F_1}{\partial y}(0, 0) \\ \frac{\partial F_2}{\partial x}(0, 0) & \frac{\partial F_2}{\partial y}(0, 0) \end{pmatrix} = \begin{pmatrix} P(a) & 0 \\ Q(a) & 1 \end{pmatrix}$$

By a rotation around a , we may assume that $P(a) \neq 0$, and so $F = (F_1, F_2) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, A)$ is an analytic isomorphism. \square

Exercise 1.3 Describe the trajectories of the following differential equations:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}, \quad \begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases}, \quad \begin{cases} \dot{x} = x \\ \dot{y} = y \end{cases}$$

These are respectively called a center, a saddle and a node (or radial) singularity.

Example 1.1 The trajectories of the differential equation

$$\begin{cases} \dot{x} = 2y + \frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + 0.9y \end{cases} \quad (1.2)$$

are depicted in Figure (1.1).

The collection of the images of the solutions of (1.1) gives us us an analytic singular foliation $\mathcal{F} = \mathcal{F}(X)_{\mathbb{R}} = \mathcal{F}(X) = \mathcal{F}_{\mathbb{R}}$ in \mathbb{R}^2 . Therefore, when we are talking

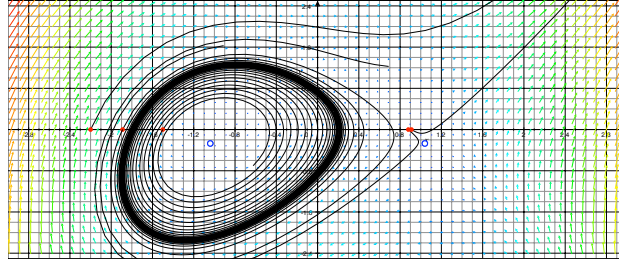


Fig. 1.1 A limit cycle crossing $(x,y) \sim (-1.79,0)$

about a foliation we are not interested in the parametrization of its leaves (trajectories). It is left to the reader to verify that:

Exercise 1.4 For a polynomial $R \in \mathbb{R}[x,y]$, a vector field X in \mathbb{R}^2 and a point $a \in \mathbb{R}^2$ with $X(a) \neq 0$ and $R(a) \neq 0$, the image of a solutions of X and $R \cdot X$ passing through a are the same.

In other words, the foliation associated to X and $R \cdot X$ in $\mathbb{R}^2 \setminus \{R = 0\}$ are the same. For this reason from the beginning we have assumed that P and Q have no common factors. Being interested only on the foliation $\mathcal{F}(X)$, we may write (1.1) in the form

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)},$$

$$\omega = 0, \text{ where } \omega = Pdy - Qdx \in \Omega_{\mathbb{R}^2}^1.$$

In the second case we use the notation $\mathcal{F} = \mathcal{F}(\omega)_{\mathbb{R}} = \mathcal{F}(\omega)$. In this case the foliation \mathcal{F} is characterized by the fact that ω restricted to the leaves of \mathcal{F} is identically zero. A systematic definition of differential 1-forms will be done in §2.1.

Definition 1.1 The singular set of the foliation $\mathcal{F}(Pdy - Qdx)$ is defined in the following way:

$$\text{Sing}(\mathcal{F}) = \text{Sing}(\mathcal{F})_{\mathbb{R}} := \{(x,y) \in \mathbb{R}^2 \mid P(x,y) = Q(x,y) = 0\}.$$

By our assumption $\text{Sing}(\mathcal{F})$ is a finite set of points. The leaves of \mathcal{F} near a point $A \in \text{Sing}(\mathcal{F})$ may be complicated.

Exercise 1.5 Using a software which draws the trajectories of vector fields, describe the solutions of (1.2) near its singularities.

By Bezout theorem we have

$$\#\text{Sing}(\mathcal{F}) \leq \deg(P) \deg(Q)$$

The upper bound can be reached, for instance by the differential equation $\mathcal{F}(Pdy - Qdx)$, where $P = (x-1)(x-2)\cdots(x-d)$, $Q = (y-1)(y-2)\cdots(y-d')$.

1.2 Poincaré first return map

From topological point of view a leaf L of $\mathcal{F} = \mathcal{F}(\omega)$ is either homeomorphic to \mathbb{R} or to the circle $S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. In the second case L is called a closed solution of \mathcal{F} (but not yet a limit cycle).

Exercise 1.6 For a foliation $\mathcal{F} = \mathcal{F}(\omega)_{\mathbb{R}}$ the curve $\{R = 0\}$, where $d\omega = Rdx \wedge dy$, intersects all closed leaves of \mathcal{F} .

Exercise 1.7 Show that if $d(Pdy - Qdx) = 0$ then there is a polynomial $R \in \mathbb{R}[x, y]$ of degree $\leq \deg(P) + 1$ and $\deg(Q) + 1$ such that $\omega = dH$.

We consider a point $p \in L$ and a transversal section Σ to \mathcal{F} at p . For any point q in Σ near enough to p , we can follow the leaf of \mathcal{F} in the anti-clockwise direction and since L is closed we will encounter a new point $h(q) \in \Sigma$. We have obtained an analytic function

$$h : \Sigma \rightarrow \Sigma,$$

which is called the Poincaré first return map. Later, in the context of holomorphic foliations we will call it the holonomy map. Usually we take a coordinate system z in Σ with $z(p) = 0$ and write the power series of h at 0:

$$h(z) = \sum_{i=1}^{\infty} \frac{h^{(i)}(0)}{i!} z^i$$

Definition 1.2 $h'(0)$ is called the multiplier of the closed solution L . If the multiplier is 1 then we say that h is tangent to the identity and we have

$$h(z) = z + \dots + a_n z^n + \dots, \quad a_n \in \mathbb{C}, \quad a_n \neq 0.$$

for some $n \in \mathbb{N}_{\geq 2}$ which is called the tangency order of h . In other words, the tangency order is n if

$$h^{(i)}(0) = 0, \quad \forall 1 < i < n, \quad \text{and } h^{(n)}(0) \neq 0.$$

A closed solution L of \mathcal{F} is called a limit cycle if its Poincaré first return map is not identity. In case the Poincaré first return map is identity then the leaves of \mathcal{F} near L are also closed. In this case we can talk about the continuous family of cycles δ_z , $z \in \Sigma$, where δ_z is the leaf of \mathcal{F} through z .

Exercise 1.8 Prove that the multiplier and order of tangency do not depend on the coordinate system z in Σ .

Proposition 1.1 *In the above situation, we have*

$$h'(0) = \exp\left(-\int_{\delta} \frac{d\omega}{\omega}\right).$$

The proof will be presented in Chapter ??

1.3 Hilbert 16-th problem

It is natural to ask whether a foliation $\mathcal{F}(Pdy - Qdx)$ has a finite number of limit cycles. This is in fact the first part of Hilbert 16-th problem:

Theorem 1.3 (Ilyashenko [Iy91], Écalle [É92]) *Each polynomial foliation $\mathcal{F}(Pdy - Qdx)$ has a finite number of limit cycles.*

The above theorem was proved by Yu. Ilyashenko and J. Écalle independently. We have associated to each foliation \mathcal{F} the number $N(\mathcal{F})$ of limit cycles of \mathcal{F} . It is natural to ask how $N(\mathcal{F})$ depends on the ingredient polynomial P and Q of \mathcal{F} .

Conjecture 1.1 (Hilbert 16'th problem) *Fix a natural number $n \in \mathbb{N}$. There is a $N(n) \in \mathbb{N}$ depending only on n such that each foliation $\mathcal{F}(Pdx - Qdy)$ with $\deg(P), \deg(Q) \leq n$ has at most $N(n)$ limit cycles.*

Of course, it would be of interest to give an explicit description of $N(n)$ and more strongly determine the nature of

$$N(n) := \max \left\{ N(\mathcal{F}(\omega)) \mid \omega = Pdy - Qdx, \deg(P), \deg(Q) \leq n \right\}.$$

One of the objective of the present text is to explain the fact that Hilbert 16'th problem is a combination of many unsolved difficult problems. We note that even the case $n = 2$ is open.

1.4 Algebraic curves invariant by foliations

Let $f \in \mathbb{R}[x, y]$. An algebraic curve over \mathbb{R} is defined to be

$$\{f = 0\} := \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}.$$

It can happen that such an algebraic curve is empty, for instance take $f = x^2 + y^2 + 1$, or it is a point, for instance take $f = x^2 + y^2$. For a moment assume that $\{f = 0\}$ near to a point look like a smooth curve, for instance take $f = x^2 + y^2 - 1$ for which the curve is a circle of radius 1.

Let $\mathcal{F}(X)$, $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ be a foliation in \mathbb{R}^2 as before. We would like to see when the smooth part of f is a part of trajectories of X (leaves of $\mathcal{F}(X)$). The gradient vector

$$\frac{\partial f}{\partial x} \frac{\partial}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial}{\partial y}$$

is perpendicular to the curve and so if we have

$$\frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q = f \cdot R. \quad (1.3)$$

then $\{f = 0\}$ in neighborhood of p is a part of a leaf of \mathcal{F} . If the equality (1.3) occurs then we say that the algebraic curve $\{f = 0\}$ is \mathcal{F} -invariant.

Chapter 2

Darboux's theorem

In this chapter we will state and prove a theorem due to G. Darboux in [Dar78]. It says that if an algebraic foliation in \mathbb{C}^2 has an infinite number of algebraic leaves then it must have a first integral. We will actually work over an algebraically closed field k of characteristic zero instead of \mathbb{C} , and this will force us to use the algebro-geometric notation \mathbb{A}_k^2 instead of \mathbb{C}^2 . This will also automatically lead the reader to think about similar problems when the characteristic of the field k is not zero.

2.1 Some algebraic notations

The set of polynomial differential 0-forms, 1-forms and 2-forms are respectively given by

$$\begin{aligned}\Omega_{\mathbb{A}_k^2}^0 &:= k[x, y], \\ \Omega_{\mathbb{A}_k^2}^1 &:= \{Pdy - Qdx \mid P, Q \in k[x, y]\}, \\ \Omega_{\mathbb{A}_k^2}^2 &:= \{Pdx \wedge dy \mid P \in k[x, y]\}.\end{aligned}$$

The wedge product is defined in the following way:

$$(P_1dx + Q_1dy) \wedge (P_2dx + Q_2dy) := (P_1Q_2 - P_2Q_1)dx \wedge dy.$$

It follows from the definition that for all $\omega_1, \omega_2 \in \Omega_{\mathbb{A}_k^2}^1$ we have $\omega_1 \wedge \omega_1 = 0$ and $\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$. We have the differential maps:

$$\begin{aligned}d_0 : \Omega_{\mathbb{A}_k^2}^0 &\rightarrow \Omega_{\mathbb{A}_k^2}^1, \quad d_0(P) = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy. \\ d_1 : \Omega_{\mathbb{A}_k^2}^1 &\rightarrow \Omega_{\mathbb{A}_k^2}^2, \quad d_1(Pdx + Qdy) = d_0P \wedge dx + d_0Q \wedge dy.\end{aligned}$$

Exercise 2.1 Show that $d_1 \circ d_0 = 0$.

From now on we do not write the sub-index of d ; being clear in the context which of d_0 or d_1 it is.

Exercise 2.2 If $d\omega = 0$ for some $\omega \in \Omega_{\mathbb{A}_k}^1$ then there is a $f \in \Omega_{\mathbb{A}_k}^0$ such that $\omega = df$. Is this true for $\text{char}(k) \neq 0$? Can you classify all ω 's which do not satisfy the mentioned property.

An easier statement is that if $df = 0$ for $f \in \Omega_{\mathbb{A}_k}^0$ then f is a constant, that is, $f \in k$. This is false if the characteristic of k is not zero. For instance, in a field of characteristic p we have $dx^p = px^{p-1}dx = 0$ but x^p is not a constant.

Let $k = \mathbb{R}$ or \mathbb{C} . Let also $\gamma = (x(t), y(t)) : (k, 0) \rightarrow k^2$ be an analytic map and $\omega = Pdx + Qdy \in \Omega_{\mathbb{A}_k}^1$. The pull-back of ω by γ is defined to be

$$\gamma^*\omega := (P(x(t), y(t)) \frac{\partial x(t)}{\partial t} + Q(x(t), y(t)) \frac{\partial y(t)}{\partial t}) dt$$

Exercise 2.3 Show that $\gamma^*\omega = 0$ is independent of the parametrization t , i.e if $a : (k, 0) \rightarrow (k, 0)$ is an analytic map and $\gamma^*\omega = 0$ then $(\gamma \circ a)^*\omega = 0$.

If $\gamma^*\omega = 0$ then we say that ω restricted to the image of γ is zero. We denote by

$$k(x, y) := \left\{ \frac{P}{Q} \mid P, Q \in k[x, y] \right\}$$

the field of rational (meromorphic) functions in \mathbb{A}_k^2 . The set of meromorphic differential i -forms is denoted by $\Omega_{\mathbb{A}_k}^i(*)$ (instead of $k[x, y]$ we have used $k(x, y)$).

Exercise 2.4 Show that if for $\omega_1, \omega_2 \in \Omega_{\mathbb{A}_k}^1(*)$ we have $\omega_1 \wedge \omega_2 = 0$ then $\omega_2 = R\omega_1$ for some $R \in k(x, y)$. Formulated in a different way, show that if for $\omega_1 = Pdy - Qdx, \omega_2 \in \Omega_{\mathbb{A}_k}^1$ we have $\omega_1 \wedge \omega_2 = 0$ and P and Q are relatively prime then $\omega_2 = R\omega_1$ for some $R \in k[x, y]$. Is this exercise true for $\text{char}(k) \neq 0$.

Definition 2.1 For $\Omega \in \Omega_{\mathbb{A}_k}^2$ and $\omega \in \Omega_{\mathbb{A}_k}^1$ we denote by $\frac{\Omega}{\omega}$ any meromorphic differential 1-form α such that

$$\Omega = \omega \wedge \alpha.$$

For instance, if $\Omega = Rdx \wedge dy$ and $\omega = Pdx + Qdy$ then we can choose

$$\alpha := \frac{R}{P} dy.$$

It follows from Exercise 12 that the difference of two such α is an element in $k(x, y)\omega$.

Exercise 2.5 (Stokes formula) Let δ be a closed anti-clockwise oriented path in \mathbb{R}^2 which does not intersect itself. Let also Δ be the region in \mathbb{R}^2 which δ encloses. Then

$$\int_{\delta} \omega = \int_{\Delta} d\omega.$$

which is called the Stokes formula. Give a proof of this using the classical books in calculus.

2.2 Invariant algebraic sets

In §1.4 we have seen that if an algebraic curve is tangent to a vector field, this amounts to a polynomial equation. In this section we take this as definition.

Definition 2.2 We say that a curve $\{f = 0\}$ given by a polynomial $f \in k[x, y]$ is $\mathcal{F}(\omega)$ -invariant if

$$\omega \wedge df = f\eta, \quad \text{for some } \eta \in \Omega_{\mathbb{A}_k^2}^1. \quad (2.1)$$

The geometric description of the equality (2.1) is as follows. Let us write $\omega = Pdy - Qdx$ and $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ as usual. We know that

$$\omega \wedge df = (Pdy - Qdx) \wedge \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) = (X \cdot \nabla f) dx \wedge dy = fRdx \wedge dy \quad (2.2)$$

where $\eta = Rdx \wedge dy$. At a smooth point of $f = 0$ we have $X \cdot \nabla f = 0$ and ∇f is perpendicular to $\{f = 0\}$. These imply that X is tangent to $\{f = 0\}$ at that point. Let us take $f \in k[x, y]$ which is not necessarily irreducible. We take the decomposition $f = \prod_{i=1}^n f_i^{n_i}$, $n_i \in \mathbb{N}$ into irreducible polynomials f_i . We have

$$\omega \wedge \frac{df}{f} = \sum_{i=1}^n n_i \omega \wedge \frac{df_i}{f_i}$$

and so in Definition 4 we do not need to assume that f is irreducible. We may extend Definition 4 to meromorphic functions, that is, $n_i \in \mathbb{N}$. Let us define

$$\text{div}(f) := \sum_{i=1}^n n_i D_i, \quad D_i := \{f_i = 0\}.$$

and assume that D_i 's are distinct, that is, for distinct i and j , f_i is not a multiple of f_j by a constant in k .

Exercise 2.6 For $f \in k(x, y)$, $\{f = 0\}$ is \mathcal{F} -invariant if and only if all D_i 's are \mathcal{F} -invariant.

2.3 First integral

Definition 2.3 We say that $f \in k(x,y)$ is a (rational) first integral of the foliation $\mathcal{F}(\omega)$ if

$$\omega \wedge df = 0. \quad (2.3)$$

If this is the case we say that $\mathcal{F}(\omega)$ has a first integral. If $f \in k[x,y]$, that is, f is a polynomial then we say that $\mathcal{F}(\omega)$ is Hamiltonian.

Exercise 2.7 Using Exercise 12, show that if $\mathcal{F}(\omega)$ has the first integral f then there is $g \in k(x,y)$ such that

$$\omega = gdf.$$

Let us assume that $f = \frac{F}{G}$ where $F, G \in k[x,y]$ have no (non-constant) common factors. We have

$$df = \frac{G.dF - F.dG}{G^2}$$

and so $\mathcal{F}(G.dF - F.dG)$ has the first integral $\frac{F}{G}$.

Proposition 2.1 *If the foliation $\mathcal{F}(\omega)$ has the first integral $f := \frac{F}{G}$ as above then all the algebraic curves $F - cG = 0, c \in k$ are $\mathcal{F}(\omega)$ -invariant.*

Proof. The polynomial F and G have no common factors, and so by Exercise 14 it is enough to show that $\frac{F-cG}{G} = f - c = 0$ is \mathcal{F} -invariant. This follows from $\omega \wedge d(f - c) = \omega \wedge df = 0$. \square

2.4 Darboux's theorem

Theorem 2.1 (G. Darboux, [Dar78]) *If the foliation \mathcal{F} has infinite number of invariant algebraic curves then \mathcal{F} has a rational first integral.*

Recall that by definition two algebraic curves $\{f_1 = 0\}, \{f_2 = 0\}$ are the same if $f_1 = c \cdot f_2$ for some $c \in k$.

Proof. The proof is classical and can be found in [LNS] page 92. Let us assume that $\mathcal{F}(\omega)$ has infinite number of invariant algebraic curves $\{f_i = 0\}, i \in \mathbb{N}$. By definition $\omega \wedge df_i = f_i \cdot \eta_i, \eta_i \in \Omega_{\mathbb{A}_k^2}$. We rewrite this

$$\omega \wedge \frac{df_i}{f_i} = p_i dx \wedge dy \quad \text{where } p_i \in k[x,y]$$

A key observation in the proof is that $\deg(p_i)$ is independent of the degree of f_i . To see this fact we write

$$(Pdx + Qdy) \wedge \left(\frac{\partial f_i}{\partial x} dx + \frac{\partial f_i}{\partial y} dy \right) = f_i \cdot p_i dx \wedge dy$$

and so

$$P \frac{\partial f_i}{\partial x} - Q \frac{\partial f_i}{\partial y} = f_i \cdot p_i$$

Let $d := \max\{\deg(P), \deg(Q)\}$. Then

$$\deg(p_i) + \deg(f_i) = \deg(f_i \cdot p_i) = \deg\left(P \frac{\partial f_i}{\partial x} - Q \frac{\partial f_i}{\partial y}\right) \leq d + \deg(f_i) - 1$$

and so $\deg(p_i) \leq d - 1$. The vector space $k[x, y]_{\leq n} = \{f \in k[x, y] \mid \deg f \leq n\}$ is finite dimensional and in fact

$$\dim k[x, y]_{\leq n} = \binom{n+2}{2}$$

We set $n = d - 1$ and define a_d to be the dimension of the k -vector space generated by p_i 's. We have

$$a_d \leq \binom{d+1}{2}$$

We choose a basis p_1, p_2, \dots, p_{a_d} for such a vector space. The element p_{a_d+1} is linearly dependent with the element of such a basis, that is, there are $r_i \in k$, $i = 1, \dots, a_d + 1$ such that

$$\sum_{i=1}^{a_d+1} r_i \cdot p_i = 0$$

and $r_{a_d+1} \neq 0$. In other words

$$\omega \wedge \sum_{i=1}^{a_d+1} r_i \cdot \frac{df_i}{f_i} = \sum_{i=1}^{a_d+1} r_i (\omega \wedge \frac{df_i}{f_i}) = 0.$$

Let $\alpha = \sum_{i=1}^{a_d+1} r_i \frac{df_i}{f_i}$ which is a closed form, that is, $d\alpha = 0$. We repeat the same argument for $p_1, p_2, \dots, p_{a_d}, p_{a_d+2}$

$$\sum_{i=1, i \neq a_d+1}^{a_d+2} \tilde{r}_i \cdot p_i = 0$$

for some $\tilde{r}_i \in k$ and $\tilde{r}_{a_d+2} \neq 0$ and we get $\omega \wedge \beta = 0$ with $\beta := \sum_{i=1, i \neq a_d+1}^{a_d+2} \tilde{r}_i \frac{df_i}{f_i}$. We have

$$\omega \wedge \alpha = \omega \wedge \beta = 0.$$

From this we conclude that $\alpha = f\beta$, $\omega = g\beta$ for some non-constant functions $f, g \in k(x, y)$ (see Exercise 12). Since $d\alpha = 0$ we conclude that $df \wedge \beta = 0$ which together with $\omega = g\beta$ implies that f is a first integral of $\mathcal{F}(\omega)$. Note that f is non-constant because $\alpha = f\beta$ and in the expression of α and β we have respectively the terms $\frac{df_{a_d+1}}{f_{a_d+1}}$ and $\frac{df_{a_d+2}}{f_{a_d+2}}$. \square

Exercise 2.8 The last step in the proof of Darboux's theorem must be rewritten in order to make it algorithmic. Write such an algorithm with the input $\omega, f_i, i = 1, 2, 3, \dots$ and the output f .

It is possible to derive refinements of the Darboux's theorem by analyzing its proof.

Theorem 2.2 *If the foliation $\mathcal{F}(\omega)$, $\omega = Pdy - Qdx$, $\max(\deg(P), \deg(Q)) = d$ has $\binom{d+1}{2} + 2$ number of invariant algebraic curves then \mathcal{F} has a rational first integral.*

Proof. Same as proof of Theorem 2.1. \square

We observe that we have new examples of foliations appearing in the proof of Darboux's theorem.

Definition 2.4 A holomorphic foliation $\mathcal{F}(\omega)$ has a logarithmic first integral if there are polynomials $f_1, f_2, \dots, f_s \in k[x, y]$ and $\lambda_1, \lambda_2, \dots, \lambda_s \in k$ such that

$$\omega \wedge \left(\sum_{i=1}^s \lambda_i \frac{df_i}{f_i} \right) = 0.$$

For $k = \mathbb{R}$ or \mathbb{C} , the level surfaces of the multi-valued functions $f_1^{\lambda_1} f_2^{\lambda_2} \dots f_s^{\lambda_s}$ are tangent to the foliations $\mathcal{F}(\omega)$. We call this a logarithmic first integral of $\mathcal{F}(\omega)$.

Theorem 2.3 *If the foliation $\mathcal{F}(\omega)$, $\omega = Pdy - Qdx$, $\max(\deg(P), \deg(Q)) = d$ has $\binom{d+1}{2} + 1$ number of invariant algebraic curves then \mathcal{F} has a logarithmic first integral.*

Proof. Same as proof of Theorem 2.1. \square

Exercise 2.9 Discuss Darboux's theorem over a field of non-zero characteristic.

2.5 Optimal Darboux's theorem

Are the lower bounds for the number of algebraic solutions in Theorem 2.2 and Theorem 2.3 optimal? More precisely, is there a foliation of degree d with $\binom{d+1}{2} + 1$ (resp. $\binom{d+1}{2}$) algebraic solutions and without a rational (resp. logarithmic) first integral? In this section we would like to discuss this issue.

Theorem 2.2 and Theorem 2.3 are valid using the projective degree. For a foliation $\mathcal{F}(\omega)$, $\omega = Pdy - Qdx$ of projective degree d the 1-form seen as meromorphic form in \mathbb{P}_k^2 has a pole of order $d + 2$ at the line at infinity. If $\omega \wedge \frac{df}{f} = p dx \wedge dy$ then we know that $dx, dy, \frac{df}{f}$ and p have respectively pole order 2, 2, 1 and $\deg(p)$ at the line at infinity. This implies that $\deg(p) \leq d - 1$. Therefore, we can re state these theorem using projective degree.

The previous observation implies that Theorem 2.3 is not optimal if we use affine degree. The reason is as follows. Let us take a foliation $\mathcal{F}(\omega)$ of affine degree d and with $\frac{d(d+1)}{2}$ algebraic leaves in \mathbb{A}_k^2 . We consider two cases:

1. The line at infinity is not invariant. In this case \mathcal{F} has projective degree $d-1$ and $(d+1)d/2 \geq d(d-1)/2 + 1$. Therefore, Theorem 2.3 in the projective case implies that \mathcal{F} is logarithmic.

2. The line at infinity is invariant. We take an affine chart of \mathbb{P}_k^1 such that the line at infinity is not invariant, and hence, \mathcal{F} has $(d+1)d/2 + 1$ invariant algebraic leaves. Theorem 2.3 in the projective case implies that \mathcal{F} is logarithmic again.

The conclusion is that the projective degree is more natural when we deal with an optimal version of Darboux's theorem.

Proposition 2.2 *Consider the foliation*

$$\mathcal{F}(\omega), \quad \omega := xPdy - yQdx, \quad P, Q \in k[x, y]_{\leq 1}. \quad (2.4)$$

For generic choice of P and Q , the lines $x = 0$, $y = 0$ and the line at infinity are the only invariant algebraic curves of (2.4). In particular, the foliation $\mathcal{F}(\omega)$ is not logarithmic.

Proof. The first part of the theorem must be worked out by analysing the separatrices of the seven singularities of $\mathcal{F}(\omega)$. For the second part we take a chart for \mathbb{P}_k^2 such that the line at infinity is not invariant and we have three invariant lines $l_i(x, y) = 0$, $i = 1, 2, 3$. If $\mathcal{F}(\omega)$ is logarithmic then $d(\frac{\omega}{l_i}) = 0$, for some product of lines l_i , as $\mathcal{F}(\omega)$ has no other algebraic leaves except $l_i = 0$. This is an algebraic relation between the coefficients of P and Q .

2.6 Projective spaces

In algebraic geometry many theorems are stated for compact/complete varieties. A typical example is the Bezout theorem on the number of intersections of two curves. Curves in \mathbb{A}_k^2 may not intersect each other at all, even if we assume that k is an algebraically closed field. For instance, take $xy - 1 = 0$ and $x = 0$. In this case there are many intersection points at infinity, and we are going explain what means infinity in this case. This will lead us into various compactifications such as the usual projective space \mathbb{P}_k^2 , the product of two lines $\mathbb{P}^1 \times \mathbb{P}^1$ and weighted projective space \mathbb{P}^{1, w_1, w_2} , $w_1, w_2 \in \mathbb{N}$. Holomorphic foliations are also best viewed in a compactification of \mathbb{A}_k^2 and it turns out that depending on the foliation \mathcal{F} in \mathbb{A}_k^2 one compactification is better than another one.

The projective space of dimension n as a complex manifold is defined as follows:

$$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\}) / \sim$$

where

$$a, b \in \mathbb{C}^{n+1} - \{0\}, a \sim b \Leftrightarrow a = kb, \text{ for some } k \in \mathbb{C} - \{0\}.$$

For the purpose of the present text, we will mainly use \mathbb{P}^1 and \mathbb{P}^2 . The projective space of dimension one \mathbb{P}^1 is covered by two charts x, x' biholomorphic to \mathbb{C} and the transition map is given by

$$x' = \frac{1}{x}.$$

The projective space of dimension two \mathbb{P}^2 is covered by three charts $(x, y), (u, v), (u', v')$ biholomorphic to \mathbb{C}^2 and the transition maps are given by

$$v = \frac{y}{x}, u = \frac{1}{x}, \quad v' = \frac{x}{y}, u' = \frac{1}{y}.$$

Considering the chart $(\mathbb{C}^2, (x, y))$, \mathbb{P}^2 becomes a compactification of \mathbb{C}^2 .

2.7 Projective spaces as schemes

In this section we define the projective space of dimension two \mathbb{P}_k^2 over an arbitrary field. We also explain the main idea behind the definition \mathbb{P}_k^2 as a scheme. By the affine scheme \mathbb{A}_k^2 , we simply think of the polynomial ring $k[x, y]$. Open subsets of \mathbb{A}_k^2 are given by the localization of $k[x, y]$. We will need two open subsets of \mathbb{A}_k^2 given respectively by

$$k[x, y, \frac{1}{y}] \text{ and } k[x, y, \frac{1}{x}]$$

By the projective scheme \mathbb{P}_k^2 we mean three copies of \mathbb{A}_k^2 , namely

$$k[x, y], k[x, z], k[y, z]$$

together with the isomorphism of affine subsets:

$$k[x, y, \frac{1}{y}] \cong k[x, z, \frac{1}{z}], x \mapsto \frac{x}{z}, y \mapsto \frac{1}{z} \tag{2.5}$$

$$k[x, y, \frac{1}{x}] \cong k[y, z, \frac{1}{z}], x \mapsto \frac{1}{z}, y \mapsto \frac{y}{z}$$

$$k[x, z, \frac{1}{x}] \cong k[y, z, \frac{1}{y}], x \mapsto \frac{1}{y}, z \mapsto \frac{z}{y}$$

The best way to see these isomorphisms is the following. We look at an element of $k[x, y]$ as a function on the k -rational points k^2 of the first chart and for $(a, b) \in k^2$, we use the identities

$$[a; b; 1] = [\frac{a}{b}; 1; \frac{1}{b}] = [1; \frac{b}{a}; \frac{1}{a}].$$

Let C be a curve in \mathbb{A}_k^2 given by the polynomial $f(x, y) \in k[x, y]$. It induces a curve \bar{C} in \mathbb{P}_k^2 in the following way. Let us define $f_1 := f$ and

$$f\left(\frac{x}{z}, \frac{y}{z}\right) = z^{-d} f_2(x, z), \quad f\left(\frac{1}{z}, \frac{y}{z}\right) = z^{-d} f_3(y, z)$$

We think of the the curve \bar{C} in the same way as \mathbb{P}_k^2 , but replacing $k[x, y]$ with $k[x, y]/\langle f_1 \rangle$ and so on. Here, $\langle f_1 \rangle$ is the ideal $k[x, y]$ generated by a single element f_1 . We can also think of C in the same way as \mathbb{P}_k^2 but with the following additional relations between variables:

$$f_1(x, y) = 0 \text{ in } k[x, y]$$

$$f_2(x, z) = 0 \text{ in } k[x, z]$$

and

$$f_3(y, z) = 0 \text{ in } k[y, z].$$

The above discussion does not use the fact that k is a field. In fact, we can use an arbitrary ring R instead of k . In this way, we say that we have an scheme C over the ring R . The function field of the projective space \mathbb{P}_k^2 is defined to be

$$k(\mathbb{P}_k^2) := k(x, y) \cong k(x, z) \cong k(y, z),$$

where the isomorphisms are given by (2.5). The field of rational functions on the curve C is the field of fractions of the ring $k[x, y]/\langle f_1 \rangle$. Using the isomorphism (2.5), this definition does not depend on the chart with (x, y) coordinates. We can also think of $k(C)$ as $k(x, y)$ but with the relation $f_1(x, y) = 0$ between the variables x, y . Any $f \in k(C)$ induces a map

$$C(k) \rightarrow k$$

that we denote it by the same letter f .

2.8 Foliations in projective spaces

A foliation $\mathcal{F}(\omega)$, $\omega = Pdy - Qdx$ extends to a holomorphic foliation in \mathbb{P}_k^2 . For instance, in the chart (u, v) we have

$$\omega = P\left(\frac{1}{u}, \frac{v}{u}\right)d\left(\frac{v}{u}\right) - Q\left(\frac{1}{u}, \frac{v}{u}\right)d\left(\frac{1}{u}\right) = \frac{\tilde{P}(u, v)dv - \tilde{Q}(u, v)du}{u^{d+2}}, \quad (2.6)$$

$$\tilde{P}, \tilde{Q} \in k[u, v].$$

Definition 2.5 The smallest number d in the equality (2.6) is called the (projective) degree of the foliation $\mathcal{F}(\omega)$.

It is also natural to define the (affine) degree of $\mathcal{F}(\omega)$:

$$\deg(\mathcal{F}) := \max\{\deg(P), \deg(Q)\}.$$

These two notions of degree are different. Working with foliations in \mathbb{P}_k^2 it is useful to use the projective degree.

Proposition 2.3 *A foliation of the projective degree d in the affine coordinate $\mathbb{A}_k^2 \subset \mathbb{P}_k^2$ is given by the differential form:*

$$Pdx + Qdy + g(xdy - ydx)$$

where either g is a non-zero homogeneous polynomial of degree d and $\deg(P), \deg(Q) \leq d$ or g is zero and $\max\{\deg(P), \deg(Q)\} = d$. In the first case the line at infinity is not invariant by \mathcal{F} and in the second case it is invariant by \mathcal{F} .

Proof?

Proof. \square

Proposition 2.4 *Let k be an algebraically closed field and let \mathcal{F} be a foliation in \mathbb{P}_k^2 of projective degree d . A line in \mathbb{P}_k^2 which does not cross any singularity of \mathcal{F} has d (counted with multiplicity) tangency points with the foliation \mathcal{F} . In particular, for a generic line we have exactly d simple tangency points.*

Proof?

Proof. \square

There are foliations in \mathbb{P}_k^2 which are naturally given by meromorphic 1-forms ω , see for instance §2.8. In homogeneous coordinates we write

$$\omega = f_1^{k_1} f_2^{k_2} \cdots f_s^{k_s} (Pdx + Qdy + Rdz), \quad k_i \in \mathbb{Z}, \quad Px + Qy + Rz = 0$$

where f_i, P, Q, R are homogeneous polynomials in $k[x, y, z]$ and P, Q, R have no non-constant common factors. We define

$$\text{Div}(\omega) := \sum_{i=1}^s k_i D_i, \quad D_i = \{f_i = 0\}.$$

We use the following in order to determine the degree of \mathcal{F} .

Proposition 2.5 *Let $\mathcal{F}(\omega)$ be a foliation in \mathbb{P}_k^2 given by the meromorphic form ω as above. The projective degree of \mathcal{F} is $\sum_{i=1}^s k_i \deg(f_i)$.*

Proof. \square

Remark 2.1 Let $\mathcal{F}(\omega)$ be a foliation in \mathbb{A}_k^2 given by a meromorphic form ω . We consider ω as a meromorphic form in \mathbb{P}_k^2 and hence ω might have zeros or poles along the line at infinity. The affine degree of \mathcal{F} is the projective degree of \mathcal{F} if the line at infinity is \mathcal{F} -invariant and it is equal to the projective degree of \mathcal{F} plus 1 if the line at infinity is not \mathcal{F} -invariant.

2.9 Jouanolou foliation

The holomorphic foliation defined in \mathbb{A}_k^2 by the 1-form

$$\omega := (y^d - x^{d+1})dy - (1 - x^d y)dx$$

is called the Jouanolou foliation of degree d . One usually compactify \mathbb{A}_k^2 inside \mathbb{P}_k^2 . Consider the group

$$G := \{\varepsilon \in \mathbb{C} \mid \varepsilon^{d^2+d+1} = 1\}.$$

It acts on \mathbb{A}_k^2 (and hence in \mathbb{P}_k^2):

$$(\varepsilon, (x, y)) \rightarrow (\varepsilon^{d+1}x, \varepsilon y) \quad \varepsilon \in G, (x, y) \in \mathbb{A}_k^2$$

It has a fixed point $p_1 = (0, 0)$ at \mathbb{A}_k^2 (and two other fixed points $p_2 = [0 : 1 : 0]$, $p_3 = [1 : 0 : 0]$ at infinity). For each $\varepsilon \in G$ we have $\varepsilon^*(\omega) = \varepsilon^{d+1}\omega$ and so G leaves the Jouanolou invariant. We have

$$\text{Sing}(\mathcal{F}_d)_{\mathbb{C}} = \{(\varepsilon, \varepsilon^{-d}) \mid \varepsilon \in G\}$$

(there is no singularity at infinity) and G acts on $\text{Sing}(\mathcal{F}_d)$ transitively.

For pictures of Jouanolou foliation see [MV09]

2.10 Riccati foliations

Another natural compactification of $\mathbb{A}_k^2 = \mathbb{A}_k^1 \times \mathbb{A}_k^1$ is $\mathbb{P}^1 \times \mathbb{P}^1$ which is useful for studying the Riccati foliations is given by:

$$\omega = q(x)dy - (p_0(x) + p_1(x)y + p_2(x)y^2)dx, \quad p_0, p_1, p_2, q \in k[x].$$

Substituting $y = \frac{1}{y}$ we have

$$\omega = \frac{1}{y^2}(-q(x)dy - (p_0(x)y^2 + p_1(x)y + p_2(x))dx)$$

and so all the projective lines $\{a \in \mathbb{C} \mid q(a) \neq 0\} \times \mathbb{P}^1$ are transversal to the foliation. This will be later used to define the global holonomy of Riccati foliations.

2.11 Weighted projective space

Let $w := (w_0, w_1, \dots, w_n)$, where w_i are natural numbers. The multiplicative group \mathbb{C}^* acts on $\mathbb{C}^{n+1} - \{0\}$ by

$$k \cdot x = (k^{w_0}x_0, k^{w_1}x_1, \dots, k^{w_n}x_n), \quad x \in \mathbb{C}^{n+1} - \{0\}, \quad k \in \mathbb{C}^*$$

The weighted projective space \mathbb{P}^w is a topological space defined by the quotient:

$$\mathbb{P}^w = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*.$$

For $w = (1, 1, \dots, 1)$ this is the usual projective space of dimension n . In general \mathbb{P}^w might have singularities, and hence, it is not necessarily a complex manifold. For the purpose of the present text, we need \mathbb{P}^{1, w_1, w_2} , that is, the first weight is 1. We have an open subset of $\mathbb{P}^w = \mathbb{P}^{1, w_1, w_2}$ given by

$$\mathbb{A}_k^2 \subset \mathbb{P}^w, \quad (x, y) \mapsto [1 : x : y].$$

In this way we consider weights $\deg(x) := w_1$, $\deg(y) := w_2$ and define the weighted degree of a polynomial with these weights: $\deg_w(x^n y^m) = nw_1 + mw_2$. For a polynomial P we write its homogeneous decomposition $P = P_0 + P_1 + \dots + P_a$, $P_a \neq 0$, where $\deg_w(P_i) = i$, and define $\deg_w(P) = a$. For a foliation $\mathcal{F}(\omega)$, $\omega = Pdy - Qdx$ we define

$$\deg_w(\mathcal{F}) := \max(\deg_w(\omega)).$$

where we have considered weights for differential forms: $\deg_w dx := w_1$, $\deg_w(dy) = w_2$.

We consider the weighted projective space $\mathbb{P}^{1, 2, d}$. A foliation \mathcal{F} of degree $2d$ is given by

$$\omega := (y + R(x))dy + (Q(x) + yP(x))dx,$$

$$\deg_w(R) \leq d, \quad \deg_w(Q) \leq 2d - 2, \quad \deg_w(P) \leq d - 2.$$

We can make the change of variable $(x, y) \mapsto (x, y + R(x))$ and arrive at the Liénard foliation in \mathbb{A}_k^2 :

$$\mathcal{F}(\omega), \quad \omega := ydy - (P(x) + yQ(x))dx, \quad \deg_w(Q) \leq 2d - 2, \quad \deg_w(P) \leq d - 2. \quad (2.7)$$

2.12 Foliations given by closed forms

Let us consider a meromorphic 1-form in \mathbb{A}_k^2 . In an affine coordinate system (x, y) we can write $\omega = \frac{P(x, y)dy - Q(x, y)}{R(x, y)}$, where $P, Q, R \in k[x, y]$. In this section we want to classify closed meromorphic 1-forms, that is, those with $d\omega = 0$. This will give us the class of foliations with Liouvillian first integrals. The following proposition in the complex context has been proved in [LNS, Propostion 2.5.1].

Theorem 2.4 *A meromorphic 1-form in \mathbb{A}_k^2 is closed if and only if it can be written as*

$$\omega = \sum_{i=1}^k \lambda_i \frac{df_i}{f_i} + d \left(\frac{g}{f_1^{r_1-1} f_2^{r_2-1} \dots f_k^{r_k-1}} \right), \quad r_i \in \mathbb{N}, \quad \lambda_i \in k, \quad f_i, g \in k[x, y]. \quad (2.8)$$

Proof. Clearly the differential form in (2.8) is closed. Let us prove the other direction. Let ω be a meromorphic closed 1-form in \mathbb{A}_k^2 . It is enough to prove the theorem over complex numbers. If ω is defined over k then we take $\{f_i = 0\}$ the irreducible components of the pole divisor of ω , and so, $f \in k[x, y]$. Moreover, r_i is the pole order of ω along $f_i = 0$ which can be read directly from the expression of ω . We claim that λ_i 's and g are also defined over k . If not we take $\sigma \in \text{Gal}(\bar{k}/k)$ which do not fix the expression in the right hand side of (2.8). After acting this on both hand side of (2.8) and taking the difference with (2.8) we get an equality of the form:

$$\sum_{i=1}^k (\lambda_i - \sigma(\lambda_i)) \frac{df_i}{f_i} + d \left(\frac{g - \sigma(g)}{f_1^{r_1-1} f_2^{r_2-1} \dots f_k^{r_k-1}} \right) = 0.$$

Since r_i is the pole order of ω along $f_i = 0$, f_i does not divide g . All these and the above equality imply that $\sigma(\lambda_i) = \lambda_i$, $\sigma(g) = g$ and so these are defined over k . Now let us prove the theorem over complex numbers. \square

proof?

Exercise 2.10 Rewrite Theorem 2.4 and its proof for a meromorphic differential 1-form in \mathbb{P}_k^n and in homogeneous coordinates, see [LNS, Proposition 2.5.1].

Definition 2.6 Let ω be as in (2.8). The following multi-valued function

$$f_1^{\lambda_1} f_2^{\lambda_2} \dots f_k^{\lambda_k} \exp \left(\frac{g}{f_1^{r_1-1} f_2^{r_2-1} \dots f_k^{r_k-1}} \right) \quad (2.9)$$

is called the Liouvillian first integral of $\mathcal{F}(\omega)$.

Definition 2.7 A foliation \mathcal{F} in \mathbb{A}_k^2 given by

$$\omega = \sum_{i=1}^k \lambda_i \frac{df_i}{f_i}, \quad r_i \in \mathbb{N}, \quad \lambda_i \in k, \quad f_i \in k[x, y].$$

is called a logarithmic foliation.

For the deformation of holomorphic foliations the projective degree is more suitable. Let \mathcal{F} be a foliation in \mathbb{P}_k^2 given by a closed meromorphic 1-form. Similar to the affine case we can write

$$\omega = \sum_{i=1}^k \lambda_i \frac{df_i}{f_i} + d \left(\frac{g}{f_1^{r_1-1} f_2^{r_2-1} \dots f_k^{r_k-1}} \right), \quad r_i \in \mathbb{N}, \quad \lambda_i \in k, \quad \sum \lambda_i \deg(f_i) = 0, \quad . \quad (2.10)$$

and f_i, g 's are homogenous polynomial in $k[x, y, z]$ and $\deg(g) = \deg(f_1^{r_1-1} f_2^{r_2-1} \dots f_k^{r_k-1})$.

By Proposition 6 we know that the projective degree of \mathcal{F} is $\sum_{i=1}^k d_i \cdot s_i$, where $d_i = \deg(f_i)$.

Proposition 2.6 (Lins Neto, [Net07], Observation 3.3.1, page 104) *The foliation \mathcal{F} as above is in the Zariski closure of $\mathcal{L}(d_1, d_2, \dots, d_k, \sum_{i=1}^k (s_j - 1)d_i)$.*

Proof.

2.13 Foliations and schemes

Most of the experts in holomorphic foliations has avoided to use the machinery of schemes for foliations. The main reason is that still for holomorphic foliations there are many open problems related to their dynamics and topology, and so producing more arithmetic problems and in this way justifying the usage of schemes, does not seem to accesible, or to be a priority. In this section by a simple example we explain why the language of schemes can be useful for deling with foliations. Of course, the content of this section is just a glance into a territory which might produce fruitful applications in the future.

We assume that k is algebraically closed field of characteristic zero. By definition $\mathbb{A}_k^2 := \text{Spec}(k[x, y])$ is the set of all prime ideals p of $k[x, y]$. It has three types of points.

1. A closed point of \mathbb{A}_k^2 is given by the ideal $\langle x - a, y - b \rangle$ for $a, b \in k$, and hence, the set of closed points is usually identified with k^2 . A geometer usually think of \mathbb{A}_k^2 as the set of its closed points.
2. The generic point p is simply given by the zero ideal, and one might identify it with whole \mathbb{A}_k^2 .
3. For an irreducible polynomial $f \in k[x, y]$ we have the prime ideal $p \in \langle f \rangle \in \mathbb{A}_k^2$. $f \in k[x, y]$.

In \mathbb{A}_k^2 we have a natural topology, see [hor77, page 70], and so we can talk about neighborhood of points, the ring of germs of regular functions \mathcal{O}_p and the ring of formal functions at p .

$$\check{\mathcal{O}}_p := \lim_{n \rightarrow \infty} \frac{\mathcal{O}_p}{p^n}.$$

For a closed point p , this is just the ring of formal power series around p . Let $\mathcal{F}(\omega)$, $\omega = Pdx - Qdy$ be a foliation in \mathbb{A}_k^2

Definition 2.8 *We say that ω has a first integral near a point $p \in \mathbb{A}_k^2$ if there is $f \in \check{\mathcal{O}}_p$ such that $\omega \wedge df = 0$.*

Let us write down this definition in simple words. By definition $\mathcal{F}(\omega)$ has a first integral around the generic point if and only if it has polynomial first integral $f \in k[x, y]$, that is, $\omega \wedge df = 0$. For a closed point p , \mathcal{F} has a first integral around $p = (a, b)$ if there is a formal power series f in $x - a, y - b$ and with coefficients in k such that $\omega \wedge df = 0$. The first fundamental theorem in ordinary differential equation says that if p is not a singularity of \mathcal{F} then \mathcal{F} has a first integral f around p . Moreover f is convergent in a small neighborhood of p if $k = \mathbb{C}$. Now assume that p is neither

a closed nor generic point. Therefore, $p = k[x, y]f$ for an irreducible polynomial $f \in k[x, y]$.

Proposition 2.7 *A foliation $\mathcal{F}(\omega)$ has a first integral around $p = \langle f \rangle$, $f \in k[x, y]$ an irreducible polynomial, if and only if there are sequence of polynomials $f_n \in k[x, y]$ such $f_{n+1} - f_n = f^n g_n$ for some $g_n \in k[x, y]$ and*

$$\omega \wedge df_n \in f^{n-1} \theta_n,$$

for some $\theta_n \in \Omega_{\mathbb{A}_k}^2$ and and for large enough n .

Proof. This is just the translation of a first integral around p . Note that

$$\omega \wedge \cdot : \mathcal{O}_p/p^{n+1} \rightarrow \mathcal{O}_p/p^{n+1} dx \wedge dy.$$

□

Proposition 2.8 *If \mathcal{F} as Proposition 8 has first integral around $p = \langle f \rangle$ then it has first integral around any closed point (a, b) with $f(a, b) = 0$.*

Chapter 3

Holomorphic foliations

In Chapter 1 we have considered foliations over the field of real numbers as this is needed in the announcement of Hilbert's 16-th problem and in Chapter 2 we have considered foliations over a fairly arbitrary field as in the announcement of Darboux's theorem we do not need a specific field. In this chapter we work with foliations over the field of complex numbers and call them holomorphic foliations. We replace \mathbb{C}^2 with \mathbb{R}^2 and hence the defining differential form ω of a foliation $\mathcal{F}(\omega)$ might have coefficients in \mathbb{C} . This is the beginning of the theory of holomorphic foliations on complex manifolds. We will also introduce the notion of holonomy which is a natural counterpart of the Poincaré's first return map introduced in §1.2

3.1 Complexification

Most of the discussion in Chapter 1 is valid replacing \mathbb{R} with \mathbb{C} . In this way, we replace the term analytic with holomorphic. In particular,

Theorem 3.1 *For $a \in \mathbb{C}^2$ if $X(a) \neq 0$ then there is a biholomorphism $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, a)$ such that the push-forward of $\frac{\partial}{\partial x}$ by F is X .*

The images of the complex solutions of the vector field X give us a (singular) holomorphic foliation $\mathcal{F} = \mathcal{F}(\omega)_{\mathbb{C}} = \mathcal{F}_{\mathbb{C}}$ in \mathbb{C}^2 . The leaves of $\mathcal{F}_{\mathbb{C}}$ are two dimensional real manifolds embedded in a real four dimensional space $\mathbb{C}^2 \cong \mathbb{R}^4$. If the differential 1-form ω is defined over \mathbb{R} , that is $P, Q \in \mathbb{R}[x, y]$, we can talk about both foliations $\mathcal{F}_{\mathbb{R}} = \mathcal{F}(\omega)_{\mathbb{R}}$ and $\mathcal{F}_{\mathbb{C}} = \mathcal{F}(\omega)_{\mathbb{C}}$ in \mathbb{R}^2 and \mathbb{C}^2 respectively. Note that $\mathbb{R}^2 \subset \mathbb{C}^2$ and

$$\mathcal{F}_{\mathbb{R}} = \mathbb{R}^2 \cap \mathcal{F}_{\mathbb{C}}, \quad (3.1)$$

that is, the intersection of a leaf of $\mathcal{F}_{\mathbb{C}}$ with \mathbb{R}^2 is a union of leaves of $\mathcal{F}_{\mathbb{R}}$. Note that $\mathcal{F}_{\mathbb{C}}$ may have more singularities which do not lie in the real plane \mathbb{R}^2 . In order to get some intuition of (3.1) we explain it for a very simple curve, that is, the equation of a circle. Let

$$C : x^2 + y^2 = 1, \quad D : xy - 1 = 0.$$

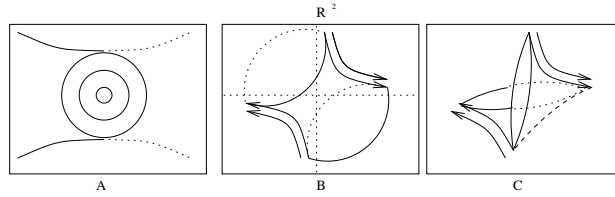


Fig. 3.1 Correct intuition

The curve $C(\mathbb{R})$ is the circle of radius 1 and $D(\mathbb{R})$ is a hyperbola and they are not isomorphic topological spaces because the first one has one connected component, whereas the second one has two. However, over complex numbers these two curves are the same and the isomorphism is given by

$$C(\mathbb{C}) \rightarrow D(\mathbb{C}), \quad (x, y) \mapsto (x + iy, x - iy),$$

where $i = \sqrt{-1}$. The curve $D(\mathbb{C})$ is parameterized in the polar coordinates by

$$x = re^{2\pi i\theta}, \quad y = r^{-1}e^{-2\pi i\theta}, \quad r \in \mathbb{R}^+, \quad \theta \in [0, 1]. \quad (3.2)$$

Since the bijection $\mathbb{R}^+ \rightarrow \mathbb{R}^+$, $x \mapsto x^{-1}$ sends 0 to ∞ , both curves $C(\mathbb{C})$ and $D(\mathbb{C})$ are cylinders with two infinities, let us say $-\infty$ and $+\infty$. A cycle δ travels from $-\infty$ to $+\infty$ and it covers the whole cylinder. We would like to make a correct intuition of this travel. This is fairly easy in the case of $C(\mathbb{C})$. This cycle is in the real four dimensional space \mathbb{C}^2 . In a certain time it fully lies in the two dimensional space $\mathbb{R}^2 \subset \mathbb{C}^2$ which is seen as a circle of radius 1 and center $0 \in \mathbb{R}^2$. It disappears from the two dimensional world and continues its travel toward $-\infty$, see Figure 3.1, A. The case of $D(\mathbb{C})$ is a little bit tricky as the first reasonable intuition turns out to be false. First of all we have to identify two connected components of the hyperbola $D(\mathbb{R})$ inside the cylinder $D(\mathbb{C})$. These are just two lines in $D(\mathbb{C})$ coming from $-\infty$ and going to $+\infty$ without touching each other. Our cycle touches each of these lines at exactly one point and it seems to make the intuition in Figure 3.1, B. However, a simple check with the parametrization (4.8) gives us the intuition in Figure 3.1, C, that is, the cycle δ near $-\infty$ is stretched along the y -axis and as it goes to $+\infty$ it becomes stretched along the x -axis.

3.2 Integrating form and transversal section

In this section we take a regular point $a \in \mathbb{C}^2$ of a foliation $\mathcal{F}(\omega)$ in the complex manifold (\mathbb{C}^2, a) which is just an small open neighborhood of a in \mathbb{C}^2 . In this context the ingredient P and Q of $\omega := Pdy - Qx$ can be holomorphic functions in (\mathbb{C}^2, a) .

We denote by $\mathcal{O}_{\mathbb{C}^2, a}$ the ring of holomorphic functions in (\mathbb{C}^2, a) and hence $P, Q \in \mathcal{O}_{\mathbb{C}^2, a}$. We denote by L_p the leaf of $\mathcal{F}(\omega)$ through $p \in (\mathbb{C}^2, a)$.

Theorem 3.2 *Assume that a is not a singularity of $\mathcal{F}(\omega)$ (regular point). There are holomorphic functions $f, g \in \mathcal{O}_{\mathbb{C}^2, a}$ such that*

$$\omega = g \cdot df$$

Further, $g(a) \neq 0$, $f(a) = 0$ and f is regular at a , that is, the derivation of f at zero is not zero.

Proof. The proof follows from Theorem 3.1. Since $\omega(X) = 0$ and the push-forward of $\frac{\partial}{\partial x}$ by F is X , the pull-back $\tilde{\omega}$ of ω under the map F necessarily satisfies $\tilde{\omega}(\frac{\partial}{\partial x}) = 0$ which implies that $\tilde{\omega} = \tilde{g}dy$, for some holomorphic function $\tilde{g} \in \mathcal{O}_{\mathbb{C}^2, 0}$. Therefore, $\omega = g \cdot df$, where g is the push-forward of \tilde{g} and f is the second coordinate of the inverse of F . \square

Definition 3.1 In Theorem 3.2, we call f a local first integral of $\mathcal{F}(\omega)$ and call g a local integrating factor of $\mathcal{F}(\omega)$.

In Theorem 3.1 we usually need the inverse $\zeta := F^{-1} = (\tilde{f}, f)$, $f, \tilde{f} \in \mathcal{O}_{\mathbb{C}^2, a}$ of F and call it a local chart for \mathcal{F} around a . Note that by our proof of Theorem 3.2, a local first integral is a second coordinate of a local chart. We would like to discuss the issue of different choices of pairs (f, g) .

Proposition 3.1 *In Theorem 3.2 let us consider two pairs (f_i, g_i) , $i = 1, 2$ such that*

$$\omega = g_1 df_1 = g_2 df_2.$$

There is a biholomorphism $h : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ such that

$$f_2 = h \circ f_1, \quad g_2 = \frac{g_1}{h'(f_1)}. \quad (3.3)$$

Proof. Since the derivative of f_i , $i = 1, 2$ at a is non-zero, we can find a holomorphic function $\tilde{f}_i \in \mathcal{O}_{\mathbb{C}^2, a}$ such that $\zeta_i := (g_i, \tilde{f}_i)$, $i = 1, 2$ is a chart of \mathcal{F} . The map $\zeta_2 \circ \zeta_1^{-1} : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ is a biholomorphism and it sends dy to $\tilde{h}dy$, for some $\tilde{h} \in \mathcal{O}_{\mathbb{C}^2, 0}$. Therefore, $\zeta_2 \circ \zeta_1^{-1}(x, y) = (\tilde{h}(x, y), h(y))$ which can be rewritten as $\zeta_2(x, y) = (\tilde{h}(\zeta_1(x, y)), h(f_1(x, y)))$. This implies that $f_2 = h \circ f_1$. The second equality in (3.3) follows from this and $g_1 df_1 = g_2 df_2$. \square

Definition 3.2 Let $\mathcal{F} = \mathcal{F}(\omega)$ be a foliation in \mathbb{C}^2 and let a be a regular point of \mathcal{F} . A transversal section to \mathcal{F} at a is

$$\Sigma_a := \{q \in (\mathbb{C}^2, a) \mid \tilde{f}(q) = 0\}$$

where $\tilde{f} \in \mathcal{O}_{\mathbb{C}^2, a}$ together with a first integral $f \in \mathcal{O}_{\mathbb{C}^2, a}$ give us a local chart $\zeta = (\tilde{f}, f)$ around a . The transversal section Σ_a has always the coordinate system given by the image of f .

Proposition 3.2 *Let $\zeta = (\tilde{f}, f) : (\mathbb{C}^2, a) \rightarrow (\mathbb{C}^2, 0)$ be a local chart for \mathcal{F} , $p, q \in (\mathbb{C}^2, a)$ be two points in the same leaf and Σ_p, Σ_q be two transversal sections to \mathcal{F} at p and q , respectively. There is a unique biholomorphism*

$$h : (\Sigma_p, p) \rightarrow (\Sigma_q, q)$$

which is characterized by the fact that $z \in (\Sigma_p, p)$ and $h(z) \in (\Sigma_q, q)$ are in the same leaf of \mathcal{F} .

The map h is called a local holonomy of \mathcal{F} .

3.3 Holonomy

In this section we take a foliation \mathcal{F} in \mathbb{C}^2 . Let $\delta : [0, 1] \rightarrow L$ be a path in a leaf L of the foliation \mathcal{F} with initial point p and end point q . Assume that δ has a finite number of self intersecting points and take two transversal sections Σ_p and Σ_q at p and q , respectively. We cover the image of δ with local charts for \mathcal{F} and since $[0, 1]$ is compact we can do this by a finite number of local charts:

$$\zeta_i : U_i \rightarrow (\mathbb{C}^2, 0), i = 0, 1, 2, 3, \dots, n.$$

Further, we can assume that $U_i \cap U_{i-1} \neq \emptyset$. We also take a transversal section Σ_i at some point p_i of the path δ in $U_{i-1} \cap U_i$. By convention, we set

$$\Sigma_0 := \Sigma_p, \Sigma_{n+1} := \Sigma_q, p_0 := p, p_{n+1} := q.$$

Using Proposition 11 we get biholomorphisms

$$h_i : (\Sigma_i, p_i) \rightarrow (\Sigma_{i+1}, p_{i+1}), i = 0, 1, 2, \dots, n$$

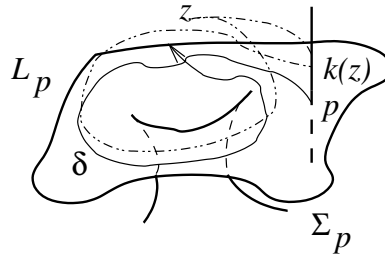


Fig. 3.2 Holonomy

Definition 3.3 The holonomy map from Σ_p to Σ_q is defined to be

$$h := h_h \circ \cdots \circ h_1 \circ h_0 : (\Sigma_p, p) \rightarrow (\Sigma_q, q).$$

The following discussion may help to have a better geometric picture of the notion of holonomy.

There is a neighborhood U_δ of the path δ such that for every $t \in [0, 1]$ and $z \in U_\delta$ near $\delta(t)$, the lifting path $\delta_{k(z),z}$ of $\delta|_{[0,t]}$ in the leaf L_z is well-defined. Roughly speaking, the path $\delta_{k(z),z}$ in the leaf L_z , connects $k(z) \in \Sigma_p$ to z in the direction of the path $\delta|_{[0,t]}$. In Figure 4.7 we have shown that in the self intersecting points of δ , depending on the choice of t , we can choose non-homotop $\delta_{k(z),z}$'s. These paths are depicted by dash-dot-dot lines. Let \tilde{U}_δ be the set of all homotopy classes $[\delta_{k(z),z}]$ in an small neighborhood U_δ of δ . The reader can easily verify that \tilde{U}_δ is a complex manifold and the natural map $\tau : \tilde{U}_\delta \rightarrow U_\delta$ may not be one to one near the self intersecting points of δ (see Figure 4.7). All functions, for example $k(z)$, that we define on the set U_δ are multivalued near such points and are one valued in \tilde{U}_δ . For simplicity, we will work with U_δ instead of \tilde{U}_δ . Let $q = \delta(t_1)$, $0 \leq t_1 \leq 1$, be a point of δ and Σ_q be a small transverse section at q to \mathcal{F} . For any point $z \in \Sigma_q$, the lifting $\delta_{k(z),z}$ of $\delta|_{[0,t_1]}$ defines the holomorphic function $k : \Sigma_q \rightarrow \Sigma_p$. The function $h = k^{-1} : \Sigma_p \rightarrow \Sigma_q$ is the holonomy of \mathcal{F} along δ from Σ_p to Σ_q .

If δ is a closed path, $q = \delta(1)$ and $\Sigma_p = \Sigma_q$, we have the holomorphic germ

$$h = h_\delta : \Sigma_p \rightarrow \Sigma_p$$

h is called the holonomy of \mathcal{F} along δ in Σ_p . In general, we get the following group morphism:

$$\pi(L, p) \rightarrow \text{Bihol}(\Sigma, p), \quad \delta \mapsto h_\delta. \quad (3.4)$$

Note that the fundamental group $\pi(L, p)$ is discrete, however, the group of biholomorphisms of $(\Sigma_p, 0)$ is not at all discrete. For instance, after choosing a coordinate system in Σ_p , we have multiplication by a constant in Σ_p .

3.4 Holonomy II (written by Olivier Thom)

The following proposition explains the dependance of the holonomy on the path δ . It is an easy consequence of the fact that in a local chart, the holonomy does not depend of the path, but only of its endpoints.

Proposition 3.3 *Suppose \mathcal{F} is smooth around an open subset $U \subset L$ of a leaf, and that $\delta_1, \delta_2 : [0, 1] \rightarrow U$ are homotopic with fixed ends inside U . Then for any transversals Σ_0, Σ_1 above $\delta_1(0), \delta_1(1)$, the holonomies $h_1, h_2 : \Sigma_0 \rightarrow \Sigma_1$ of \mathcal{F} along δ_1 and δ_2 are equal.*

Thus for any leaf L , any point $p_0 \in L \setminus \text{Sing}(\mathcal{F})$ and any transversal Σ above p_0 , the holonomy induces an anti-representation

$$\rho_{\mathcal{F}} : \pi_1(L \setminus \text{Sing}(\mathcal{F}), p_0) \rightarrow \text{Diff}(\Sigma, p_0).$$

If δ is a closed loop based at p , and φ is a parametrization $\varphi : (\mathbb{C}, 0) \rightarrow (\Sigma, p_0)$ of a transversal Σ , we can use this parametrization to express the holonomy as a diffeomorphism of $(\mathbb{C}, 0)$:

$$\varphi^{-1} \circ h \circ \varphi \in \text{Diff}(\mathbb{C}, 0).$$

If $\psi \in \text{Diff}(\mathbb{C}, 0)$, then $\varphi \circ \psi$ is another parametrization of Σ and the holonomy in this new coordinate writes $\psi^{-1} \circ (\varphi^{-1} h \varphi) \circ \psi$. Thus the holonomy written in a coordinate is not well-defined, but is only defined modulo conjugacy.

There is another way to define the holonomy. Consider again a path δ in a leaf L , covered by local charts U_i . On each U_i , choose a local first integral f_i of \mathcal{F} such that $L \cap U_i = \{f_i = 0\}$. On an intersection $U_{i-1} \cap U_i$, both functions f_{i-1} and f_i are local first integrals of \mathcal{F} so by Proposition 10, there exists a germ of diffeomorphism $\tilde{h}_i \in \text{Diff}(\mathbb{C}, 0)$ such that $f_i = \tilde{h}_i \circ f_{i-1}$.

Note that the function $\tilde{h}_1^{-1} \circ \dots \circ \tilde{h}_i^{-1} \circ f_i$ extends f_0 to U_i . If δ is closed we can take $U_n = U_0$ and $f_0 = f_n$; the construction above then gives another first integral $\tilde{h}_n^{-1} \circ \dots \circ \tilde{h}_1^{-1} \circ f_0$ of \mathcal{F} . Denote by

$$\tilde{h}_{\delta} = h_1 \circ \dots \circ h_n.$$

This diffeomorphism is in fact equal to the holonomy of \mathcal{F} along δ in the following sense:

Proposition 3.4 *In the context above, let $p \in L \cap U_0$, Σ a transversal above p and $\varphi : (\mathbb{C}, 0) \rightarrow (\Sigma, p)$ the parametrization of Σ satisfying $f_0 \circ \varphi = \text{id}$. Then $\tilde{h}_{\delta} = \varphi^{-1} h_{\delta} \varphi$.*

Proof. Let $y \in \mathbb{C}$ and L_y be the leaf of \mathcal{F} passing through $\varphi(y)$. On U_0 the function f_0 is constant on L_y , equal to y . On $U_0 \cap U_1$, we have $f_1 = \tilde{h}_1 \circ f_0$ so that $f_1(L_y \cap U_1) = \tilde{h}_1(y)$. It follows that $\tilde{h}_1^{-1} \circ f_1$ is constant equal to y on L_y . We carry on until U_n to obtain that $\tilde{h}_n^{-1} \circ f_n$ takes the value y on this leaf. But the leaf L_y might cross Σ at a different point after the loop δ , let $\varphi(z)$ be this point.

Thus we get

$$\tilde{h}_{\delta}^{-1} \circ f_0(\varphi(z)) = y,$$

but by definition, $\varphi(z) = h_{\delta} \circ \varphi(y)$ and the result follows.

3.5 A formula for integrating factor

Let $\mathcal{F}(\omega)$ be a holomorphic foliation in \mathbb{C}^2 and let p be a regular point of \mathcal{F} . There is a Zariski neighborhood U of p and a regular 1-form α defined in U such that

$$d\omega = \omega \wedge \alpha$$

For example, if \mathcal{F} is given by a 1-form $Pdy - Qdx$, then we can define α as follows:

$$\alpha = -\frac{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}{P} dx, \quad \text{or} \quad \alpha = \frac{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}{Q} dy, \quad (3.5)$$

This is defined in the Zariski open set $P \neq 0$. For every two such 1-forms α_1 and α_2 , we have:

$$d\omega = \omega \wedge \alpha_1 = \omega \wedge \alpha_2 \Rightarrow \omega \wedge (\alpha_1 - \alpha_2) = 0.$$

This implies that for a rational function $f \in \mathbb{C}(x, y)$ we have $\alpha_2 - \alpha_1 = f\omega$ and so

$$\alpha_1|_L = \alpha_2|_L \quad \text{for any leaf } L \text{ of } \mathcal{F}$$

Therefore, α_i 's coincide in the leaves of \mathcal{F} . Using two choices of α in (3.5), we know $\alpha|_L$ is actually holomorphic in L . We denote it by

$$\frac{d\omega}{\omega}$$

If two 1-forms ω and ω' induce the same foliation \mathcal{F} , then there is a rational function $f = f(x, y)$ such that $\omega' = f\omega$ and therefore:

$$\begin{aligned} d\omega' &= d(f\omega) = df \wedge \omega + fd\omega = \omega' \wedge \left(-\frac{df}{f} + \omega_1\right) \Rightarrow \\ \alpha' &= \alpha - \frac{df}{f} \end{aligned} \quad (3.6)$$

Definition 3.4 The integrating factor g and first integral f along the path δ are defined as follows:

$$\begin{aligned} g, f &: U_\delta \rightarrow \mathbb{C} \\ g(z) &= \exp\left(\int_{\delta_{k(z), z}} -\frac{d\omega}{\omega}\right), \\ f(z) &= \int_{\delta_{k(z), z}} \frac{\omega}{g}. \end{aligned}$$

The following proposition justifies the names.

Proposition 3.5 *At each point q along the path δ , f and g are local first integral and integrating factor, that is, $\omega = gdf$ in U_δ , $f(q) = 0$ and the derivative of f at q is non-zero.*

Proof. Let us take a local chart, first integral \tilde{f} and integrating factor \tilde{g} around p . Since $\omega = \tilde{g}d\tilde{f}$, we have $-\frac{d\omega}{\omega} = \frac{d\tilde{g}}{\tilde{g}}$ and so up to multiplication with a constant $g = \tilde{g}$ and $f = \tilde{f}$. By analytic continuation we have $\omega = gdf$ in U_δ . Since $f(p) = 0$, we know that f is zero in the leaf passing through p and since $\omega = gdf$ and ω does not vanish in U_δ , the derivative of f in any point of U_δ is non-zero. \square

Now assume that the path δ is closed and consider a transversal section Σ to \mathcal{F} at p . We get the holonomy map $h : \Sigma \rightarrow \Sigma$. In a coordinate system $z \in (\mathbb{C}, 0) \cong \Sigma$ we can write the Taylor series $h = h_1z + h_2z^2 + \dots$ and compute $h_1 = h'(0)$. The number $h'(0)$ does not depend on the choice of coordinate system z in Σ , and so by abuse of notation, we also denote it by $h'(p)$. It is sometimes called the multiplier of \mathcal{F} along δ .

Theorem 3.3 (Poincaré formula) *Let δ be a closed path in a leaf L of the foliation \mathcal{F} , Σ be a transverse section at $p \in \delta$ to the foliation and $h : \Sigma \rightarrow \Sigma$ be the holonomy along δ . Then*

$$h'(p) = \exp \left(\int_\delta \frac{d\omega}{\omega} \right) \quad (3.7)$$

Proof. This is a direct consequence of Proposition 14. Let us consider f, g as holomorphic functions in a neighborhood of p and \tilde{f}, \tilde{g} be the analytic continuation of f, g along δ . All four functions are defined in a neighborhood of p . We choose the restriction of f on Σ as a coordinate system. The holonomy written in this coordinate system and viewed as $h : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ satisfies $\tilde{f}(z) = h \circ f(z)$ for all $z \in \Sigma$. Therefore, by Proposition 10 we have

$$\exp \left(- \int_\delta \frac{d\omega}{\omega} \right) = \tilde{g}(p) = \frac{g(p)}{h'(f(p))} = \frac{1}{h'(0)}.$$

This finishes the proof. \square

The author learned Proposition 14 and Theorem 3.3 in a course in complex dynamical system with S. Shahshahani in Iran.

3.6 Minimal set

For a holomorphic foliation in $\mathbb{P}_\mathbb{C}^2$ one may formulate many problems related to the accumulation of its leaves. The most simple one which is still open is the following:

Problem 3.1. Is there a foliation \mathcal{F} in $\mathbb{P}_\mathbb{C}^2$ with a leaf L which does not accumulate in the singularities of \mathcal{F} .

For instance the above problem for Jouanolou foliation is proved numerically for $d \leq 4$ and it is still open for general d . Let us suppose that such an \mathcal{F} and L exist and set $M := \bar{L}$, where the closure is taken in \mathbb{P}^2 . It follows that M is a union of leaves of \mathcal{F} . We may suppose that M does not contain a proper \mathcal{F} -invariant subset. In this case we call M a minimal set.

Proposition 3.6 *A foliation in \mathbb{P}^2 with algebraic leaf has not a minimal set.*

For many other useful statement on minimal sets see [CLNS88].

Chapter 4

Singularities of holomorphic foliations (written by Hossein Movasati and Olivier Thom)

In this chapter we collect some local aspect of holomorphic foliations. We would like to study

$$\mathcal{F}(\omega), \quad \omega = P(x,y)dy - Q(x,y)dx, \quad P, Q \in \mathcal{O}_{(\mathbb{C}^2, 0)}.$$

with $P(0) = Q(0) = 0$. This study will be important for the algebraic aspects of holomorphic foliations. For instance, the fact that many holomorphic foliations do not have algebraic invariant curves is closely related to the analysis of their singularities.

4.1 Singularities of multiplicity one

The following discussion can be found partially in [CS87] page 40 page 44–48. Let $\omega = P(x,y)dy - Q(x,y)dx$, with $P, Q \in \mathcal{O}_{(\mathbb{C}^2, 0)}$, be a germ of a holomorphic foliation at $0 \in \mathbb{C}^2$. We assume that 0 is a singularity of $\mathcal{F}(\omega)$, this is, $P(0) = Q(0) = 0$. Writing the Taylor series of ω at 0 we get

$$\omega = \omega_m + \omega_{m+1} + \dots$$

with $\omega_i = P_i(x,y)dy - Q_i(x,y)dx$ such that P_i, Q_i are homogeneous polynomials of degree i . The number m is called the multiplicity of ω at $0 \in \mathbb{C}^2$. If $m = 1$ then we say that ω_1 is the linear part of ω . We will also use the notation $\mathcal{F}(X)$, where X is the vector field $X := P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}$.

In this section we are mainly interested in the germ of holomorphic foliations with a non-zero linear part. To each vector field X we can associate the jacobian matrix $J(X; x, y)$ in the basis (x, y) given by

$$J(X; x, y) = \begin{bmatrix} \frac{\partial P}{\partial x}(0) & \frac{\partial P}{\partial y}(0) \\ \frac{\partial Q}{\partial x}(0) & \frac{\partial Q}{\partial y}(0) \end{bmatrix}.$$

We can use Jordan canonical form for a 2×2 matrix with complex coefficients and get the following result.

Proposition 4.1 *Let $\mathcal{F}(X)$ be a germ of holomorphic foliation at 0 and let 0 be a singularity of $\mathcal{F}(X)$. Then, up to biholomorphisms $h : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$, X can be written in one of the following formats:*

1. $y \frac{\partial}{\partial x} + \dots$
2. $(ax + y) \frac{\partial}{\partial x} + ay \frac{\partial}{\partial y} + \dots$ with $a \neq 0$.
3. $ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + \dots$ with $(a, b) \neq (0, 0)$

Proof. This comes from the fact that a diffeomorphism $h \in \text{Diff}(\mathbb{C}^2, 0)$ acts on jacobians matrices by conjugation. Indeed, suppose $Y = h^*X$ so that $(dh)(Y) = X \circ h$, we then get

$$J(h^*X; x, y) = dh^{-1}J(X; x, y)dh.$$

Exercise 4.1 *State and prove a similar proposition as Proposition 16 over the field of real numbers. One have to use the Jordan canonical form of two times two matrices over real numbers. Note that the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a, b \in \mathbb{R}$, $b \neq 0$ over complex numbers has two complex eigenvalues $a \pm ib$ and it cannot be diagonalized over real numbers.*

Definition 4.1 *We say that the foliation $\mathcal{F}(X)$ has a simple singularity at the origin if the jacobian matrix $J(X; x, y)$ has two distinct eigenvalues $a \neq b$ with $a/b \notin \mathbb{Q}^{+*}$.*

We will see in the chapter about blowing-up that we can reduce the study of singular foliations to the study of foliations with only simple singularities, so in this chapter we will mainly focus on these. Note that if $X = ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y}$, the corresponding ordinary differential equation and its solution passing through (x_0, y_0) are

$$\begin{cases} \dot{x} = ax \\ \dot{y} = by \end{cases} \Rightarrow \begin{cases} x(t) = x_0 e^{at} \\ y(t) = y_0 e^{bt} \end{cases}.$$

Note also that around a point $p \in \{xy \neq 0\}$, the foliation $\mathcal{F}(X)$ admits the first integral

$$f(x, y) = y^a x^{-b}.$$

This function is holomorphic on every simply connected subset of $\{xy \neq 0\}$; the other first integrals $f^{1/a}$ and $f^{-1/b}$ can be extended respectively to simply connected subsets of $\{x \neq 0\}$ and $\{y \neq 0\}$, but in general no first integral of $\mathcal{F}(X)$ extends to a holomorphic function at the origin.

Exercise 4.2 *For a leaf L of $\mathcal{F}(X)$, describe the topological closure \bar{L} of L (Hint: See [CS87] pages 44-46)*

Let us calculate some holonomies. From the above equation we see that the x - and y -axes are leaves of $\mathcal{F}(X)$. We will name them L_1 and L_2 , respectively. Let

$p \neq 0$ be in L_1 , that is $p = (x_0, 0)$. Let also δ be the circle through p turning around 0 in L_1 anti clockwise and $\Sigma = \{(x_0, y), y \in \mathbb{C}\}$. We take a point $z = (x_0, y) \in \Sigma$ and would like to compute the action of the holonomy on z . We can parametrize δ by $\delta(s) = (x_0 e^{2i\pi s}, 0)$ for $s \in [0, 1]$. The analytic continuation of the leaf L of $\mathcal{F}(X)$ passing through z and along δ is of the form $\tilde{\delta}(s) = (x_0 e^{2i\pi s}, y e^{2i\pi \frac{b}{a} s})$. For $s = 1$ we get the holonomy map

$$h : \Sigma \rightarrow \Sigma$$

$$(x_0, y) \mapsto (x_0, y e^{2i\pi \frac{b}{a}})$$

If we parametrize Σ by y this is simply

$$(\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$$

$$y \mapsto e^{2i\pi \frac{b}{a}} y \tag{4.1}$$

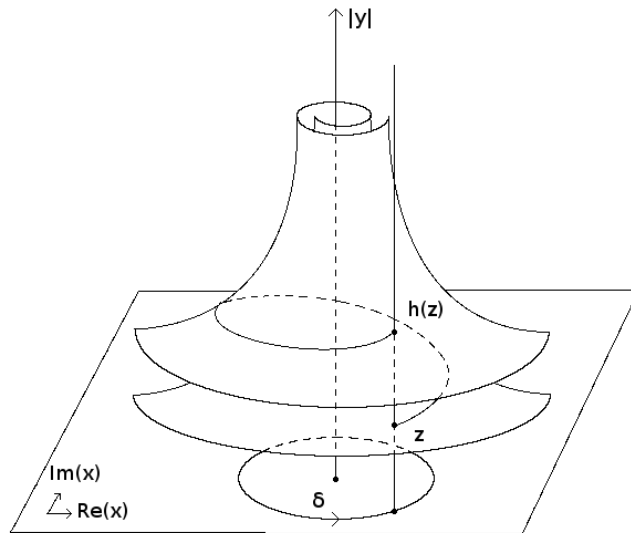


Fig. 4.1 Holonomy around a singularity

4.2 Examples

Let us collect some examples of singularities of foliations. If the foliation is simple, we will suppose that its linear part is already in Jordan canonical form:

$$X = y \frac{\partial}{\partial y} + \alpha x \frac{\partial}{\partial x} + \dots \quad (4.2)$$

with $\alpha \in \mathbb{C}$. For each type of singular point, the questions are always the same: is the foliation diffeomorphic to its linear part? If not, how can we classify the set of such germs of foliation modulo diffeomorphisms? Is it true that X has two invariant curves tangent to the axes?

4.2.1 Hyperbolic singularities: $\alpha \notin \mathbb{R}$

A foliation $\mathcal{F}(X)$ is said to have an hyperbolic singular point at the origin if the exponent α in equation (4.2) is not real.

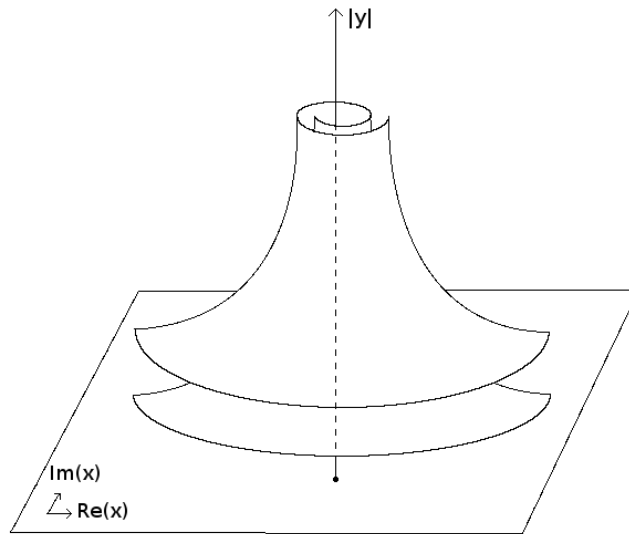


Fig. 4.2 The foliation $y \frac{\partial}{\partial y} - (1 + \varepsilon i)x \frac{\partial}{\partial x}$

Let X be a linear vector field with an hyperbolic singularity of exponent α . Note that $|e^{2i\pi\alpha}| \neq 1$ so that the holonomy of each axis is either contracting or dilating. As a consequence, each leaf accumulates both the x -axis and the y -axis.

4.2.2 Saddle singularities: $\alpha \in \mathbb{R}^-$

The foliation $\mathcal{F}(X)$ is said to have a saddle point at the origin if in equation (4.2), we have $\alpha \in \mathbb{R}^-$.

Note that although there are no differences between them over \mathbb{C} , this class re-groups both real saddles and real centers.

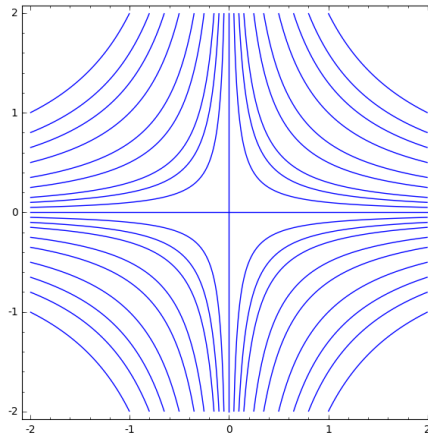


Fig. 4.3 The real saddle $\{xy = cst\}$

The situation is quite different depending on whether $\alpha \in \mathbb{Q}^-$ or not. Indeed, if X is a linear saddle with exponent $\alpha \in \mathbb{Q}^-$, then $\mathcal{F}(X)$ admits a holomorphic first integral. In particular the holonomy is periodic and every leaf is closed.

In contrast, if X is a linear saddle but $\alpha \in \mathbb{R}^- \setminus \mathbb{Q}^-$, the holonomy of the x -axis writes $h(y) = e^{2i\pi\alpha^{-1}}y$. So h is an irrational rotation and the adherence of a generic leaf is a 3-dimensional real manifold contained in $\{xy \neq 0\}$.

4.2.3 Node singularities: $\alpha \in \mathbb{R}^+$

The foliation $\mathcal{F}(X)$ is said to have a node at the origin if in equation (4.2), we have $\alpha \in \mathbb{R}^+$.

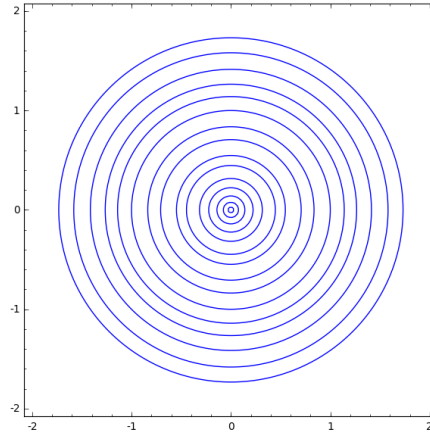


Fig. 4.4 The real center $\{x^2 + y^2 = cst\}$

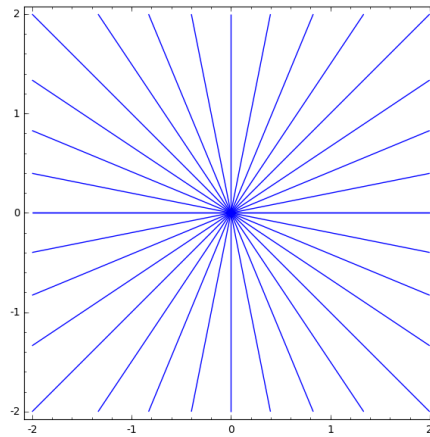


Fig. 4.5 The foliation $\{y/x = cst\}$

Once again, the situation depends on whether the exponent α is rational or not. If $\alpha = \frac{p}{q} \in \mathbb{Q}^+$, then the linear model X has a meromorphic first integral $f(x, y) = \frac{y^p}{x^q}$, so each leaf is closed and passes through the origin. On the other side, is $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the real function $g(x, y) = \frac{|y|^\alpha}{|x|}$ is constant on each leaf, and in fact the adherence of a leaf L of $\mathcal{F}(X)$ is exactly $\bar{L} = g^{-1}(g(L))$. Since each manifold \bar{L} passes through the origin, it follows that \bar{L} separates the space \mathbb{C}^2 in two components: $\mathbb{C}^2 \setminus \bar{L} = U_1 \cup U_2$ and each U_i is a neighborhood of an axis.

4.2.4 Saddle-node singularities: $\alpha = 0$

Note that if X is a vector field with linear part $y \frac{\partial}{\partial y}$, the foliation associated to the linear part is smooth so we cannot expect X to behave like its linear part.

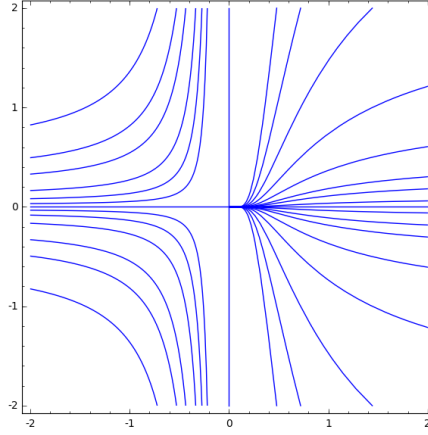


Fig. 4.6 The saddle-node $y \frac{\partial}{\partial y} + x^2 \frac{\partial}{\partial x}$

Take as an example the vector field $X = y \frac{\partial}{\partial y} + x^2 \frac{\partial}{\partial x}$. The foliation $\mathcal{F}(X)$ is also defined by the closed meromorphic 1-form $\frac{dy}{y} - \frac{dx}{x^2}$ which can be integrated to give the first integral $f(x, y) = ye^{\frac{1}{x}}$.

We see in figure 4.2.4 that $\mathcal{F}(X)$ behaves as a saddle for negative x and as a node for positive x , hence the name "saddle-node".

4.3 Separatrices

Let $\mathcal{F}(X)$, $X := P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}$ a germ of holomorphic foliation in $(\mathbb{C}^2, 0)$, also described by the 1-form $\omega := P(x, y)dy - Q(x, y)dx$ and let $0 \in \mathbb{C}^2$ be a singularity of $\mathcal{F}(\omega)$.

Definition 4.2 For $f \in \mathcal{O}_{(\mathbb{C}^2, 0)}$ with $f(0) = 0$, the curve $\{f = 0\}$ is a separatrix of $\mathcal{F}(\omega)$ if $\omega \wedge df = f \cdot \eta$ for some $\eta \in \Omega^2_{(\mathbb{C}^2, 0)}$.

For a formal series $f \in \mathbb{C}[[x, y]]$ with $f(0) = 0$, the formal curve $\{f = 0\}$ is a formal separatrix of $\mathcal{F}(\omega)$ if $\omega \wedge df = f \cdot \eta$ for some $\eta \in \Omega^2_{(\mathbb{C}^2, 0)} \otimes_{\mathcal{O}_{(\mathbb{C}^2, 0)}} \mathbb{C}[[x, y]]$.

If $\{f = 0\}$ admits the local parametrization $t \mapsto \gamma(t)$ (ie. $f \circ \gamma \equiv 0$ and $\gamma \neq 0$), then this equation is equivalent to the existence of a function $r(t)$ with

$$X \circ \gamma(t) = r(t) \cdot d\gamma \left(\frac{\partial}{\partial t} \right).$$

In the global (algebraic) context we say that $\{f = 0\}$ is $\mathcal{F}(\omega)$ -invariant .

Proposition 4.2 *Let $\mathcal{F}(X)$ be a germ of foliation with a singularity of multiplicity 1 at the origin. Suppose $J(X; x, y)$ has two eigenvalues $a, b \in \mathbb{C}$ with $b/a \notin \mathbb{Z}$ and consider an eigenvector v associated to a . Then \mathcal{F} has an holomorphic separatrix tangent to v .*

Proof. Suppose that the linear part of X is already diagonal:

$$X = (ax + \varepsilon_1) \frac{\partial}{\partial x} + (by + \varepsilon_2) \frac{\partial}{\partial y},$$

with $a, b \in \mathbb{C}$ and $\varepsilon_1, \varepsilon_2$ functions vanishing at order 2. Consider the dilatation $\lambda_t : (x, y) \in \mathbb{C}^2 \mapsto (tx, ty)$ and the vector field $X_t = \lambda_t^* X$:

$$X_t = (ax + \frac{1}{t} \varepsilon_1(tx, ty)) \frac{\partial}{\partial x} + (by + \frac{1}{t} \varepsilon_2(tx, ty)) \frac{\partial}{\partial y}.$$

Remark that the family (X_t) is a holomorphic deformation of the linear part X_0 of X . Consider the loop $\gamma(s) = (re^{2i\pi s}, 0)$, the curve $T = \{x = r\}$ and a disk $D = D(0, r) \subset T$ for r small enough. Note also that the fibration $\{x = cst\}$ is transverse to X_0 above γ so will be transverse to X_t above γ for t small enough. Thus the holonomy h_t of X_t on T along γ will be well-defined for small t and gives an holomorphic application

$$h_t : D \rightarrow T.$$

By continuity, h_t is a deformation of $h_0(y) = e^{2i\pi b/a} y$. Now, the holomorphic variety $V = \{h_t(y) - y = 0\} \subset \mathbb{C}_t \times \mathbb{C}_y$ contains the point $(0, 0)$ and is smooth and transverse to $\{t = 0\}$ at $(0, 0)$ since $\frac{\partial}{\partial y}(h_t(y) - y) \neq 0$ by hypothesis. Hence the existence of a unique fixed point y_t of h_t for every small t , which can also be seen as a fixed point of the holonomy of X on $\lambda_t(T)$ along $\lambda_t(\gamma)$. The collection of points $F := \{(rt, y_t)\} \subset \mathbb{C}^2$ is thus a closed leaf of $\mathcal{F}(X)$ passing through the origin, that is, a separatrix. Note finally that by construction the tangent of F at the origin is the tangent of the horizontal separatrix of X_0 at the origin.

Remark 4.1 *To complete this proposition, let us mention what happens when $b/a \in \mathbb{Z}$. If $b/a \in \mathbb{N}^*$ or if $b = 0$, there exists an holomorphic separatrix tangent to v . If $a = 0$, there also exists a separatrix tangent to v , but in general it is only formal. This is the case for the following example given by Euler:*

$$x^2 \frac{\partial}{\partial x} + (y - x) \frac{\partial}{\partial y}$$

which has the formal separatrix $y = \sum_{n \geq 0} n! x^{n+1}$.

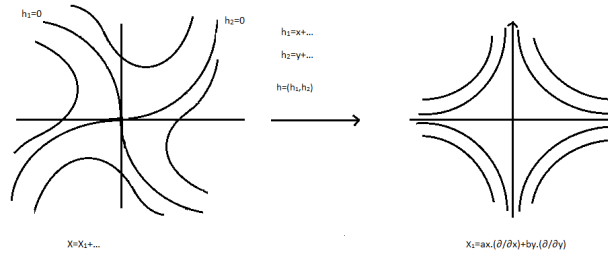


Fig. 4.7 Poincaré linearization theorem

The existence of formal separatrices can be proven by writing the equation for a curve $\gamma(t)$ to be a separatrix and solving it term by term. The convergence must then be proven by estimating the coefficients.

If $b/a \in \mathbb{N}^+$, it might happen that there are no formal separatrix tangent to v , as for example for

$$x \frac{\partial}{\partial x} + (2y + x^2) \frac{\partial}{\partial y}.$$

For a general foliation, we have the following result.

Theorem 4.1 (Camacho-Sad) *The germ of any holomorphic foliation $F(\omega)$ in $(\mathbb{C}^2, 0)$ has a separatrix.*

This will be proved after introducing the notion of "blow up" of singularities.

4.4 Poincaré theorem I

Definition 4.3 We say that de foliation $\mathcal{F}(ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + \dots)$ belongs to the Poincaré domain if

1. $\frac{a}{b} \notin \mathbb{R}^-$
2. $\frac{a}{b} \notin \{2, 3, 4, \dots, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

Theorem 4.2 *Let us assume that the holomorphic foliation $\mathcal{F}(X)$, $X = ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + \dots$ is in the Poincaré domain. Then there exists a biholomorphism $h : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ such that the pull-back of X by h is its linear part $ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y}$.*

We will give a proof for the special case when the foliation is hyperbolic (ie. $a/b \notin \mathbb{R}$); the general case can be shown using a proof similar to that of Dulac's theorem thereafter.

We already know by Proposition 17 that $\mathcal{F}(X)$ has two separatrices tangent to the axes so we can look at the holonomy of the horizontal separatrix: by hyperbolicity this is a diffeomorphism which is either strictly contracting or strictly dilating. The following lemma shows that in this case the holonomy is linearizable; we will then use this fact to construct a diffeomorphism between X and its linear part.

Lemma 4.1 (Poincaré Theorem) *Let $h : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ be the germ of a diffeomorphism and suppose that $|h'(0)| \neq 1$. Then there is a unique diffeomorphism $\varphi \in \text{Diff}(\mathbb{C}, 0)$ tangent to the identity such that $\varphi \circ h \circ \varphi^{-1}(z) = h'(0)z$ for all z .*

Proof. Put $\lambda = h'(0)$. Even if it means replacing h by h^{-1} , we can suppose $|\lambda| < 1$. Choose a $\mu \in \mathbb{R}$ such that $0 < \mu^2 < |\lambda| < \mu < 1$ and a small disk $D \subset \mathbb{C}$ such that

$$\mu^2|z| < |h(z)| < \mu|z| \quad \forall z \in D.$$

In particular, $h(D) \subset D$. Let $\delta \in \mathbb{R}$ be such that $|h(z) - \lambda z| \leq \delta|z|^2$ for $z \in D$. Then

$$\begin{aligned} |h^n(z) - \lambda^n z| &\leq \sum_{k=1}^n |\lambda|^{n-k} |h^k(z) - \lambda h^{k-1}(z)| \\ &\leq \sum_{k=1}^n |\lambda|^{n-k} \delta |h^{k-1}(z)|^2 \\ &\leq \sum_{k=1}^n |\lambda|^{n-k} \delta \mu^{2(k-1)} |z|^2 \\ &\leq \delta |\lambda|^{n-1} |z|^2 \sum_{k=1}^{+\infty} \left(\frac{\mu^2}{|\lambda|} \right)^{k-1} \\ &\leq C |\lambda|^n |z|^2 \end{aligned}$$

where we defined $C = \frac{\delta}{|\lambda| - \mu^2}$. In particular, the application

$$\varphi : z \in D \mapsto \lim_{n \rightarrow \infty} \frac{h^n(z)}{\lambda^n} \in \mathbb{C}$$

is well-defined.

Moreover, for each integers n, p ,

$$\begin{aligned} \left| \frac{h^{n+p}(z)}{\lambda^{n+p}} - \frac{h^n(z)}{\lambda^n} \right| &\leq \frac{1}{|\lambda|^{n+p}} |h^p(h^n(z)) - \lambda^p h^n(z)| \\ &\leq C \frac{1}{|\lambda|^n} |h^n(z)|^2 \\ &\leq C \left(\frac{\mu^2}{|\lambda|} \right)^n |z|^2 \\ &\leq C \left(\frac{\mu^2}{|\lambda|} \right)^n. \end{aligned}$$

The sequence (h^n/λ^n) thus satisfies Cauchy's uniform criterium and its limit φ is holomorphic on D . Note that $\varphi(0) = 0$, $\varphi'(0) = 1$ and for each $z \in D$,

$$\varphi \circ h(z) = \lim_{n \rightarrow \infty} \lambda \frac{h^{n+1}(z)}{\lambda^{n+1}} = \lambda \varphi(z).$$

Therefore the diffeomorphism φ is the one we were seeking.

Suppose $\tilde{\varphi}$ is another such diffeomorphism. Note that $\psi := \tilde{\varphi} \circ \varphi^{-1}$ satisfies $\psi(\lambda z) = \lambda \psi(z)$ for all z . If $\psi(z) = z + \sum_{n \geq 2} a_n z^n$, the latter equation gives at order n the equality $a_n \lambda^n z^n = \lambda a_n z^n$. Since λ is not a root of unity, the only possibility is $a_n = 0$, so that $\tilde{\varphi} = \varphi$.

Proof (Proof of Theorem 4.2). By Proposition 17 the foliation $\mathcal{F}(X)$ has two separatrices, and without loss of generality we can suppose that they are the axes C_x and C_y . Fix a transversal $T = \{x = x_0\}$ to C_x , and $h(y)$ the holonomy of the loop $\gamma(s) = (e^{2i\pi s} x_0, 0) \subset C_x$ on T . We have already noticed that when \mathcal{F} is hyperbolic, $h'(0) \neq 1$ so by Lemma 1, there exists $\varphi \in \text{Diff}(\mathbb{C}, 0)$ tangent to the identity with $\varphi \circ h = h'(0)\varphi$.

Let U be the universal cover of $(\mathbb{C}^2, 0) \setminus C_y$: it comes equipped with the pullback of $\mathcal{F}(X)$ and the vertical fibration $\{x = cst\}$. Since U is simply connected, \mathcal{F} has a first integral f on U equal to $\varphi(y)$ on T . Similarly, the linear part X_0 of X has a first integral f_0 on U equal to y on T . Consider the diffeomorphisms

$$\psi : (x, y) \in U \rightarrow (x, f(x, y)) \in \mathbb{C}^2,$$

$$\psi_0 : (x, y) \in U \rightarrow (x, f_0(x, y)) \in \mathbb{C}^2$$

and

$$\Phi = \psi_0^{-1} \circ \psi.$$

Now remark that $\psi(e^{2i\pi} x, h(y)) = (e^{2i\pi} x, f(x, y))$ and $\psi_0(e^{2i\pi} x, h'(0)y) = (e^{2i\pi} x, y)$ by definition of the holonomy. Thus $\Phi(x_0, y) = (x_0, \varphi(y))$ and

$$\begin{aligned} \Phi(e^{2i\pi} x_0, y) &= \psi_0^{-1}(e^{2i\pi} x_0, f(x_0, h^{-1}(y))) \\ &= \psi_0^{-1}(e^{2i\pi} x_0, \varphi \circ h^{-1}(y)) \\ &= (e^{2i\pi} x_0, h'(0)\varphi \circ h^{-1}(y)) \\ &= (e^{2i\pi} x_0, \varphi(y)). \end{aligned}$$

This proves that the diffeomorphism $\Phi : U \rightarrow U$ descends to a diffeomorphism $\Phi : (\mathbb{C}^2, 0) \setminus C_y \rightarrow (\mathbb{C}^2, 0) \setminus C_y$ which by construction sends the leaves of X to the leaves of X_0 . By construction the diffeomorphism Φ is fibered over x and we can consider it as a family of diffeomorphisms φ_x of the vertical transversals T_x . The diffeomorphism φ_x is a conjugation between the holonomies of X and X_0 on T_x computed in the variable y . By unicity, it is the diffeomorphism given by Lemma 1. We want to prove that φ_x has a limit φ_0 when $x \rightarrow 0$.

Since $\mathcal{F}(X)$ has two separatrices C_x and C_y , it can be written $X = ax(1 + \varepsilon_1(x, y)) \frac{\partial}{\partial x} + by(1 + \varepsilon_2(x, y)) \frac{\partial}{\partial y}$. Introduce the operator $\lambda_t(x, y) = (tx, y)$ and $X_t = \lambda_t^* X = ax(1 + \varepsilon_1(tx, y)) \frac{\partial}{\partial x} + by(1 + \varepsilon_2(tx, y)) \frac{\partial}{\partial y}$. The holonomy h_x of \mathcal{F} on T_x computed in the variable y is the holonomy of X_t on T_{x_0} computed in the variable y for $t = x/x_0$; it is clear that h_x is a holomorphic family with a limit h_0 at $x = 0$. Notice that in this context, the majorations in Lemma 1 can be done in family so that the conjugacies φ_x can be defined on a common disk D and $\varphi_x \rightarrow \varphi_0$ when $x \rightarrow 0$ on D . Thus Φ can be extended by φ_0 on C_y and gives the sought diffeomorphism.

Theorem 4.3 (Dulac) *let $\mathcal{F}(X)$, $X = ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + \dots$ be a holomorphic foliation in $(\mathbb{C}^2, 0)$ with $a = kb$, $k \in \mathbb{N}$ and $k \geq 2$. Then either X is holomorphically linearizable, or there is a unique biholomorphic function $h : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ tangent to the identity such that the pull-back of X by h is*

$$(ax + cy^n) \frac{\partial}{\partial x} + (by \frac{\partial}{\partial y}). \quad (4.3)$$

Proof. The theorem admits generalizations for vector fields in $(\mathbb{C}^n, 0)$ (see [Arnold]). For this reason, we adopt the notation $(x_1, x_2) = (x, y)$ and $(a, b) = (\lambda_1, \lambda_2)$. Let

$$h = (u_1, u_2) = (x_1 + \xi_1(x_1, x_2), x_2 + \xi_2(x_1, x_2)) \quad (4.4)$$

where ξ_1, ξ_2 are two formal power series $\xi_j = \sum_{|n| \geq 2} \xi_{j,n} x^n$, where $n = (n_1, n_2)$ is a multi index, $|n| = n_1 + n_2$ and $x^n = x_1^{n_1} x_2^{n_2}$. Write $X = (\lambda_1 x_1 + \phi_1(x_1, x_2)) \frac{\partial}{\partial x_1} + (\lambda_2 x_2 + \phi_2(x_1, x_2)) \frac{\partial}{\partial x_2}$ and $L = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \lambda_2 x_2 \frac{\partial}{\partial x_2}$ its linear part. Let us try to linearize X formally, that is, find h as before such that

$$L \cdot u_j = \lambda_j u_j + \phi_j(u_1, u_2) \quad (4.5)$$

where $L \cdot u_j = du_j(L)$ denotes the derivative of u_j in the direction L . This is the same as to say that the pull-back of X is L . The equalities (4.4) and (4.5) imply that

$$L \cdot x_j + \frac{\partial \xi_j}{\partial x_1} L \cdot x_1 + \frac{\partial \xi_j}{\partial x_2} L \cdot x_2 = \lambda_j (x_j + \xi_j) + \phi_j(x_1 + \xi_1, x_2 + \xi_2) \quad (4.6)$$

we have $L \cdot x_j = \lambda_j x_j$ so

$$\sum_{|n| \geq 2} (\lambda_j - n_1 \lambda_1 - n_2 \lambda_2) \xi_{j,n} x^n = -\phi_j(x_1 + \xi_1, x_2 + \xi_2). \quad (4.7)$$

By hypothesis, $\lambda_1 = k\lambda_2$ so we have

$$\lambda_2 - n_1 \lambda_1 - n_2 \lambda_2 = (1 - kn_1 - n_2) \lambda_2 \neq 0, \quad \text{and}$$

$$\lambda_1 - n_1 \lambda_1 - n_2 \lambda_2 = (k(1 - n_1) - n_2) \lambda_2$$

which can only be zero for $(n_1, n_2) = (0, k)$. We can solve the equalities in (4.7) recursively for each multi-index (n_1, n_2) except $(0, k)$. The equation for $\xi_{1,(0,k)}$ is either $0 = 0$, in which case we can determine the coefficients of ξ_1, ξ_2 , and they are unique except for $\xi_{1,(0,k)}$, or this equation is $0 = c$ for some $c \in \mathbb{C}$. In this case we have already found a polynomial biholomorphism h such that $h^*X = N + \dots$, where $N = L + cx_2^k \frac{\partial}{\partial x_1}$. The equation $N \cdot h = X \circ h$, consists of two equations

$$N \cdot u_j = \lambda_j u_j + \phi_j(u_1, u_2);$$

the second one is the same as before, and the first one becomes

$$\sum_{|n| \geq 2} (\lambda_1 - n_1 \lambda_1 - n_2 \lambda_2) \xi_{1,n} x^n + cx_2^k = -\phi_1(x_1 + \xi_1, x_2 - \xi_2).$$

It follows that we can determine the coefficients of ξ_1, ξ_2 and they are unique except for $\xi_{1,(0,k)}$.

Now, let us check that these series are in fact convergent. Given two series $A(x_1, x_2)$ and $B(x_1, x_2)$ with positive coefficients we say that $A < B$ if $A_n < B_n \forall n \in \mathbb{N}^2$. We denote by $\hat{C}(x_1, x_2)$ the series $C(x_1, x_2)$ replacing its coefficients by their norm and by $\hat{C}(x)$ the series $\hat{C}(x_1, x_2)$ by taking $x_1 = x_2 = x$. We know that \hat{C} is convergent in $|x_1| < R$ and $|x_2| < R$ if \hat{C} is convergent for $|x| < R$. Let us prove now that $\hat{\xi}_1 + \hat{\xi}_2$ is convergent. The hypotheses on λ_1, λ_2 imply that there exists a $\delta > 0$ such that

$$\delta < |\lambda_j - n_1 \lambda_1 - n_2 \lambda_2|, \quad \forall |n| \geq 2, n \neq (0, k). \quad (4.8)$$

From (4.7), if we choose $\xi_{1,(0,k)} = 0$, we get

$$\delta \hat{\xi}_j < \hat{\phi}_j(x_1 + \hat{\xi}_1, x_2 + \hat{\xi}_2) \quad (4.9)$$

$$\Rightarrow \hat{\xi}_1 + \hat{\xi}_2 < \delta^{-1} [\hat{\phi}_1(x + \hat{\xi}_1 + \hat{\xi}_2) + \hat{\phi}_2(x + \hat{\xi}_1 + \hat{\xi}_2)] \quad (4.10)$$

Our problem is reduced to the following one. Let $F(x) \in \mathcal{O}_{(\mathbb{C}, 0)}$ be a convergent series with positive coefficients and assume that its multiplicity at $x = 0$ is ≥ 2 . By implicit function theorem, there exists a holomorphic function $y(x) \in \mathcal{O}_{(\mathbb{C}, 0)}$ such that

$$y(x) = F(x + y(x)), \quad (4.11)$$

the multiplicity of y at $x = 0$ is ≥ 2 and y is the unique solution, even as a formal series. Consider the application $z \in \mathbb{R}[[x]] \mapsto F(x + z) \in \mathbb{R}[[x]]$, we see easily that if $z_1 - z_2$ cancels at order $k \geq 1$ at the origin, then $F(x + z_1) - F(x + z_2)$ cancels at order $k + 1$. It follows that for every $z_0 \in x\mathbb{R}[[x]]$, the sequence $z_{k+1} = F(x + z_k)$ converges to the unique solution y of $z = F(x + z)$ satisfying $z(0) = 0$. In particular, y has real positive coefficients. Now, F is increasing so if $z(x)$ is a formal series with

$$z(x) \leq F(x + z(x)), \quad (4.12)$$

then denoting again $z_{k+1} = F(x + z_k)$, we get $z_k(x) \leq F(x + z_k(x))$ for each k , and so $z \leq z_1 \leq \dots \leq y$. This proves that $z(x)$ converges.

Exercise 4.3 Use a computer and draw $\mathcal{F}(X)$ with X as in (4.3) for $a = n = 2, b = 1$.

Definition 4.4 A biholomorphism $h : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ is tangent to the identity if $h = (x + \zeta_1(x, y), y + \zeta_2(x, y))$ where the multiplicity of ζ_1, ζ_2 at 0 is ≥ 2 .

4.5 Siegel domain

In the Poincaré theorem we have excluded a class of holomorphic foliations and it is natural to ask whether they are also linearizable. Let us start with the definition of such a class.

Definition 4.5 The foliation $\mathcal{F}(X)$, $X = ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + \dots$ is in the Siegel domain if

$$ab \neq 0, \frac{a}{b} \in \mathbb{R}^-$$

We say that $\mathcal{F}(X)$ has resonance if $\frac{a}{b} \in \mathbb{Q}^-$.

For a holomorphic foliation $\mathcal{F}(X)$ in the Siegel domain and without resonance, we have still the formal power series h conjugating X with its linear part. However, it can happen that this formal power series is not convergent.

Definition 4.6 We say that $\mathcal{F}(X)$ is of type (c, v) , $c, v > 0$ if

$$\begin{cases} |a - n_1 a - n_2 b| \\ |b - n_1 a - n_2 b| \end{cases} > \frac{c}{(n_1 + n_2)^v} \quad \forall n_1, n_2 \geq 0, n_1, n_2 \in \mathbb{N}$$

Theorem 4.4 (Siegel) If $\mathcal{F}(X)$ is of type (c, v) then there exist a local biholomorphism $h : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ such that the pulled back of X by h is the linear part of X , that is, $ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y}$.

Proof. See [Arn80].

It is natural to define the sets:

$$\begin{aligned} SD &:= \{(a, b) \in \mathbb{C}^2 \mid \frac{a}{b} \in \mathbb{R}^-\} \\ \widetilde{SD} &:= \{(a, b) \in SD \mid (a, b) \text{ of type } (c, v) \text{ for some } (c, v), c, v > 0\} \end{aligned}$$

Exercise 4.4 Is \widetilde{SD} dense in SD ? Give examples of elements of \widetilde{SD} and $SD \setminus \widetilde{SD}$. See [CS87] and the references therein.

4.6 Singularities with resonance

Recall the definition of a germ of holomorphic foliation $\mathcal{F}(X)$ with resonance in Definition (19). In the resonance case note that if we write $\frac{a}{b} = -\frac{n}{m}$ ($m, n = 1, n, m \in \mathbb{N}$) then we have

$$\begin{cases} a - a(m+1) - bn = 0 \\ b - am - b(n+1) = 0 \end{cases} \quad n + m + 1 \geq 2$$

In this case, the coefficients $\zeta_{1,(m+1,n)}, \zeta_{2,(m+1,n)}$ cannot be determined in the recursion given in the proof of Theorem 4.2.

Theorem 4.5 *Let $\mathcal{F}(X)$, $X := ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} + \dots$ be a germ of holomorphic foliation in $(\mathbb{C}^2, 0)$ and assume that $\frac{a}{b} \in \mathbb{Q}^-$ (the resonance case). Then there is a biholomorphism $h := (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ such that the pull-back of X by h is of the format*

$$\tilde{X} := (ax + xyA(x, y))\frac{\partial}{\partial x} + (by + xyB(x, y))\frac{\partial}{\partial y}$$

where $A, B \in \mathcal{O}_{(\mathbb{C}^2, 0)}$ with $A(0) = B(0) = 0$.

In the above theorem the foliation $\mathcal{F}(X)$ has at least two separatrices because $\mathcal{F}(\tilde{X})$ has two Separatrics $\{x = 0\}$ and $\{y = 0\}$.

Proof. Proceeding as in theorem 4.2 we write X as

$$\dot{u}_j = \lambda_j u_j + \phi_j(u_1, u_2) \quad (4.13)$$

where

$$u_j = x_j + \xi_j(x_1, x_2) \quad (4.14)$$

We need to find $\xi_j(x_1, x_2)$ such that h^*X is of the form $\dot{x}_j = \lambda_j x_j + \psi_j(x_1, x_2)$ where $\psi_j(x_1, x_2) \in (x_1 \cdot x_2)$. Here, $(x_1 \cdot x_2)$ denotes the ideal of analytic functions generated by $x_1 x_2$. Making the same substitutions we get that

$$\sum_{|n| \geq 2} (n_1 \lambda_1 + n_2 \lambda_2 - \lambda_j) \xi_{j,n} x^n + \sum_{|n| \geq 2} \psi_{j,n} x^n = \phi_j(x_1 + \xi_1, x_2 + \xi_2) - \frac{\partial \xi_j}{\partial x_1} \psi_1 - \frac{\partial \xi_j}{\partial x_2} \psi_2 \quad (4.15)$$

We define

- if $x^n \notin (x_1 \cdot x_2)$ take $\psi_{j,n} = 0$
- if $x^n \in (x_1 \cdot x_2)$ take $\xi_{j,n} = 0$

If $x^n \notin (x_1 \cdot x_2)$ then the coefficient $n_1 \lambda_1 + n_2 \lambda_2 - \lambda_j \neq 0$. It follows that we can calculate ξ_1, ξ_2 formally. To see that they are convergent we claim that if $x^n \notin (x_1 \cdot x_2)$ then $\exists \delta > 0$ such that $|n_1 \lambda_1 + n_2 \lambda_2 - \lambda_j| > \delta$. From this the calculation of $\xi_{j,n}$ is done by

$$\sum_{|n| \geq 2} (n_1 \lambda_1 + n_2 \lambda_2 - \lambda_j) \xi_{j,n} x^n = \phi_j(x_1 + \xi_1, x_2 + \xi_2) \bmod (x_1 \cdot x_2)$$

and so $\delta_{\hat{\xi}_j} < \hat{\phi}_j(x_1 + \hat{\xi}_1, x_2 + \hat{\xi}_2)$, which imply

$$\delta(\hat{\xi}_1 + \hat{\xi}_2) < \hat{\phi}_1(x + \hat{\xi}_1 + \hat{\xi}_2) + \hat{\phi}_2(x + \hat{\xi}_1 + \hat{\xi}_2)$$

and so we proceed as in theorem 4.6.

Chapter 5

Camacho-Sad theorem

In this chapter we explain one of the main index theorems is holomorphic foliations, namely the Camacho-Sad index theorem. This together with Baum-Bott index theorem and a local analysis of holomorphic foliations around singularities, are our main tools in order to study the non-existence of invariant algebraic curves for holomorphic foliations.

5.1 Camacho-Sad index

Let $\mathcal{F}(\omega)$, $\omega := Pdy - Qdx, P, Q \in \mathcal{O}_{(\mathbb{C}^2, 0)}$ be a germ of holomorphic foliation in $(\mathbb{C}^2, 0)$ and assume that $0 \in \mathbb{C}^2$ is an isolated singularity of \mathcal{F} , that is, $P(0) = Q(0) = 0$ and P and Q do not have common factors. Let also $f \in \mathcal{O}_{(\mathbb{C}^2, 0)}$ and $\{f = 0\}$ is a separatrix of \mathcal{F} , that is,

$$df \wedge \omega = f \cdot \eta \text{ where } \eta \in \Omega_{(\mathbb{C}^2, 0)}^2.$$

Proposition 5.1 *There exist holomorphic functions $g, h \in \mathcal{O}_{(\mathbb{C}^2, 0)}$ and $\eta \in \Omega_{(\mathbb{C}^2, 0)}^1$ such that h is not divisible by f and*

$$g\omega = h \cdot df + f\eta.$$

Proof. Since $f = 0$ is a separatrix, we have $df \wedge \omega = f \cdot \eta$ and so $f_x \cdot P + f_y \cdot Q = fS$ for some $S \in \mathcal{O}_{(\mathbb{C}^2, 0)}$. Then

$$f_y \cdot \omega = f_y(Pdy - Qdx) = (f_y \cdot P)dy - (f \cdot S - f_x \cdot P)dx = Pdf - f(Sdx)$$

The same statement is true if one replaces $\mathcal{O}_{(\mathbb{C}^2, 0)}$ with $k[x, y]$ and "separatrix" with "invariant algebraic curve".

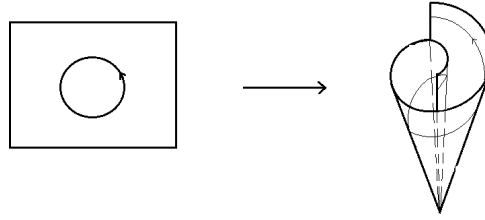


Fig. 5.1 A singular separatrix

Theorem 5.1 (*Puiseux parametrization*) Let $C = \{f(x, y) = 0\}$, $f \in \mathcal{O}_{(\mathbb{C}^2, 0)}$ be a germ of a curve in $(\mathbb{C}^2, 0)$. There is a holomorphic map $\gamma: (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$ such that $f(\gamma(t)) = 0$ and γ is a bijection between $(\mathbb{C}, 0)$ and $\{f(x, y) = 0\}$.

We will prove this theorem later when we introduce the notion of a blow-up. For now, we only mention that the above theorem is trivial for smooth curves. If $\{f = 0\}$ is smooth at 0, that is $(\frac{\partial f}{\partial x}(0), \frac{\partial f}{\partial y}(0)) \neq (0, 0)$ then one can find γ using implicit function theorem. Another example is the singular curve given by $f = y^2 - x^3$. It has the parametrization given by $\gamma(t) = (t^2, t^3)$. From now on let γ be a path in $C = \{f = 0\}$ which is the image of a path in $(\mathbb{C}, 0)$ turning around 0 anti-clockwise and under the map γ . Recall the definition of $\frac{d\omega}{\omega}$ from §3.5.

Definition 5.1 The Camacho-Sad index of $(\mathcal{F}, C, 0)$ is

$$I(F, C, 0) := \frac{-1}{2\pi i} \int_{\gamma} \frac{\eta}{h}.$$

Note that $\lambda = e^{2i\pi I(F, C)}$ is the multiplier of the holonomy h of $\mathcal{F}(\omega)$ along the path γ .

We can reinterpret Proposition 18 in the following way. There is a meromorphic 1-form Ω in $(\mathbb{C}^2, 0)$ which induces the foliation \mathcal{F} and

$$\tilde{\eta} := \Omega - \frac{df}{f}$$

has no poles along $f = 0$. Actually, this 1-form $\tilde{\eta}$ is unique restricted to $f = 0$. With the notation of Proposition 18 we write $g\omega = h \cdot df + f \cdot \eta$ and we have $\Omega = \frac{g}{fh}$, $\tilde{\eta} = \frac{\eta}{h}$. Note also that if we define $\tilde{\omega} = f\Omega = df + f\tilde{\eta}$ then

$$\frac{d\tilde{\omega}}{\tilde{\omega}} = \tilde{\eta}, \text{ restricted to } f = 0.$$

This follows from

$$\begin{aligned} \frac{d\tilde{\omega}}{\tilde{\omega}} &= \frac{d(df + f\tilde{\eta})}{\tilde{\omega}} = \frac{d(f\tilde{\eta})}{\tilde{\omega}} = \frac{(df \wedge \tilde{\eta}) + f \cdot d(\tilde{\eta})}{\tilde{\omega}} \\ &\stackrel{(1)}{=} \frac{(df \wedge \tilde{\eta})}{\tilde{\omega}} = \frac{(df + f\tilde{\eta}) \wedge \tilde{\eta}}{\tilde{\omega}} = \tilde{\eta} \end{aligned}$$

For (1) we restrict to $f = 0$. Note that it makes sense to say that the restriction of $f \frac{d\tilde{\eta}}{\tilde{\omega}}$ to $\{f = 0\}$ is zero, because $\tilde{\eta}$ has no poles along $f = 0$.

If we take $\omega = Pdy - Qdx$ then we have

$$\frac{d\omega}{\omega} = -\frac{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}{P} dx = -\frac{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}{Q} dy \quad (5.1)$$

The second equality is valid when it is restricted to the leaves of $\mathcal{F}(\omega)$. Note that the residue of $\frac{d\omega}{\omega}$ in a separatrix differs from the Camacho-Sad index by an integer. However, if we take the differential 1-form

$$\hat{\omega} = \frac{1}{P}\omega = dy - \frac{Q}{P}dx \quad (5.2)$$

then we have we have

$$\frac{d\hat{\omega}}{\hat{\omega}} = -\frac{\partial\left(\frac{Q}{P}\right)}{\partial y} dx \quad (5.3)$$

and

Proposition 5.2 *If the curve $f = 0$ is smooth and it is not tangent to the y axis at 0 then the Camacho-Sad index can be computed using $\frac{d\hat{\omega}}{\hat{\omega}}$, that is,*

$$I(F, C, 0) := \frac{-1}{2\pi i} \int_{\gamma} \frac{d\hat{\omega}}{\hat{\omega}}.$$

Proof. From the hypothesis it follows that f_y has not zeros in $(\mathbb{C}^2, 0)$. From another side we have $\tilde{\omega} = f_y \hat{\omega}$ which follows from the explicit construction of η in Proposition 18. Therefore,

$$\frac{d\tilde{\omega}}{\tilde{\omega}} = -\frac{df_y}{f_y} + \frac{d\hat{\omega}}{\hat{\omega}}$$

and the proof follows.

Exercise 5.1 *Let $\omega = Pdy - yQdx$ and so $y = 0$ is a separatrix of $\mathcal{F}(\omega)$ calculate $I(F, 0)$.*

Sometimes we write $I(F, C) = I(F, C, 0)$, being clear in the context which singularity we are dealing with.

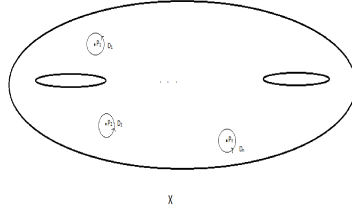


Fig. 5.2 Tangency

5.2 Residue formula

The notion of a residue is purely algebraic and we can avoid integrals in its definition, see for instance [Tat68] and Serre's book in this article. Therefore, the Camacho-Sad index can be defined for foliations in \mathbb{P}_k^2 for arbitrary field k .

The residue formula for smooth curves.

Theorem 5.2 *Let $C \subset \mathbb{P}_k^2$ be a smooth curve and let ω be a meromorphic differential 1-form in C . We have*

$$\sum_{p \in C} \text{residue}_p(\omega) = 0.$$

Proof. We prove this for $k = \mathbb{C}$. The curve C over \mathbb{C} is naturally a Riemann surface. By definition $\text{residue}_p(\omega) = \frac{1}{2\pi i} \oint_p \omega$. Since $d\omega = 0$, by the Stokes theorem

$$\sum_{p_i} \oint_{p_i} \omega = \int \int_{X \setminus \bigcup_{i=1}^n D_i} d\omega = 0$$

5.3 Camacho-Sad theorem

Theorem 5.3 *Let \mathcal{F} be a holomorphic foliation in \mathbb{P}_k^2 and let C be a smooth algebraic \mathcal{F} -invariant curve of degree d in \mathbb{P}_k^2 , then*

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap C} I(\mathcal{F}, C, p) = d^2.$$

Proof. First of all note that we can choose a line $\mathbb{P}_k^1 \subset \mathbb{P}_k^2$ and we can write the foliation $\mathcal{F}(\omega)$, $\omega = P(x, y)dy - Q(x, y)dx$, in the coordinates (x, y) of the affine chart $\mathbb{A}_k^2 = \mathbb{P}_k^2 \setminus \mathbb{P}_k^1$ such that

1. The smooth algebraic curve $C \subset \mathbb{P}_k^2$ intersects \mathbb{P}_k^1 transversely in d points.

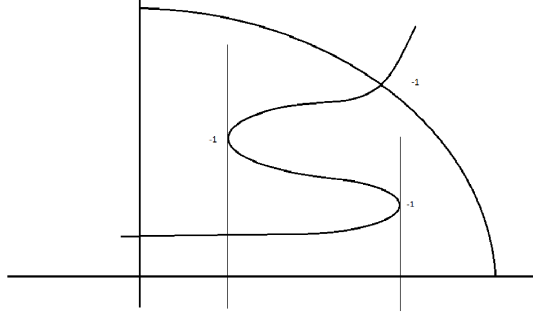


Fig. 5.3 Tangency

2. In the affine chart \mathbb{A}_k^2 , the vertical lines $x = c$ are either transversal or have tangency of order two with the curve C . In addition, all the tangency points are regular points of the foliation \mathcal{F} . By the Bezout theorem the number of such tangency points is $d(d - 1)$.

Now consider the differential form (5.2) which induces the foliation \mathcal{F} and let η be the differential 1-form in (5.3) multiplied with -1 . The poles of the 1-form η restricted to C are divided in three groups: 1. Singularities of \mathcal{F} in C 2. The tangency points of the curve with vertical lines 3. The intersections of C with the line at infinity. We compute the residue of η around all these points and use the residue formula in Theorem 5.2 and we get the proof.

For a singular point $p \in C$ of \mathcal{F} , by definition we have $\text{Residue}(\eta, p) = I(\mathcal{F}, C, p)$. Therefore, we do not need to compute it. For tangency points, we can locally parameterized a leaf tangent to a vertical line by $L : x = g(y) = t \cdot y^2 + \dots$. For simplicity we assume that such a tangency point is at $(0, 0)$. Since $\frac{P(x,y)}{Q(x,y)} = \frac{dx}{dy}$, we have $\frac{P(g(y),y)}{Q(g(y),y)} = g'(y)$. Therefore,

$$\frac{\partial \left(\frac{Q(g(y),y)}{P(g(y),y)} \right)}{\partial y} = -\frac{g''}{(g')^2}.$$

From this we get

$$\begin{aligned} \eta|_L &= \frac{\partial \left(\frac{Q(g(y),y)}{P(g(y),y)} \right)}{\partial y} g'(y) dy \\ &= -\frac{g''}{g'} dy \end{aligned}$$

This has residue -1 at the tangency point p and so in total we get $-d(d - 1)$.

Now let us calculate the residue of η for a point $p \in C$ in the third group. The differential form (5.2) has a pole order -1 at infinity. Using the formula (3.6), we conclude that the residue of $\frac{d\hat{\omega}}{\hat{\omega}} = -\eta$ at p is $+1$ and so in total we get $-d$ for residues of η for the third group. Finally, by residue formula we have

$$\sum_p I(\mathcal{F}, C, p) - d(d-1) - d = 0.$$

Exercise 5.2 Discuss the Camacho-Sad index and theorem for arbitrary field k instead of \mathbb{C} .

Exercise 5.3 If we use $\frac{d\omega}{\omega} = -\frac{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}{P} dx$ in the definition of Camacho-Sad index what would be the corresponding Camacho-Sad theorem. Repeat the same proof as in Theorem 5.3.

Chapter 6

Baum-Bott index and Theorem

In this chapter we introduce the Baum-Bott index of holomorphic vector fields. The original articles [?] deals with vector fields in arbitrary dimensions, however, in the present chapter we focus on dimension two.

6.1 Baum-Bott index

Let $\mathcal{F}(\omega)$, $\omega = Pdy - Qdx$ be a holomorphic foliation in $U := (\mathbb{C}^2, 0)$ with a singularity at $0 \in \mathbb{C}^2$. The differential form $\frac{d\omega}{\omega}$ that we defined in §4.6 is well-defined in the leaves of \mathcal{F} , however, any realization of it in $(\mathbb{C}^2, 0)$ might be meromorphic. There is a way to avoid meromorphicity, allowing complex conjugate of holomorphic functions (real analytic functions). This is as follows.

We define a C^∞ differential $(1, 0)$ -form η in $U \setminus \{0\}$:

$$\eta := (P_x + Q_y) \left(\frac{\bar{P}dx + \bar{Q}dy}{|P|^2 + |Q|^2} \right) \quad (6.1)$$

Proposition 6.1 *The differential form η satisfies the following properties:*

1. $\eta \wedge \omega = d\omega$.
2. $\eta \wedge d\eta$ is a closed form, that is, $d(\eta \wedge d\eta) = 0$.
3. If $\hat{\omega} = R\omega$ and $\hat{\eta}$ is defined as in (6.1) using $\hat{\omega}$ then $(\hat{\eta} \wedge d\hat{\eta} - \eta \wedge d\eta)$ is an exact form.

Proof. The first and second item are easy and are left to the reader. For the third item we proceed as follows. We have

$$\hat{\eta} - \eta = \frac{dR}{R} + f\omega$$

where f is given by

$$f := \frac{R_x \bar{Q} - R_y \bar{P}}{R(|P|^2 + |Q|^2)},$$

and so

$$\begin{aligned} \hat{\eta} \wedge d\hat{\eta} - \eta \wedge d\eta &= d \left(f\eta \wedge \omega + \frac{dR}{R} \wedge \eta + f \frac{dR}{R} \wedge \omega \right) \\ &= d \left(\frac{dR}{R} \wedge \eta - f\omega \wedge \hat{\eta} \right) \end{aligned} \quad (6.2)$$

The second and third item in Proposition 20 as above are equivalent to say that η induces an element in $H_{\text{dR}}^3(U \setminus \{0\})$ which depends only on the foliation \mathcal{F} and not the differential 1-form ω . where $\alpha = f\omega$

Definition 6.1 The Baum-Bott index of a foliation \mathcal{F} defined in $(\mathbb{C}^2, 0)$ is defined to be

$$BB(\mathcal{F}, 0) := \text{Residue}(\eta \wedge d\eta, 0) = \frac{1}{4\pi^2} \int_{\mathbb{S}} \eta \wedge d\eta$$

Here, $\mathbb{S} = \mathbb{S}^3(0, r)$ is the sphere of dimension three, radius r and the center $0 \in \mathbb{C}^2$, for a small positive number r .

It is an easy exercise to show that if η_i , $i = 1, 2$ are two C^∞ differential $(1, 0)$ -forms in $U \setminus \{0\}$ such that they satisfy the item 1 and 2 of Proposition 20 then $\eta_2 \wedge d\eta_2 - \eta_1 \wedge d\eta_1$ is exact and so in the definition of the Baum-Bott index we can use any C^∞ differential $(1, 0)$ -form in $U \setminus \{0\}$ satisfying item 1 and 2 of Proposition 20. In particular, this implies that the Baum-Bott index is invariant under coordinates change. This fact is going to be used in the next proposition.

Proposition 6.2 Let \mathcal{F} be a foliation in an open subset U of \mathbb{C}^2 and let A be an open sub set of U such that the topological closure of A in \mathbb{C}^2 is inside U and the boundary of A is compact and smooth. Further, assume that there is no singularity of \mathcal{F} in the boundary of A . Then

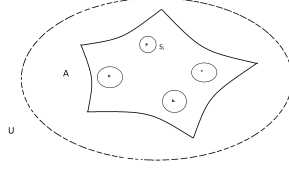
$$\sum_{p \in A \cap \text{Sing}(\mathcal{F})} BB(F, p) = \frac{1}{4\pi^2} \int_{\partial A} \eta \wedge d\eta$$

Proof. This follows from Stokes' theorem and the fact that $\eta \wedge d\eta$ is closed:

$$\int_{\partial A} \eta \wedge d\eta - \sum \int_{\partial S_i} \eta \wedge d\eta = \int_A d(\eta \wedge d\eta) = 0$$

where \mathbb{S}_i^3 are small spheres around the singularity p_i of \mathcal{F} in A .

Proposition 6.3 Let $X = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}$ be a vector field in $U = (\mathbb{C}^2, 0)$ with an isolated singular point at $0 \in \mathbb{C}^2$ and let $\mathcal{F} = \mathcal{F}(X)$ be the induced foliation. Assume that



2.png

Fig. 6.1 U

$$A = DX_0 = \begin{pmatrix} P_x(0) & P_y(0) \\ Q_x(0) & Q_y(0) \end{pmatrix}.$$

has a non-zero determinant. We have

$$BB(F, p) = \frac{1}{4\pi^2} \int_{\mathbb{S}^3} \eta \wedge d\eta = \frac{\text{trace}(A)^2}{\det(A)},$$

Proof. An explicit calculation shows that

$$\eta \wedge d\eta = \frac{(P_x + Q_y)^2}{|P|^2 + |Q|^2} (\bar{Q}d\bar{P} - \bar{P}d\bar{Q}) \wedge dx \wedge dy$$

Let $D := \det(A)$ and $T := \text{trace}(A)$. We consider two cases.

1- X is a linear vector field. By our hypothesis, $\varphi(x, y) = (P, Q) = (u, v)$ is a bi-homeomorphism (a coordinate change). We have

$$\theta = \varphi^*(\eta \wedge d\eta) = \frac{T^2}{(|u|^2 + |v|^2)} (-\bar{u}d\bar{v} + \bar{v}d\bar{u}) \wedge \left(\frac{1}{D}(du \wedge dv)\right)$$

We integrate θ over $\mathbb{S}^3 = (|U|^2 + |V|^2 = 1)$ and use the Stokes theorem. We have

$$\int_{\mathbb{S}^3} \theta = \frac{T^2}{D} \int_{\mathbb{S}^3} (\bar{u}d\bar{v} - \bar{v}d\bar{u}) \wedge du \wedge dv = \frac{T^2}{D} \int_B 2du \wedge d\bar{u} \wedge dv \wedge d\bar{v}$$

where $B = B(0, 1)$ is the unit ball with the center 0. Since $du \wedge d\bar{u} \wedge dv \wedge d\bar{v} = 4dV$, where dV is the Euclidean volume form of \mathbb{C}^2 , we get

$$BB(\mathcal{F}, 0) = \frac{1}{4\pi^2} \int_{\mathbb{S}^3} \theta = \frac{T^2}{D}$$

2-The general case. Let us write $P = P_1 + R$ and $Q = Q_1 + S$, where P_1, Q_1 are linear and the vanishing order of R and S at 0 is ≥ 2 . Consider the function

$$H_t : \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad H_t(p) = tp$$

For $0 < t \leq 1$ we have $\mathbb{S}^3 \subseteq H_t^{-1}(B(0, 2)) = B(0, \frac{2}{t})$ where $\theta_t = H_t^*(\eta \wedge d\eta)$ and so

$$BB(F, 0) = \frac{1}{4\pi t} \int_{\mathbb{S}^3} \theta_t$$

We have

$$H_t^*(\eta \wedge d\eta) = \frac{(\Delta t)^2}{(|P_1 + R_t|^2 + |Q_1 + S_t|^2)^2} [(\bar{Q}_1 + \bar{S}_t)d(\bar{P}_1 + \bar{R}_t) - (\bar{P}_1 + \bar{R}_t)d(\bar{Q}_1 + \bar{S}_t)] \wedge dx \wedge dy$$

where $\Delta t := T + R_t \circ H_t + S_y \circ H_t$ and $R_t := t^{-1}(S \circ H_t)$, $S_t = t^{-1}(S \circ H_t)$. We take the limit $t \rightarrow 0$ and we see that Δt uniformly converges to T and R_t, S_t uniformly converges to zero. Therefore, θ_t converges uniformly to θ_0 which is derived from the linear part of X . Using the first case we get the result.

Theorem 6.1. (Baum-Bott in $\mathbb{P}_{\mathbb{C}}^2$). *Let \mathcal{F} be a foliation of degree k in $\mathbb{P}_{\mathbb{C}}^2$ with isolated singularities. Then*

$$\sum_{p \in \text{Sing}(\mathcal{F})} BB(\mathcal{F}, p) = (k+2)^2 \quad (6.3)$$

Proof. After taking a proper affine chart $E_0 = \{[1; x; y] | x, y \in \mathbb{C}\} \subset \mathbb{P}_{\mathbb{C}}^2$ and a multiplication of X with a constant, we can assume that $\text{Sing}(\mathcal{F}) \subset P_0$, where

$$P_0 := \{[1, x, y] | |x| < 1, |y| < 1\}.$$

Let us consider the other affine charts $E_1 = \{[u; 1; v] | u, v \in \mathbb{C}\}$ and $E_2 = \{[z; w; 1] | z, w \in \mathbb{C}\}$ and the corresponding polydisks

$$P_1 = \{[u, 1, v] | |u| < 1, |v| < 1\}, \quad P_2 = \{[u', v', 1] | |z| < 1, |w| < 1\}.$$

We notice that

$$\mathbb{P}_{\mathbb{C}}^2 = \bar{P}_0 \cup \bar{P}_1 \cup \bar{P}_2, \quad \partial P_i = \bigcup_{j \neq i} (\bar{P}_i \cap \bar{P}_j).$$

Let X_i be a polynomial vector field which induces the foliation \mathcal{F} in E_i , $i = 0, 1, 2$ and let ω_i be the polynomial 1-form such that $\omega_i(X_i) = 0$. Let also $\phi_{ij} : E_i \rightarrow E_j$ be the change of coordinates between E_i and E_j . We have

$$\phi_{10}^*(\omega_0) = u^{-(k+2)} \omega_1, \quad \phi_{20}^*(\omega_0) = (u')^{-(k+2)} \omega_2$$

In other words in the intersection $E_i \cap E_j$ we have $\omega_i = f_{ij} \omega_j$, where $f_{ij} = \frac{1}{f_{ji}}$ and

$$f_{01}|_{E_0} = x^{k+2}, \quad f_{02}|_{E_0} = y^{k+2}, \quad f_{12}|_{E_0} = \frac{y^{k+2}}{x^{k+2}}.$$

We have $f_{ij} f_{jk} f_{ki} = 1$ which implies that

$$\frac{df_{ij}}{f_{ij}} + \frac{df_{jk}}{f_{jk}} + \frac{df_{ki}}{f_{ki}} = 0, \quad \forall i, j, k \in \{1, 2, 3\}.$$

Now let us consider η_j such that $d\omega_j = \eta_j \wedge \omega_j$. We have

$$\eta_i \wedge \omega_i = d\omega_i = d(f_{ij}\omega_j) = df_{ij}\omega_j + f_{ij}d\omega_j = \left(\frac{df_{ij}}{f_{ij}} + \eta_j\right) \wedge \omega_i \quad (6.4)$$

and in a similar way as in the proof of the third part of Proposition 20, there are C^∞ functions g_{ij} in $E_i \cap E_j \setminus \text{Sing}(\mathcal{F})$ such that

$$\eta_i = \eta_j + \frac{df_{ij}}{f_{ij}} + g_{ij}\omega_i$$

Let $\alpha_{ij} := g_{ij}\omega_i$ and so $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0$. Using (6.2) we get

$$\Theta_i - \Theta_j = d\left(\frac{df_{ij}}{f_{ij}} \wedge \eta_j + \eta_i \wedge \alpha_{ij}\right) \quad (6.5)$$

where $\Theta_i := \eta_i \wedge d\eta_i$. Using these and the fact that $\text{Sing}(\mathcal{F}) \subset P_0$ we have that

$$\begin{aligned} 4\pi^2 \sum_{p \in \text{Sing}(\mathcal{F})} BB(\mathcal{F}, p) &= \int_{\partial P_0} \Theta_0 = \int_{\partial P_0} \Theta_i + \int_{\partial P_1} \Theta_1 + \int_{\partial P_3} \Theta_3 \\ &= \int_{P_{01}} \Theta_0 + \int_{P_{02}} \Theta_0 + \int_{P_{10}} \Theta_1 + \int_{P_{12}} \Theta_1 + \int_{P_{20}} \Theta_2 + \int_{P_{21}} \Theta_2 \\ &= \int_{P_{01}} (\Theta_0 - \Theta_1) + \int_{P_{12}} (\Theta_1 - \Theta_2) + \int_{P_{20}} (\Theta_2 - \Theta_0) \\ &= \int_T \alpha \end{aligned}$$

where

$$\alpha = -\eta_1 \wedge \frac{df_{01}}{f_{01}} + \eta_0 \wedge \alpha_{01} - \eta_2 \wedge \frac{df_{12}}{f_{12}} + \eta_1 \wedge \alpha_{12} - \eta_0 \wedge \frac{df_{20}}{f_{20}} + \eta_2 \wedge \alpha_{20} \quad (6.6)$$

and $P_{ij} = \bar{P}_i \cap \bar{P}_j$ and T is the two dimensional torus which is the boundary of all P_{ij} 's. For the last equality we have used the Stokes's theorem. We substitute $\eta_i = \eta_0 + \frac{df_{i0}}{f_{i0}} + \alpha_{i0}$ in the expression of α and we have

$$\begin{aligned} \alpha &= -\left(\eta_0 + \frac{df_{10}}{f_{10}} + \alpha_{10}\right) \wedge \frac{df_{01}}{f_{01}} + \eta_0 \wedge \alpha_{01} - \left(\eta_0 + \frac{df_{20}}{f_{20}} + \alpha_{20}\right) \wedge \frac{df_{12}}{f_{12}} + \\ &\quad \left(\eta_0 + \frac{df_{10}}{f_{10}} + \alpha_{10}\right) \wedge \alpha_{12} - \eta_0 \wedge \frac{df_{20}}{f_{20}} + \left(\eta_0 + \frac{df_{20}}{f_{20}} + \alpha_{20}\right) \wedge \alpha_{20} \\ &= -\frac{df_{20}}{f_{20}} \wedge \frac{df_{12}}{f_{12}} - \alpha_{01} \wedge \frac{df_{01}}{f_{01}} - \alpha_{20} \wedge \frac{df_{12}}{f_{12}} + \frac{df_{10}}{f_{10}} \wedge \alpha_{12} + \frac{df_{20}}{f_{20}} \wedge \alpha_{20} \end{aligned}$$

Now, we just note that

$$\begin{aligned}
& \alpha_{01} \wedge \frac{df_{01}}{f_{01}} + \alpha_{20} \wedge \frac{df_{12}}{f_{12}} - \frac{df_{10}}{f_{10}} \wedge \alpha_{12} - \frac{df_{20}}{f_{20}} \wedge \alpha_{20} \\
&= \alpha_{20} \wedge \left(\frac{df_{12}}{f_{12}} + \frac{df_{20}}{f_{20}} \right) + \alpha_{10} \wedge \frac{df_{01}}{f_{01}} - \frac{df_{10}}{f_{10}} \wedge \alpha_{12} \\
&= \alpha_{20} \wedge \left(-\frac{df_{01}}{f_{01}} \right) + \alpha_{10} \wedge \frac{df_{01}}{f_{01}} - \frac{df_{10}}{f_{10}} \wedge \alpha_{12} \\
&= \frac{df_{01}}{f_{01}} \wedge (\alpha_{01} + \alpha_{12} + \alpha_{20}) = 0
\end{aligned}$$

Hence $\alpha = -\frac{df_{20}}{f_{20}} \wedge \frac{df_{12}}{f_{12}} = (k+2)^2 \frac{dx}{x} \wedge \frac{dy}{y}$. Using the parametrization $(x, y) = (e^{i\theta}, e^{i\psi})$ with $\theta, \psi \in (0, 2\pi)$ we obtain that

$$4\pi^2 \sum_{p \in \text{Sing}(\mathcal{F})} BB(\mathcal{F}, p) = \int_T \alpha = (k+2)^2 \int_T \frac{dx}{x} \wedge \frac{dy}{y} = 4\pi^2 (k+2)^2$$

6.2 Applications of Baum-Bott and Camacho-Sad index theorems

Let \mathcal{F} be a foliation of degree k in \mathbb{P}^2 with finite singularities, let X be a vector field respect to \mathcal{F} and for each singularity P_i of \mathcal{F} then DX_{p_i} has non zero eigenvalues a_j, b_j and $\frac{a_j}{b_j} \notin \mathbb{Q}^+$. We say that $F \in A_k$.

Let X be a vector field and define a foliation \mathcal{F} and p_i is singularity of \mathcal{F} such that DX_{p_i} has two non zero eigenvalues such that $\frac{a_j}{b_j} \notin \mathbb{Q}^+$ then at this point there is exactly two separatrix .?

Proposition 6.4 *Let \mathcal{F} be a non degenerate foliation of degree k over \mathbb{P}^2 then*

$$|\text{Sing}(\mathcal{F})| = 1 + k + k^2$$

this means the number of singularities of \mathcal{F} is $k^2 + k + 1$.

Suppose that α_j, β_j are separatrix of \mathcal{F} at p_j , let X be a vector field respect to \mathcal{F} and at neighborhood of p_j , DX_{p_j} has two eigenvalues a_j, b_j such that

$$I(\mathcal{F}, \alpha_j) = \frac{b_j}{a_j} \text{ and } I(\mathcal{F}, \beta_j) = \frac{a_j}{b_j}$$

Definition 6.2 *A configuration associated to \mathcal{F} is a subset of all separatrix of \mathcal{F}*

$$\text{sep}(\mathcal{F}) = \{a_j, b_j | j = 1, \dots, N\}$$

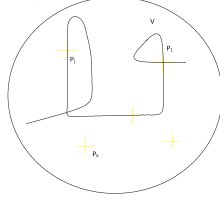


Fig. 6.2 separatrix

We say a configuration C is proper if $C \neq \text{sep}(\mathcal{F})$

Given a configuration $C \subsetneq \text{sep}(\mathcal{F})$ then we use the notation

$$I(\mathcal{F}, C) = \sum_{\delta \in C} I(\mathcal{F}, \delta).$$

Observe that $I(\mathcal{F}, C)$ is a

$$I(\mathcal{F}, C) = \sum_{\alpha_j \in C} I(\mathcal{F}, \alpha_j) + \sum_{\beta_j \in C} I(\mathcal{F}, \beta_j)$$

If $V = \{f = 0\}$ is a \mathcal{F} -invariant, we can define a configuration associated to \mathcal{F} and V

$$C(\mathcal{F}, V) = \{\delta \in \text{sep}(\mathcal{F}) : S \subset V\}$$

Proposition 6.5 Let $\mathcal{F} \in A_k$, $k \geq 2$. suppose that $I(\mathcal{F}, C)$, for all proper configuration $C \subsetneq \text{sep}(\mathcal{F})$ is not positive integer then \mathcal{F} has not \mathcal{F} -invariant (algebraic solution).

Proof. Proof by contradiction, let V be \mathcal{F} -invariant by the Comacho-Sad index we have $I(\mathcal{F}, V) = I(\mathcal{F}, C(\mathcal{F}, V))$ is a positive integer so by the hypothesis $C(\mathcal{F}, S) = \text{sep}(\mathcal{F})$. let us compute $I(\mathcal{F}, \text{sep}(\mathcal{F}))$ use the Baum-Bott theorem on \mathbb{P}^2 , according to this theorem we have :

$$\sum_{p \in \text{Sing}(\mathcal{F})} BB(\mathcal{F}, p) = \sum_{j=1}^N BB(\mathcal{F}, p_j) = (k+2)^2$$

since \mathcal{F} at each its singularity is non-degenerate then $BB(\mathcal{F}, p_j) = \frac{T_j^2}{D_j}$ (by the example)

$$BB(\mathcal{F}, p_j) = \frac{(a_j + b_j)^2}{a_j \cdot b_j} = \frac{a_j}{b_j} + \frac{b_j}{a_j} + 2 = I(\mathcal{F}, \beta_j) + I(\mathcal{F}, \alpha_j) + 2$$

then

$$(k+2)^2 = \sum_{j=1}^N (I(\mathcal{F}, \beta_j) + I(\mathcal{F}, \alpha_j) + 2) = I(\mathcal{F}, \text{sep}(\mathcal{F})) + 2(k^2 + k + 1)$$

$$I(\mathcal{F}, \text{sep}(\mathcal{F})) = -k^2 + k + 1$$

If $k \geq 3$ then $(-k^2 + k + 2 < 0)$, $\text{sep}(\mathcal{F})$ can not be a configuration of an algebraic curve (\mathcal{F} -invariant). for $k = 2$ then $(-k^2 + k + 2 = 2)$ then so by Comacho-Sad index $\text{sep}(\mathcal{F})$ can not be a configuration of an algebraic curve.

Proposition 6.6 For $k \geq 2$ the Jouanolou $J(2, k)$ foliation has not algebraic solution (J -invariant) curve.

Proof. Suppose that $X = (y^k - x^{k+1}) \frac{\partial}{\partial x} + (1 - x^k y) \frac{\partial}{\partial y}$ and $J(2, k) = \mathcal{F}(X)$. For every singularities p_j of \mathcal{F} , \mathcal{F} at p_j is non degenerate (?) and

$$DX(1, 1) = \begin{pmatrix} -(k+1) & k \\ -k & -1 \end{pmatrix}$$

Therefore, the quotients of eigenvalue of DX at p_j are the roots of the equation

$$Z + \frac{1}{Z} = \frac{T^2}{D} = \frac{(k+1)^2}{N} \text{ where } N = k^2 + k + 1 \text{ then the roots are}$$

$$z_1 = \frac{-k^2 + 2k + 2 + k(k+2)\sqrt{3}i}{2N}$$

$$z_2 = \frac{-k^2 + 2k + 2 - k(k+2)\sqrt{3}i}{2N}$$

In particular α_1, β_1 are separatrices of \mathcal{F} at $(1, 1)$, we have $I(J(2, k), \alpha_1) = z_1, I(J(2, k), \beta_1) = z_2$ if C a proper configuration then

$$I(J(2, k), C) = m.z_1 + n.z_2 = (m+n) \left(\frac{-k^2 + 2k + 2}{2N} \right) + (m-n) \left(\frac{k(k+2)\sqrt{3}i}{2N} \right)$$

? where $0 < m + n < 2N$, note that $I(J(2, k), C)$ is real, so $m=n$ then $I(J(2, k), C) = (m) \left(\frac{-k^2 + 2k + 2}{N} \right)$. if $k \geq 3$ then $I(J(2, k), C) \notin \mathbb{R}$ or $I(J(2, k), C) \leq 0$, for $k = 2$, $I(J(2, k), C) \notin \mathbb{R}$ or $I(J(2, k), C) = \frac{2m}{7}$ can not be positive integer while $m < 7$. by the last proposition $J(2, k)$ has not algebraic solution.

Chapter 7

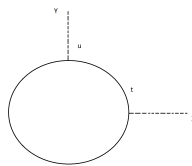
Blow up

Similar to the case of singularities of algebraic varieties, the notion of blow-up is essential in order to understand how complicated is the singularities of holomorphic foliations. The final result is a Theorem of Seidenberg which says that after a finite number of blow-ups any singularity of a holomorphic foliation, we get the so-called reduced singularities. As in the case of previous chapters, we work in the algebraic context of foliations in \mathbb{A}_k^2 , however, the whole discussion can be done for germs of foliations in $(\mathbb{C}^2, 0)$.

7.1 Blow-up of a point

Let us consider the affine variety \mathbb{A}_k^2 with coordinates (x, y) . Let also fix a point $p \in \mathbb{A}_k^2$. For simplicity we take $p = (0, 0)$. The blow-up of \mathbb{A}_k^2 at p is the variety $\tilde{\mathbb{A}}_k^2$ obtained by glueing

$$U_0 = \text{Spec}(k[x, t]), \quad U_1 := \text{Spec}(k[u, y])$$



up 1.png

Fig. 7.1 blowing up

via

$$ut = 1, \quad y = tx.$$

we have the following well-defined map

$$\varphi : \tilde{\mathbb{A}}_k^2 \rightarrow \mathbb{A}_k^2,$$

which is obtained by gluing the maps

$$\varphi_0 : U_0 \rightarrow \mathbb{A}_k^2, \quad (x, t) \mapsto (x, xt)$$

$$\varphi_1 : U_1 \rightarrow \mathbb{A}_k^2, \quad (u, y) \mapsto (uy, y)$$

We will frequently use Figure 7 to visualize a blow-up.

7.2 Blow-up and foliations

Let consider a foliation $\mathcal{F}(\omega)$, $\omega = Pdx + Qdy$ in \mathbb{A}_k^2 . Let p be an isolated singular point of \mathcal{F} . For simplicity we assume that $p = (0, 0) \in \mathbb{A}_k^2$. We write the homogeneous decomposition of ω

$$\omega = \sum_{i=k} \omega_i, \quad \omega_i = P_i(x, y)dx + Q_i(x, y)dy, \quad (P_i, Q_i) \neq (0, 0)$$

where P_i, Q_i are homogeneous polynomials of degree i and the sum is finite. In the local context of holomorphic foliations in $(\mathbb{C}^2, 0)$ the above sum can be infinite.

Definition 7.1 *The natural number $m_p(\mathcal{F}) := k$ is called the algebraic multiplicity of \mathcal{F} at p .*

Let us now analyze the foliation \mathcal{F} after a blow-up at p . In the chart U_0 with (x, t) coordinates we have

$$\begin{aligned} \varphi_0^* \omega &= \sum_{i=k} P_i(x, tx)dx + Q_i(x, tx)(xdt + tdx) \\ &= \sum_{j=k} x^j (P_j(1, t) + tQ_j(1, t))dx + x^{j+1} Q_j(1, t)dt \end{aligned}$$

In a similar way in the chart U_1 with the coordinates (u, y) we have

$$\begin{aligned} \varphi_1^* \omega &= \sum_{i=k} P_i(uy, y)(udy + ydu) + Q_i(uy, y)(dy) \\ &= \sum_{j=k} y^{j+1} P_j(u, 1)du + y^j (Q_j(u, 1) + uQ_j(u, 1))dy \end{aligned}$$

Let

$$C_i := xP_i(x, y) + yQ_i(x, y), \quad C := C_k$$

We consider two cases:

1-dicritical case $C(x, y) \equiv 0$. In this case $\varphi_0^* \omega$ and $\varphi_1^* \omega$ are divisible by x^{k+1} and y^{k+1} , respectively. In other words, $\varphi^* \omega$ has a zero divisor of order $k+1$ along $\varphi^{-1}(0)$. Let us define

$$\begin{aligned}\hat{\omega} &= \frac{1}{x^{k+1}} \varphi_0^* \omega \\ \check{\omega} &= \frac{1}{y^{k+1}} \varphi_1^* \omega\end{aligned}$$

We write

$$\begin{aligned}\hat{\omega} &= [(P_{k+1}(1, t) + tQ_{k+1}(1, t))dx + Q_k(1, t)dt] + x\alpha \\ \check{\omega} &= [P_k(u, 1)du + (Q_{k+1}(u, 1) + uP_{k+1}(u, 1))dy] + y\beta\end{aligned}\quad (7.1)$$

where α and β are 1-forms in \mathbb{A}_k^2 , and so

$$\check{\omega} = u^{k+1} \hat{\omega} \quad (7.2)$$

Now, we analyze the foliation $\varphi^* \mathcal{F}$ near $\varphi^{-1}(0)$. For simplicity, we work in the chart U_0 with (x, t) coordinates. Since $P_k(1, t) + tQ_k(1, t) = 0$, the polynomial Q_k is not identically zero. For points $q := (0, t)$ with $Q_k(0, t) \neq 0$ the leaf of \mathcal{F} passing through q is transversal to $x = 0$. For a point $q = (0, t)$ with $Q_k(0, t) = 0$ and $C_{k+1}(1, t) \neq 0$, we still have a regular point of \mathcal{F} , however, the leaf of \mathcal{F} through q is tangent to $x = 0$. The singularities of \mathcal{F} in $x = 0$ are given by $(0, t)$'s where t is a solution of

$$Q_k(1, t) = 0, \quad C_{k+1}(1, t) = 0$$

The point $(u, y) = (0, 0)$ in the chart U_1 must be treated separately. The foliation \mathcal{F} is transversal to $\varphi^{-1}(0)$ at this point if $P_k(0, 1) \neq 0$, that is, the homogeneous polynomial $P_k(x, y)$ has y^k term. In a similar way, \mathcal{F} is tangent to $\varphi^{-1}(0)$ at $(u, y) = (0, 0)$ if $P_k(0, 1) = 0$ and $C_{k+1}(0, 1) = Q_{k+1}(0, 1) \neq 0$. If both $P_k(0, 1)$ and $C_{k+1}(0, 1)$ vanish then we have a singularity of \mathcal{F} at the point $(u, y) = (0, 0)$.

2-Non-dicritical case $C(x, y) \neq 0$. In this case $\varphi_0^* \omega$ and $\varphi_1^* \omega$ are divisible by x^k and y^k , respectively. We define

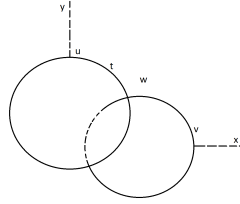
$$\begin{aligned}\hat{\omega} &= \frac{1}{x^k} \varphi_0^* \omega = [(P_k(1, t) + tQ_k(1, t))dx + xQ_k(1, t)dt] + x\alpha \\ \check{\omega} &= \frac{1}{y^k} \varphi_1^* \omega = [(Q_k(u, 1) + uP_k(u, 1))dy + yP_k(u, 1)du] + y\beta\end{aligned}$$

where α are 1-forms. These are related by the equality $\check{\omega} = u^k \hat{\omega}$. In this case the Projective line $\varphi^{-1}(0)$ is invariant by the foliation \mathcal{F} . The singularities of \mathcal{F} in $\{x = 0\} = \varphi^{-1}(0) \cap U_0$ are given by $(0, t)$ with $C(1, t) = 0$. It has a singularity at $(u, y) = (0, 0)$ in the chart U_1 if $Q_k(0, 1) = 0$.

Now, we consider some examples.

Example 7.1 Let

$$\omega = (y^k + 2yx^{2k-2})dx - x^{2k-1}dy, \quad k \geq 2.$$



up 2.png

Fig. 7.2 blowing up

We have $m_0(\mathcal{F}) = k$ and $C(x, y) = xy^k \neq 0$. In the chart U_0 the foliation $\varphi^* \mathcal{F}$ is given by $\hat{\omega} = (t^k + tx^{k-1})dx - x^k dt$, and so, it has a singularity of multiplicity k at $(x, t) = (0, 0)$. In U_1 we have

$$\check{\omega} = (u + u^{2k-1}y^{k-1})dy + (y^k + 2y^k u^{2k-2})du$$

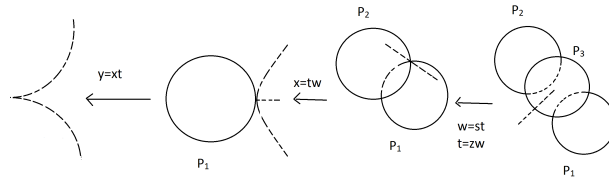
and so $\varphi^* \mathcal{F}$ has a singularity of multiplicity 1 at $(u, y) = (0, 0)$. We do one more blow-up at $(x, t) = 0$. For simplicity we redefine $(x, y) := (x, t)$, reuse t, u as before and redefine

$$\omega = (y^k + yx^{k-1})dx - x^k dy$$

Given $y = tx$ then we have $\hat{\eta}(x, t) = t^k dx - x dt$, it shows that at $x = 0$ and $t = 0$ has algebraic multiplicity 1. Finally, when $x = uy$, then $\hat{\eta}(u, y) = u dy - u^{k-1} y du$ again we found the algebraic multiplicity at $u = 0, y = 0$ is 1.

Example 7.2 Determination of typical leaf $\omega = d(x^3 - y^2) = 3x^2 dx - 2y dy = 0$. The point $p = (0, 0)$ is singularity of $\mathcal{F}(\omega)$ with algebraic multiplicity one ($m_p(\mathcal{F}) = 1$). Since $C_1 = -2y^2 \neq 0$, $\varphi^{-1}(0)$ is \mathcal{F} -invariant and $\hat{\omega} = (3x - 2t^2)dx - (2xt)dt$ has singularity at $(x, t) = (0, 0)$. We redefine $(x, y) := (x, t)$ so will be $\omega = (3x - 2y^2)dx - 2xy dy$ and $\mathcal{F} = \mathcal{F}(\omega)$, it has algebraic multiplicity one at the new point $p = (0, 0)$, i.e., $m_p(\mathcal{F}) = 1$ and again since $c_1 = 3x^2 \neq 0$ so $\varphi^{-1}(0)$ is \mathcal{F} -invariant and $\check{\omega} = (3u^2 - 4yu)dy + (3uy - 2y^2)du$, it has singularity at $p = (0, 0)$, so by again we redefine $(x, y) := (u, y)$ then $\omega = (3xy + 2y)dx + (3x^2 + 4x)dy$ and given $\mathcal{F} = \mathcal{F}(\omega)$ it has a singularity at $p = (0, 0)$ with $m_p(\mathcal{F}) = 2$. note that $C_1 = 6x^2 y - 6xy^2 \neq 0$ so $\varphi^{-1}(0)$ is \mathcal{F} -invariant and after the pulled back of ω we have that

$$\begin{aligned} \hat{\omega} &= (6t - 6t^2)dx + (3x - 4xt)dt \\ \check{\omega} &= (6u^2 - 6u)dy + (3uy - 2y)du \end{aligned}$$



up 3.png

Fig. 7.3 blowing up

7.3 Multiplicity along an invariant curve

Let \mathcal{F} be a foliation in \mathbb{A}_k^2 with an isolated singularity at $0 \in \mathbb{A}_k^2$. Let us assume that $S := \{y = 0\}$ is \mathcal{F} -invariant, and hence,

$$\omega = y\alpha(x, y)dx + \beta(x, y)dy, \quad \beta(0, 0) = 0$$

The differential 1-form ω evaluated at $(x, 0)$ is of $\beta(x, 0)dy$.

Definition 7.2 The multiplicity of \mathcal{F} along S at the singular point $p = (0, 0) \in S$ is defined to be the multiplicity of $\beta(x, 0)$ at $x = 0$. It will be denoted by the $m_p(\mathcal{F}, S)$.

Proposition 7.1 $m_p(\mathcal{F}) \leq m_p(\mathcal{F}, S)$

Proof. This follows immediately from the definitions of $m_p(\mathcal{F})$ and $m_p(\mathcal{F}, S)$.

We would like to know how the number $m_p(\mathcal{F}, S)$ behaves after a blow-up. Recall the notations of §7.1. We have a transform S' of S by the blow-up map which is uniquely determined by the fact that it is irreducible and

$$\varphi^{-1}(S) = \varphi^{-1}(0) + S'$$

The curve S' intersects the exceptional divisor $\varphi^{-1}(0)$ in a unique point q . Let \mathcal{F}_1 be the pull-back of the foliation \mathcal{F} by the blow-up map φ .

Proposition 7.2 We have

$$m_q(\mathcal{F}_1, S') = m_p(\mathcal{F}, S) - (m_p(\mathcal{F}) - 1)$$

Proof. We have

$$\begin{aligned}
\hat{\omega} &:= \frac{1}{x^k} \varphi_0^* \omega \\
&= \frac{1}{x^k} [tx\alpha(x, tx)dx + \beta(x, tx)(tdx + xdt)] \\
&= \frac{1}{x^k} [(tx\alpha(x, tx) + t\beta(x, tx))dx + x\beta(x, tx)dt]
\end{aligned}$$

where $k = m_p(F)$. The proposition follows immediately. Note that in the chart U_0, S' is given by $t = 0$.

7.4 Sequences of blow-ups

We are going to apply a sequence of blow-ups in a point $p = (0, 0) \in \mathbb{A}_k^2$. We fix the notations as follows. In the $(m-1)$ -th step we have a surface M_{m-1} with a divisor $D^{(m-1)} = \bigcup_{j=1}^{m-1} P_j^{(m-1)}$ such that each $P_j^{(m-1)}$ is isomorphic to the projective line \mathbb{P}_k^1 .

We have also a point $p_{m-1} \in D^{(m-1)}$. The variety M_m is obtained by a blow-up at p_{m-1} . Let $\varphi_{m-1} : M_m \rightarrow M_{m-1}$ be the blow-up map. For the new divisor $D^{(m)}$ we have

$$\begin{aligned}
P_m^{(m)} &= \varphi_{m-1}^{-1}(p_{m-1}), \\
P_j^{(m)} &= \varphi_{m-1}^{-1}(P_j^{(m-1)}), \quad j = 1, 2, \dots, m-1.
\end{aligned}$$

Definition 7.3 We define the weight $\rho(P_j^{(m)})$ in the following way. By definition

$$\rho(P_1^{(m)}) = 1, \quad \rho(P_j^{(m)}) = \rho(P_j^{(j)})$$

and

$$\rho(P_j^{(j)}) = \sum_{\substack{i < j \\ P_i^{(j)} \cap P_j^{(j)} \neq \emptyset}} \rho(P_i^{(j)}).$$

In other words, the weight of $P_m^{(m)}$ is the sum of the weights of the projective lines containing p_{m-1} .

Now, let us consider a foliation \mathcal{F} in $M_0 = \mathbb{A}_k^2$. We denote by \mathcal{F}_m the pull-back of \mathcal{F} by the blow-up map $\varphi_0 \circ \varphi_1 \circ \dots \circ \varphi_{m-1}$. We will choose the point p_m from the singular set of the foliation \mathcal{F}_m .

In the discussion below we will assume that all $P_j^{(m)}$ are \mathcal{F}_m -invariant.

Definition 7.4 Let $q \in P_j^{(m)} \cap \text{Sing}(\mathcal{F}_m)$. If q is a smooth point of $D^{(m)}$ then we define

$$m_q^*(\mathcal{F}_m, P_j^{(m)}) = m_q(\mathcal{F}_m, P_j^{(m)})$$

otherwise

$$m_q^*(\mathcal{F}_m, P_j^m) = m_q(\mathcal{F}_m, P_j^{(m)}) - 1.$$

Proposition 7.3 *We have*

$$m_p(\mathcal{F}) + 1 = \sum_{\substack{q \in \text{Sing}(\widehat{\mathcal{F}}_m) \cap D^{(m)} \\ j=1, \dots, m}} \rho(P_j^{(m)}) m_q^*(\mathcal{F}_m, P_j^{(m)}). \quad (7.3)$$

Proof. We prove the proposition by induction on m . For $m = 1$ we have to show that

$$m_p(\mathcal{F}) + 1 = \sum_{q \in P_1^{(1)} \cap \text{Sing} \mathcal{F}_1} m_q(\mathcal{F}_1, P_1^{(1)})$$

We know that the foliation \mathcal{F}_1 in U_0 is given by

$$\hat{\omega}(x, t) = [(P_k(1, t) + tQ_k(1, t))dx + xQ_k(1, t)dt] + x\alpha.$$

We can assume that $Q_k(0, 1) \neq 0$, or equivalently $Q_k(1, t)$ is of degree $k = m_p(\mathcal{F})$. Therefore, all the singularities of \mathcal{F}_1 are in the U_0 chart and so

$$\sum_{q \in \text{Sing}(\mathcal{F})} m_q(\mathcal{F}, P_1^{(1)}) = \text{deg}(P_k(1, t) + tQ_k(1, t)) = m_p(\mathcal{F}) + 1$$

Now, assume that (7.3) is true for after m blow-ups. We consider two cases:

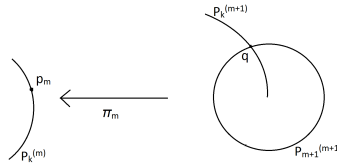
1. $p_m \in P_k^{(m)}$ is a regular point of $D^{(m)}$. By definition

$$\rho(P_{m+1}^{(m+1)}) = \rho(P_k^{(m)}), \quad m_{p_m}^*(\mathcal{F}_m, P_k^{(m)}) = m_{p_m}(\mathcal{F}_m, P_k^{(m)})$$

Let $\{q\} = P_k^{(m+1)} \cap P_{m+1}^{(m+1)}$. We have

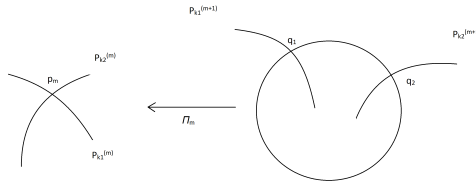
$$\begin{aligned} \rho(P_k^{(m)}) m_{p_m}(\mathcal{F}_m, P_k^{(m)}) &= \rho(P_k^{(m+1)}) [m_q(\mathcal{F}_{m+1}, P_k^{(m+1)}) + m_{p_m}(\mathcal{F}_m) - 1] + \\ \rho(P_k^{(m+1)}) (m_q(\mathcal{F}_{m+1}, P_k^{(m+1)}) - 1) &+ \rho(P_{m+1}^{(m+1)}) \left(\sum_{\substack{r \in P_{m+1}^{(m+1)} \\ r \in \text{Sing}(\mathcal{F}_{m+1})}} m_r(\mathcal{F}_{m+1}, P_{m+1}^{(m+1)}) \right) = \\ \rho(P_k^{(m+1)}) m_q^*(\mathcal{F}_{m+1}, P_k^{(m+1)}) &+ \rho(P_{m+1}^{(m+1)}) (m_q(\mathcal{F}_{m+1}, P_{m+1}^{(m+1)}) - 1) + \\ \rho(P_{m+1}^{(m+1)}) \left(\sum_{\substack{r \in P_{m+1}^{(m+1)} \\ r \in \text{Sing}(\mathcal{F}_{m+1}) \setminus \{q\}}} m_r(\mathcal{F}_{m+1}, P_{m+1}^{(m+1)}) \right) \end{aligned}$$

Now, in (7.3) for m we replace $\rho(P_k^{(m)}) m_{p_m}(\mathcal{F}_m, P_k^{(m)})$ with the sum obtained in the above equality and we get (7.3) for $m + 1$.



up 4.png

Fig. 7.4 blowing up



up 5.png

Fig. 7.5 blowing up

2- $p_m \in P_{k_1}^{(m)} \cap P_{k_2}^{(m)}$ and so it is not a smooth point of $D^{(m)}$. In this case we will replace the terms

$$\rho(P_{k_1}^{(m)})m_{p_m}^*(\mathcal{F}_m, P_{k_1}^{(m)}) + \rho(P_{k_2}^{(m)})m_{p_m}^*(\mathcal{F}_m, P_{k_2}^{(m)})$$

of (7.3) for m with appropriate sums and we will get (7.3) for $m + 1$. Let $\{q_i\} = P_{k_i}^{(m+1)} \cap P_{m+1}^{(m+1)}$, $i = 1, 2$. We have

$$\begin{aligned}
& \rho(P_{k_1}^{(m)})m_{p_m}^*(\mathcal{F}_m, P_{k_1}^{(m)}) + \rho(P_{k_2}^{(m)})m_{p_m}^*(\mathcal{F}_m, P_{k_2}^{(m)}) = \\
& \rho(P_{k_1}^{(m+1)})(m_{q_1}^*(\mathcal{F}_{m+1}, P_{k_1}^{(m+1)}) + m_{p_m}(\mathcal{F}_m) - 1) + \\
& \rho(P_{k_2}^{(m+1)})(m_{q_2}^*(\mathcal{F}_{m+1}, P_{k_2}^{(m+1)}) + m_{p_m}(\mathcal{F}_m) - 1) = \\
& \rho(P_{k_1}^{(m+1)})m_{q_1}^*(\mathcal{F}_{m+1}, P_{k_1}^{(m+1)}) + \rho(P_{k_2}^{(m+1)})m_{q_2}^*(\mathcal{F}_{m+1}, P_{k_2}^{(m+1)}) + \\
& \rho(P_{m+1}^{(m+1)})(m_{p_m}(\mathcal{F}_m) - 1).
\end{aligned}$$

We have also

$$\begin{aligned}
m_{p_m}(\mathcal{F}_m) - 1 &= \sum_{\substack{r \in P_{m+1}^{(m+1)} \\ r \in \text{Sing}(\mathcal{F}_{m+1})}} m_r(\mathcal{F}_{m+1}, P_{m+1}^{(m+1)}) - 2 \\
&= \sum_{\substack{r \in P_{m+1}^{(m+1)} \\ r \in \text{Sing}(\mathcal{F}_{m+1}) \\ r \neq q_1, q_2}} m_r(\mathcal{F}_{m+1}, P_{m+1}^{(m+1)}) + m_{q_1}(\mathcal{F}_{m+1}, P_{m+1}^{(m+1)}) + m_{q_2}(\mathcal{F}_{m+1}, P_{m+1}^{(m+1)}) - 2 \\
&= \sum_{\substack{r \in P_{m+1}^{(m+1)} \\ r \in \text{Sing}(\mathcal{F}_{m+1})}} m_r^*(\mathcal{F}_{m+1}, P_{m+1}^{(m+1)}).
\end{aligned}$$

7.5 Milnor number

For a holomorphic foliation $\mathcal{F}(\omega)$, $\omega = Pdy - Qdx$ in \mathbb{A}_k^2 with an isolated singularity at $p = 0$ we define the Milnor number

$$\mu_p(\mathcal{F}) := \dim_k \frac{\mathcal{O}_{\mathbb{A}_k^2, p}}{\langle P, Q \rangle} \quad (7.4)$$

we also define $l_p(\mathcal{F})$ to be the zero order of $\varphi^*(\omega)$ along the exceptional divisor $\varphi^{-1}(0)$. According to our discussion in §7.3, $l_p(\mathcal{F}) = m_p(\mathcal{F})$ if $\varphi^{-1}(0)$ is \mathcal{F}_1 -invariant and $l_p(\mathcal{F}) = m_p(\mathcal{F}) + 1$ otherwise.

Theorem 7.1 *If p is a dicritical singular point of \mathcal{F} then*

$$\mu_p(\mathcal{F}) = l_p(\mathcal{F})^2 + l_p(\mathcal{F}) - 1 + \sum_{q \in \varphi^{-1}(p)} \mu_q(\mathcal{F}_1) \quad (7.5)$$

and if p is a non-dicritical point of \mathcal{F} then

$$\mu_p(\mathcal{F}) = l_p(\mathcal{F})^2 - l_p(\mathcal{F}) - 1 + \sum_{q \in \varphi^{-1}(p)} \mu_q(\mathcal{F}_1) \quad (7.6)$$

For a proof see Mattei-Moussu or Soares-Mol

7.6 Seidenberg's theorem

Let \mathcal{F} be a foliation in \mathbb{A}_k^2 .

Definition 7.5 We say that p is a reduced singularity of $\mathcal{F}(X)$ if its linear part is not zero and it is of the form $\lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y}$, where

1. $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ and $\frac{\lambda_1}{\lambda_2} \notin \mathbb{Q}^+$ or
2. One of the λ_i 's is zero and the other is not.

We will use the following proposition in Theorem 11.1.

Proposition 7.4 *If a singularity 0 of a foliation in \mathbb{A}_k^2 is reduced and has a meromorphic first integral f then f is actually holomorphic at 0.*

Proof. If 0 is an indeterminacy point of f then we have a singularity of $\tilde{\mathcal{F}}$ with infinitely many separatrices, and so it is not reduced.

In this section we prove the following.

Theorem 7.2 *There is finite sequence of blow-ups such that pull-back foliation has only reduced singularities.*

Proof. We first prove that after a sequence of blow-ups all the singularities of the pull-back foliation have multiplicity m_p equal to 1. We use the fact that if l_p (resp. μ_p) equals to 1 then m_p equals to 1. If $l_p(\mathcal{F}) = 1$ then $m_p(\mathcal{F}) = 1$ and we are done. Otherwise, we perform a blow-up at p and use (7.5) and (7.6) and we conclude that $\mu_q(\mathcal{F}_1) < \mu_p(\mathcal{F})$. This means that after a finite number of blow-ups we have singularities with either l_p or μ_p equal to 1.

Now, assume that p is a singularity of $\mathcal{F}(X)$ with a non-zero linear part. Using Proposition 16 we can consider only the following cases:

1. $y \frac{\partial}{\partial x} + \dots$
2. $(ax + y) \frac{\partial}{\partial x} + ay \frac{\partial}{\partial y} + \dots$ with $a \neq 0$.
3. $ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + \dots$ with $(a, b) \neq (0, 0)$

Chapter 8

Center conditions

One of the most simple singularities of holomorphic foliations in dimension two is the center singularity. In the real plane it appears as two topologically different singularity: circles of different radius centered at the origin and families of hyperbola converging to the union of x and y axes, see Figure 3.1. Despite such a simplicity, its presence for a holomorphic foliation in \mathbb{C}^2 induces an algebraic set in any parameter space of holomorphic foliations, and the classification of the irreducible components of such an algebraic set is still far from being understood. This has fruitful applications in Hilbert's 16-th problem and in the literature of planar differential equations is known as "Center conditions". For further discussion of this topic in the context of holomorphic foliations in surfaces see [Mov04a], and for its application for codimension one foliations in higher dimensions see [Net07]

8.1 Foliations with a center

Let \mathcal{F} be a foliation in \mathbb{C}^2 given by $\omega = P(x,y)dy - Q(x,y)dx$. The points in $\text{Sing}(\mathcal{F}) := \{P(x,y) = Q(x,y) = 0\}$ are called the singularities of \mathcal{F} . We assume that P and Q have no common factor and so $\text{Sing}(\mathcal{F})$ is a finite set of points on \mathbb{C}^2 .

Definition 8.1 A singularity p of \mathcal{F} is called a center singularity, or center for simplicity, if

1. The linear part of ω at p has non-zero determinant. In other words, $(P_x Q_y - P_y Q_x)(p) \neq 0$.
2. There exists a germ of holomorphic function $f \in \mathcal{O}_{(\mathbb{C}^2, p)}$ which has non-degenerate critical point at $p \in \mathbb{C}^2$ and

$$\omega \wedge df = 0.$$

that is, the leaves of \mathcal{F} near p are given by the level curves of f .

Exercise 8.1 (Morse Lemma in dimension two) Let f be as in Definition 29. There is a holomorphic coordinate system (\tilde{x}, \tilde{y}) around p with $\tilde{x}(p) = \tilde{y}(p) = 0$ and such that in this coordinate system, we have $f = \tilde{x}^2 + \tilde{y}^2$.

A center is also called a Morse singularity. Near the center the leaves of \mathcal{F} are homeomorphic to a cylinder, therefore each leaf has a nontrivial closed cycle. Note that the two curves $x^2 + y^2 = 1$ and $xy = 1$ are different in the real plane \mathbb{R}^2 but isomorphic in the complex plane \mathbb{C}^2 , see Figure 3.1. In Definition 29 we can replace f with a formal power series.

Theorem 8.1 (Mattei-Moussu, [MM80], Theorem A, page 472) Let \mathcal{F} be a foliation in \mathbb{C}^2 , p be a singularity of \mathcal{F} with a formal power series first integral, that is, there is $f \in \hat{\mathcal{O}}_{\mathbb{C}^2, p}$ such that $\omega \wedge df = 0$. Then there is a formal power series $g \in \hat{\mathcal{O}}_{\mathbb{C}, 0}$ in one variable and with $g'(0) \neq 0$ such that $g \circ f$ is convergent, and hence, \mathcal{F} has a convergent first integral $g \circ f$.

Proof?

Proof.

Let $\mathcal{F}(d)$ be the space of foliations of degree d . For this one can take any definition of degree: affine degree, projective degree or weighted degree. This is a Zariski open subset in some affine space \mathbb{C}^N and it is defined over \mathbb{Q} . We write $t = (t_\alpha, \alpha \in I) \in \mathbb{C}^N$ and t_α 's are coefficients of monomials used in the expression of ω . We denote by $\mathcal{M}(d)$ the closure of the subset of $\mathcal{F}(d)$ containing foliations with a center. We have learned the statement and proof of the following proposition from A. Lins Neto in [Net07]. This must go back to Poincaré and Dulac.

Proposition 8.1 $\mathcal{M}(d)$ is an algebraic subset of $\mathcal{F}(d)$ defined over \mathbb{Q} , that is, it is given by the zero set of polynomials in t and with rational coefficients.

Proof. Let $\mathcal{M}_0(d)$ be the set of all foliations in $\mathcal{M}(d)$ with a center at the origin $(0, 0) \in \mathbb{C}^2$ and with a local first integral of the type

$$f = xy + f_3 + f_4 + \cdots + f_n + \cdots \quad (8.1)$$

where f_i is homogeneous polynomials of degree i and \cdots means higher order terms. Let us prove that $\mathcal{M}_0(d)$ is an algebraic subset of $\mathcal{F}(d)$. Let $\mathcal{F}(\omega) \in \mathcal{M}_0(d)$ and $\omega = \omega_1 + \omega_2 + \omega_3 + \cdots + \omega_{d+1}$ be the homogeneous decomposition of ω . We have

$$\omega \wedge df = (\omega_1 + \omega_2 + \omega_3 + \cdots + \omega_{d+1}) \wedge (d(xy) + df_3 + df_4 + \cdots) = 0.$$

Putting the homogeneous parts of the above equation equal to zero, we obtain

$$\begin{cases} \omega_1 \wedge d(xy) = 0 \Rightarrow \omega_1 = k \cdot d(xy), \text{ } k \text{ is constant,} \\ \omega_1 \wedge df_3 = -\omega_2 \wedge d(xy), \\ \dots \\ \omega_1 \wedge df_n = -\omega_2 \wedge df_{n-1} - \cdots - \omega_{n-1} \wedge d(xy). \\ \dots \end{cases} \quad (8.2)$$

Dividing the 1-form ω by k , we can assume that $k = 1$. Let $\mathbb{C}[x, y]_n$ denote the set of homogeneous polynomials of degree n . Define the operator

$$S_n : \mathbb{C}[x, y]_n \rightarrow \mathbb{C}[x, y]_n dx \wedge dy,$$

$$S_n(g) = \omega_1 \wedge d(g).$$

We have

$$\begin{aligned} S_{i+j}(x^i y^j) &= d(xy) \wedge d(x^i y^j) \\ &= (xdy + ydx) \wedge (x^{i-1} y^j - x^i y^{j-1})(jxdy + iydx) \\ &= (j-i)x^i y^j dx \wedge dy. \end{aligned}$$

This implies that when n is odd S_n is bijective and so in (8.2), f_n is uniquely defined by the terms f_m, ω_m 's $m < n$, and when n is even

$$\text{Image}(S_n) = A_n dx \wedge dy,$$

where A_n is the subspace generated by the monomials $x^i y^j$, $i \neq j$. When n is even the existence of f_n implies that the coefficient of $(xy)^{\frac{n}{2}}$ in

$$-\omega_2 \wedge df_{n-1} - \cdots - \omega_{n-1} \wedge d(xy)$$

which is a polynomial, say P_n , with variables

$$\text{coefficients of } \omega_2 \dots \omega_{n-1}, f_2, \dots, f_{n-1}$$

is zero. The coefficients of $f_i, i \leq n-1$ is recursively given as polynomials in coefficients of $\omega_i, i \leq n-1$ and so the algebraic set

$$\mathcal{M}_0(d) : P_4 = P_6 = \cdots = P_n = \cdots = 0 \dots$$

consists of all foliations \mathcal{F} in $\mathcal{F}(d)$ which have a formal first integral of the type (8.1) at $(0, 0)$. It follows from Mattei-Moussu theorem, see Theorem 8.1, that \mathcal{F} has a holomorphic first integral of the type (8.1). This implies that $\mathcal{M}_0(d)$ is algebraic. Note that by Hilbert nullstellensatz theorem, a finite number of P_i 's defines $\mathcal{M}_0(d)$. The set \mathcal{M} is obtained by the action of the group of automorphisms of \mathbb{C}^2 on $\mathcal{M}_0(d)$.

Remark 8.1 For n even f_n is not unique as we can replace it with $f_n + a \cdot (xy)^{\frac{n}{2}}$, $a \in \mathbb{k}$. One may put further constrain on f_n by assuming that f_n does not contain the monomial $(xy)^{\frac{n}{2}}$. In this way, f_n and hence the formal power series f , is uniquely determined. Is this f convergent?

8.2 Dulac's classification

Let $\mathcal{F}(d)$ is the space of holomorphic foliation of degree d in \mathbb{P}_k^2 . Let also $\mathcal{M}(d)$ be the closure of the subset of $\mathcal{F}(d)$ containing $\mathcal{F}(\omega)$'s with at least one center.

Theorem 8.2 (Lins Neto, Cerveaux [CLN96] Theorem E') *Let \mathcal{F} be a foliation of projective degree 2 in \mathbb{P}_k^2 which has a center singularity p . Then there exists a line L in \mathbb{P}_k^2 which is invariant by \mathcal{F} and such that $p \notin L$.*

Exercise 8.2 Prove Theorem 8.2 for degree one foliations in \mathbb{P}_k^2 . Put such a line at infinity and write a degree 1 foliation $\mathcal{F}(\omega)$ in an affine chart \mathbb{A}_k^2 in the format $\omega = (t_{00} + t_{10}x + t_{01}y)dy - (s_{00} + s_{10}x + s_{01}y)dx$. Write down the equation of $\mathcal{M}(1)$ in t_{ij} and s_{ij} variables and conclude that it is irreducible.

Theorem 8.3 (H. Dulac, [Dul08], [CLN91]) The algebraic set $\mathcal{M}(2)$ has four irreducible components: $\mathcal{L}(1, 1, 1, 1)$, $\mathcal{L}(3, 1)$, $\mathcal{L}(2, 1, 1)$ and an exceptional component.

a precise statement and proof

Proposition 8.2 We have

$$\mathcal{L}(2, 2) \subset \mathcal{L}(1, 1, 1, 1).$$

Proof. For any family of quadriacs $F - tG = 0$, $t \in \mathbb{P}_k^1$, there are exactly three values of t that it becomes a product of two lines. \square

Chapter 9
Brieskorn and Petrov modules

Chapter 10

Ilyashenko's theorem

Let $\mathcal{F}^a(d)$ be the space of foliations of affine degree d in \mathbb{A}_k^2 and let $\mathcal{M}^a(d)$ be the closure of space of foliation of affine degree d and with a center. In this chapter, we aim to prove the following theorem. Let us write $d+1 = d_1 + d_2 + \dots + d_s$, $d_i \in \mathbb{N}$ and define $\mathcal{L}^a(d_1, d_2, \dots, d_s)$ be the space of logarithmic foliations as in Definition 6.

Theorem 10.1 (H. Movasati [Mov04b]) *For $d \geq 2$ $\mathcal{L}(d_1, d_2, \dots, d_s)$ is a component of $\mathcal{M}(d)$, where $d = d_1 + d_2 + \dots + d_s - 1$.*

For $s = 1$ we can set $\lambda_1 = 1$ and $\mathcal{L}(d)$ is the space of Hamiltonian foliations in \mathbb{A}_k^2 and it is the founding stone of the topic of this chapter.

Theorem 10.2 (Yu. Ilyashenko [Ily69],) *The space of Hamiltonian foliations $\mathcal{F}(df)$, $f \in k[x, y] \leq d+1$ is an irreducible component of $\mathcal{M}^a(d)$.*

The variety $\mathcal{L}^a(d_1, \dots, d_s)$ is parameterized by

$$\begin{aligned} \tau : \mathbb{A}_k^s \times \mathcal{P}_{d_1} \times \dots \times \mathcal{P}_{d_s} &\rightarrow \mathcal{F}^a(d) \\ \tau(\lambda_1, \dots, \lambda_s, f_1, \dots, f_s) &= f_1 \cdots f_s \sum_{i=1}^s \frac{df_i}{f_i} \end{aligned} \quad (10.1)$$

and so it is irreducible. Let J be an equivalence relation in $I = \{0, 1, \dots, d\}$ with s equivalence classes, namely J_1, \dots, J_s . Let also $f = l_0 l_1 \dots l_d \in \mathbb{Q}[x, y]$, where l_i 's are lines in \mathbb{A}_k^2 in general position, $\mathcal{F}_0 = \mathcal{F}(df)$. In a neighborhood of \mathcal{F}_0 in $\mathcal{F}^a(d)$, $\mathcal{L}^a(d_1, \dots, d_s)$ has many irreducible components (branches) corresponding to the J 's as follows: The above parameterization near $(1, \dots, 1, \prod_{i \in J_1} l_i, \dots, \prod_{i \in J_s} l_i)$ determines an irreducible component, namely $\mathcal{L}^a(d_1, \dots, d_s)_J$, of $(\mathcal{L}(d_1, \dots, d_s), \mathcal{F}_0)$ corresponding to J . Theorem 10.1 follows from

Theorem 10.3 *The local analytic variety $\mathcal{L}^a(d_1, \dots, d_s)_J$ is smooth at the point $\mathcal{F}(df)$ and the tangent cone of $\mathcal{M}^a(d)$ at $\mathcal{F}(df)$ is the union of tangent spaces of $\mathcal{L}(d_1, \dots, d_s)_J$ where J runs through all equivalence relations as above.*

10.1 The projective case

It is natural to ask whether Theorem 10.1 is valid using the projective degree instead of affine degree. Let us redefine $\mathcal{F}^a(d)$ to be the space of foliations of projective degree d in \mathbb{P}_k^2 and with an invariant line. Its subvariety $\mathcal{M}^a(d)$ parametrizes foliations with a center which does not lie on the invariant line. According to Exercise 28 we can only talk about $\mathcal{L}^a(d_1, d_2, \dots, d_s) \subset \mathcal{F}^a(d)$ if one of d_i 's is equal to 1.

Exercise 10.1 *The only algebraic leaves of a generic logarithmic foliation*

$$\sum_{i=1}^s \lambda_i \frac{df_i}{f_i}, \quad s \geq 2,$$

in \mathbb{P}_k^2 are $\{f_i = 0\}$, $i = 1, 2, \dots$,

Let $a+1, b+1$ be natural numbers, $c = \gcd(a+1, b+1)$, $q := \frac{a+1}{c}$ and $p := \frac{b+1}{c}$. We take lines $l_1, l_2, \dots, l_{a+1}, \check{l}_1, \check{l}_2, \dots, \check{l}_{b+1}$ in general position in \mathbb{P}_k^2 . Let also

$$F := l_1 l_2 \cdots l_{a+1}, \quad G := \check{l}_1 \check{l}_2 \cdots \check{l}_{b+1}.$$

We consider the foliation \mathcal{F}_0 given by

$$\omega := p \sum_{i=1}^{a+1} \frac{dl_i}{l_i} - q \sum_{j=1}^{b+1} \frac{d\check{l}_j}{\check{l}_j} = \frac{pGdF - qFdG}{FG}. \quad (10.2)$$

which is of degree $d := a+b$.

Exercise 10.2 The space of logarithmic foliations $\mathcal{L}^p(d_1, d_2, \dots, d_s)$, $d := d_1 + d_2 + \dots + d_s - 2$ contains the point \mathcal{F}_0 given by (10.2) with $d = a+b$. Determine the number of branches of $\mathcal{L}^p(d_1, d_2, \dots, d_s)$ near \mathcal{F}_0 .

The following conjecture seems to be quite accessible following the same line of arguments as in Theorem 10.1.

Conjecture 10.1 *For $d \geq 2$ $\mathcal{L}^p(d_1, d_2, \dots, d_s)$ is a component of $\mathcal{M}^p(d)$, where $d = d_1 + d_2 + \dots + d_s - 2$.*

For $s = 2$ this conjecture has been proved in [Mov04a, Mov00].

Chapter 11

Picard-Lefschetz theory

The most simple foliations are fibrations. These foliations lack dynamics, as all the leaves are algebraic. However, they enjoy a beautiful topological theory which is known as Picard-Lefschetz theory. In this chapter we deal with fibrations by curves in \mathbb{P}^2 , however, the theory can be developed for fibrations on projective varieties of arbitrary dimension. Our main reference for this Chapter are Arnold, Gusein-Zade and Varchenko's book [AGZV88] which is mainly suitable for local study of fibrations. An adaptation for fibrations in \mathbb{C}^2 has been done in the author's book [Mov19, Chapter 6].

11.1 Fibration

Let $f \in k(x, y)$ be rational function in x, y . In homogeneous coordinates $[x : y : z]$ we can write $f = \frac{F(x, y, z)}{G(x, y, z)}$, where F and G are two homogeneous polynomial of the same degree and with no common factor.

Definition 11.1 The indeterminacy set \mathcal{R} of f is the set of points in \mathbb{P}^2 in which f has the form $\frac{0}{0}$. This is namely

$$\mathcal{R} := \{F = 0, G = 0\}.$$

Let $X = \mathbb{P}_k^2$ we have a map $f : X \setminus \mathcal{R} \rightarrow \mathbb{P}_k^1$ and we sometimes write it $f : X \dashrightarrow \mathbb{P}_k^1$ knowing that it is not defined in \mathcal{R} .

Theorem 11.1 *There is a smooth algebraic variety \tilde{X} , regular maps $\tilde{f} : \tilde{X} \rightarrow \mathbb{P}^1$ and $\pi : \tilde{X} \rightarrow X$, all defined over k , such that*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi} & X \\ & \searrow \tilde{f} & \downarrow f \\ & & \mathbb{P}_k^1 \end{array} \quad (11.1)$$

commutes, that is, $f \circ \pi = \tilde{f}$.

Proof. We use the desingularization theorem for the holomorphic foliation $\mathcal{F} = \mathcal{F}(df)$ in X . The indeterminacy points of f are singular points of \mathcal{F} . We perform a sequence of blow-ups at \mathcal{R} and obtain $\pi : \tilde{X} \rightarrow X$ such that the singularities of $\tilde{\mathcal{F}} := \pi^{-1}(\mathcal{F})$ over \mathcal{R} are reduced singularities. Let $\tilde{f} := f \circ \pi$. We claim that \tilde{f} induces a morphism $\tilde{f} : \tilde{X} \rightarrow \mathbb{P}_k^1$ such that the diagram (11.1) is commutative. For this we have to prove that \tilde{f} is regular, that is, it has no indeterminacy points. This follows from Proposition 29. \square

11.2 Ehresmann's fibration theorem

In this section we need the following version of Ehresmann's fibration theorem.

Theorem 11.2 (*Ehresmann's Fibration Theorem [Ehr47]*). *Let $f : Y \rightarrow B$ be a proper submersion between the C^∞ manifolds Y and B . Then f fibers Y locally trivially, that is, for every point $b \in B$ there are a neighborhood U of b and a C^∞ -diffeomorphism $\phi : U \times f^{-1}(b) \rightarrow f^{-1}(U)$ such that*

$$f \circ \phi = \pi_1 = \text{the first projection.}$$

Moreover, if $N \subset Y$ is a closed submanifold (not necessarily connected) such that $f|_N$ is still a submersion then f fibers Y locally trivially over N , that is, the diffeomorphism ϕ above can be chosen to carry $U \times (f^{-1}(b) \cap N)$ onto $f^{-1}(U) \cap N$.

The map ϕ is called the fiber bundle trivialization map.

Let f be a rational function in $\mathbb{P}^2 = \mathbb{P}_{\mathbb{C}}^2$. In this section we work over complex numbers and we deal with the topology of fibers of f . The indeterminacy set \mathcal{R} is discrete and the following holomorphic function is well-defined: $f : X - \mathcal{R} \rightarrow \mathbb{P}^1$. We use the following notations

$$L_K = f^{-1}(K), X_K = \overline{L_K}, K \subset \mathbb{P}^1$$

For any point $c \in \mathbb{P}^1$ by L_c and X_c we mean the set $L_{\{c\}}$ and $X_{\{c\}}$, respectively. Throughout the text by a compact f -fiber we mean X_t and by a f -fiber only we mean L_t . We cannot use Ehresmann's fibration theorem directly to f , as it is not proper map.

Theorem 11.3 (Ehresmann's fibration theorem for rational functions) *There exists a finite subset $C = \{c_1, c_2, \dots, c_r\}$ of \mathbb{P}^1 such that f fibers $X - \mathcal{R}$ locally trivially over $B = \mathbb{P}^1 - C$, that is, for every point $b \in B$ there is a neighborhood U of b and a C^∞ -diffeomorphism $\phi : U \times f^{-1}(c) \rightarrow f^{-1}(U)$ such that*

$$f \circ \pi = \pi_1 = \text{the projection on the first coordinate .}$$

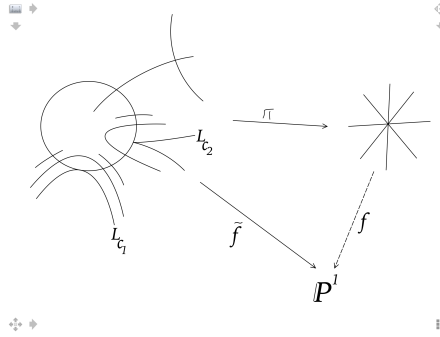


Fig. 11.1 Atypical fibers due to their behaviour at infinity

Proof. The main ingredients of the proof are Ehresmann's fibration theorem and Seidenberg's desingularization theorem. We use Theorem 11.1 and we have a commutative diagram (11.1). The surface \tilde{X} is compact and \tilde{f} is regular. Therefore \tilde{f} is a proper map. Let P_1, P_2, \dots, P_s , all isomorphic to \mathbb{P}^1 , be the set of all blow-up divisors such that \tilde{f} restricted to P_i is not a constant map. The set C is the union of the critical values of \tilde{f} and $\tilde{f}|_{P_i}$, where $P = \cup_{i=1}^s P_i$. We apply Theorem 11.2 to the pair $(\tilde{X} \setminus \tilde{f}^{-1}(C), P \setminus (\tilde{f}|_P)^{-1}(C))$ and get the result. \square

Remark 11.1 In general the set C in Theorem 11.3 is larger than the set of critical values of $f : X - \mathcal{R} \rightarrow \mathbb{P}^1$. This is also clear in the proof of this theorem. Fibers of f which become tangent to a blow-up divisor P_i must be also excluded in order to state Ehresmann's fibration theorem.

Chapter 12

Coronavirus and differential equations

The main objective of the present chapter is to find a possible differential equation of the function of number of infected people, let us say $x(t)$, in a virus outbreak. The idea is to insert all the possible parameters, such as closing schools, using masks etc, as mathematical parameters (numbers) into this differential equation, and to see whether it can be of some use to the society. It is written by the author in a self-quarantine at home and during a coronavirus outbreak. We will start with the most simple model/differential equation, and by inserting new parameters and variables, we will try to make it as near to reality as possible.

12.1 Exponential growths

It is already part of our daily language that a virus expands exponentially. The mathematical formulation of this is as follows. The unit of time is a day and $x(t)$ is the number of infected people in the day t . We assume that each infected person transmits the virus each day to another c_1 persons (for the moment a constant/independent of time). The variation of infected people after ε days is hence

$$x(t + \varepsilon) - x(t) = c_1 \varepsilon x(t)$$

After letting ε goes to zero it becomes

$$\dot{x} = c_1 \cdot x,$$

which implies $x(t) = x(0) \cdot e^{c_1 t}$. Therefore, the number of infected people growth exponentially.

12.2 When the number of infected people increases new transmissions decrease

Since a population of a country is limited, it is better to take $x(t)$ the percentage of the infected people in a country and hence

$$0 \leq x(t) \leq 1.$$

If all the population is infected then there will be no new transmission of the virus. Therefore, it is reasonable to define a new variable $0 \leq y(t) \leq x(t)$ which is the number of infected people who can actually transmit the virus. The variation of $x(t)$ is therefore $c_1 y(t)$ and the variation of $y(t)$ must decrease as $x(t)$ gets near 1. The first suggestion is

$$\begin{cases} \dot{x} = c_1 y \\ \dot{y} = y(1-x) \end{cases} \quad (12.1)$$

Remark 12.1 We have used the multiplication of two variables in an ingenuine way: For a bounded quantity y , the quantity $y(1-x)$ goes to zero when $1-x$ goes to zero.

Exercise 12.1 Show that for some constant c we have

$$y = c_1 \left(x - \frac{x^2}{2} \right) + c$$

In other words the foliation \mathcal{F} has the first integral $f := c_1 \left(x - \frac{x^2}{2} \right) - y$.

12.3 After c_2 days an infected person is cured

Under this hypothesis we are looking for the variation of infected people after $c_2 \varepsilon$ days. We have $c_1 c_2 \varepsilon y(t)$ new infected individuals and $\varepsilon x(t)$ cured individuals. The differential equation becomes:

$$\begin{cases} \dot{x} = c_1 c_2 y - x \\ \dot{y} = y(1-x) \end{cases} \quad (12.2)$$

Exercise 12.2 The foliation \mathcal{F} has two singularities $(0,0)$ and $(1,a)$ in the affine chart \mathbb{C}^2 , where $a = 1/c_1 c_2$. Find the other singularities at infinity and determine whether they are reduced or not. If a singularity is not reduced apply the Seidenberg resolution of singularities!

Remark 12.2 There are many meaningful and meaningless features of Figure 12.1. First, it tells us no matter the initial values (x_0, y_0) with $y_0 \leq x_0 \leq 1$, at the end

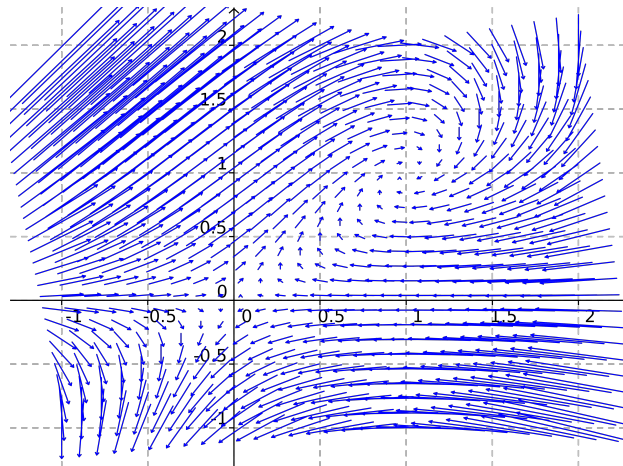


Fig. 12.1 $(y-x)\frac{\partial}{\partial x} + y(1-x)\frac{\partial}{\partial y}$

everybody will get infected. The spiral behaviour near the singularity $(1, a)$ means that once almost everybody is infected then there will be infection-and-getting-cured oscillation: once an epsilon number of people are cured they will be infected again, and on and on. Hopefully this is not the differential equation representing the reality and we still have to insert a lot of parameters into our differential equation. The fact that the number of infected people surpasses the population ($x(t) > 1$) and the number of infected people who can transmit the virus becomes bigger than the number of infected people ($y(t) > x(t)$) do not match to reality.

Remark 12.3 The unit of time, one day, and the unit of person inside a fixed population, in infinitesimal levels suggest that the constants c_1 and c_2 must be small numbers. Recall that the total population is assume to be 1. Therefore, the number $a = 1/c_1 c_2$ is big, and so, the singularity $(1, a)$ is much above the x -axis. Having a look at Figure 12.1 one observes that even in the early stages of outbreak we must have $y(t) > x(t)$. This does not combine with our initial assumption. If we want to make sense out of this we might consider a scenario in which we have to assume that there are people who are not infected but transmit the virus. One has to rewrite the differential equation once again! If $c_1 = 0$ or $c_2 = 0$ then Figure 12.2 says that the number of infected people will decrease even in the first stages of the virus outbreak, however, those who transmit will increase.

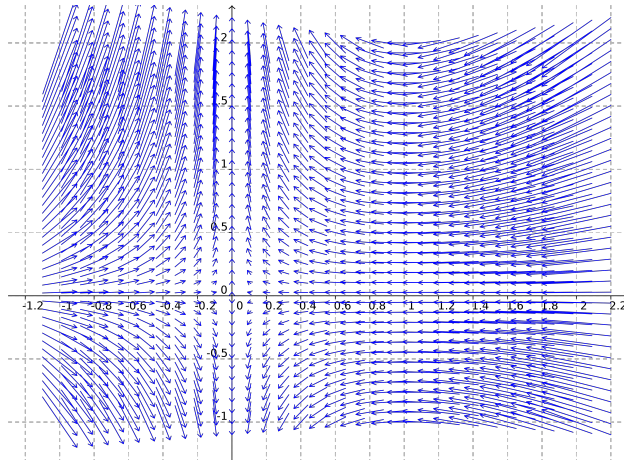


Fig. 12.2 $-x \frac{\partial}{\partial x} + y(1-x) \frac{\partial}{\partial y}$

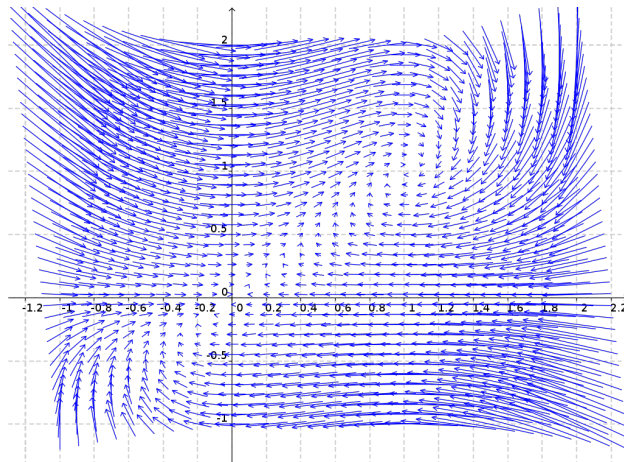


Fig. 12.3 $(y-x) \frac{\partial}{\partial x} + y(1-x)x \frac{\partial}{\partial y}$

12.4 y is low at the early stages of the outbreak

We have assumed that the variation of y , is low when x is near to 1. The same must be true for x near to zero. This gives us the contradictory fact that variation of the number of people who infectes others at the begining is low. Anyway, we are leaded to the differential equation in Figure 12.3.

Remark 12.4 Multiplication of a quantity with x so that it goes to zero when x goes to zero, can be also done by x^n . The same is also true for $(1-x)$. This means that in

general we must consider

$$\begin{cases} \dot{x} = ay - x \\ \dot{y} = y(1-x)^n x^m, \quad n, m \in \mathbb{N}. \end{cases} \quad (12.3)$$

Chapter 13

Discussions with Camacho

13.1 Poincaré theorem II

We need to discuss the Poincaré theorem for vector fields defined over a field k which is not necessarily algebraically closed such as the field of rational numbers. Therefore, we might not be able to diagonalize it in k . We take

$$\mathcal{F}(X) : (a_{11}x_1 + a_{12}x_2 + \cdots) \frac{\partial}{\partial x_1} + (a_{21}x_1 + a_{22}x_2 + \cdots) \frac{\partial}{\partial x_2}$$

The equality (4.6) and $\dot{x}_j = (a_{j1}x_1 + a_{j2}x_2)$ imply that

$$(a_{11}x_1 + a_{12}x_2) \frac{\partial \xi_j}{\partial x_1} + (a_{21}x_1 + a_{22}x_2) \frac{\partial \xi_j}{\partial x_2} - a_{j1}\xi_1 - a_{j2}\xi_2 = \phi_j(x_1 + \xi_1, x_2 + \xi_2) \quad (13.1)$$

This is a recursion in the coefficients of ξ_1, ξ_2 .

13.2 An example of Poincaré's linearization

Let $F := (F_1, F_2) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ with $F_1, F_2 \in \mathbb{C}[x, y]$ and $F_1(0, 0) = F_2(0, 0) = 0$. Let also $\mathcal{F}_1, \mathcal{F}_2$ be foliations in \mathbb{C}^2 given by respectively $d(\frac{F_1}{F_2})$ and $d(\frac{x_1}{x_2})$. The foliation \mathcal{F}_1 is mapped to \mathcal{F}_2 under F . Let

$$X := \frac{-(F_2 F_{1,x_2} - F_1 F_{2,x_2}) \frac{\partial}{\partial x_1} + (F_2 F_{1,x_1} - F_1 F_{2,x_1}) \frac{\partial}{\partial x_2}}{\begin{vmatrix} F_{1,x_1} & F_{1,x_2} \\ F_{2,x_1} & F_{2,x_2} \end{vmatrix}} \quad (13.2)$$

and

$$Y = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}.$$

Proposition 13.1 *We have*

$$DF(X(x)) = Y(F(x)), \quad \forall x \in \mathbb{C}^2.$$

Proof. This is a direct computation. \square

Proposition 32 is our first example of Poincaré's linearization. For this we have to assume that the linear parts of F_1 and F_2 are linearly independent. In other words, the derivation of F at 0 has non-zero determinant. This implies that 0 is not the pole locus of X . In this way F turns out to be the composition of a linear transformation in \mathbb{C}^2 with the linearization of X around 0.

Remark 13.1 It is easy to show that

$$\left\{ f \in \text{Biho}(\mathbb{C}^2, 0) \mid f^* Y = Y \right\} = \text{GL}(2, \mathbb{C}).$$

Therefore, if in Proposition 32 the map F is tangent to identity at 0 then it is unique, in the sense, that any two such linearizations tangent to identity are equal.

13.3 Space of foliations with a radial singularity

Proposition 13.2 *The space $\mathcal{F}(d)_{\text{radial}} \subset \mathcal{F}(d)$ of foliations with a radial singularity is an irreducible algebraic subset of $\mathcal{F}(d)$.*

Proof. Without loss of generality we only consider foliations with a radial singularity at $(0, 0)$. If the foliation is given by $\omega := P(x, y)dy - Q(x, y)dx$ then the condition of having a radial singularity at $(0, 0)$ is that the linear part of ω is $\lambda(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})$. \square

13.4 Deformation of a linearization

Let us consider the linearization in Proposition 32. We consider deformations

$$X_\varepsilon = X + \varepsilon X_1 + \cdots, \quad F_\varepsilon = F + \varepsilon F_1 + \cdots, \quad Y_\varepsilon = Y + \varepsilon Y_1 + \cdots$$

such that

$$DF_\varepsilon(X_\varepsilon(x)) = Y_\varepsilon(F_\varepsilon(x)). \quad (13.3)$$

where $x = (x_1, x_2)$. We further assume that the singularity p_ε of \mathcal{F}_ε near to $p_0 = (0, 0)$ is radial.

The equality corresponding to the coefficient of ε^1 is

$$DF_1(X) + DF(X_1) = Y_1(F(x)) + Y(F_1(x)). \quad (13.4)$$

We write this in the following format

$$DF_1(X) - Y(F_1(x)) = Y_1(F(x)) - DF(X_1). \quad (13.5)$$

The right hand side of this equality consists of known data, and F_1 is unknown. Restricting to a leaf of \mathcal{F}_1 we arrive at the following type of phenomena.

Let S be a Riemann surface and v and p be respectively a vector field and a meromorphic function on S . We are interested on the solution f of the following non-linear differential equations

$$df(v) - f = p.$$

Let us analyze this differential equations in a local chart z for S such that

$$v = z^a \frac{\partial}{\partial z}, \quad f = \sum_{i=b}^{\infty} f_i z^i, \quad p = \sum_{i=c}^{\infty} p_i z^i$$

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Chapter 14

Holonomy II

14.1 Second derivative of the holonomy map

We know that for a holonomy map $h : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ the number $h'(0)$ is independent of the choice of coordinates in $(\mathbb{C}, 0)$. Even if we did not know the formula (3.7), based on this observation we may suspect about its existence. Now if $h'(0) = 1$ we may suspect in a similar way that there is a formula for $h''(0)$. In this section we find such a formula.

Recall that $\frac{\omega}{I}$ is closed and so there is a holomorphic function F_ω such that

$$\omega = Idf.$$

Let also write

$$\omega_2 := \frac{d\omega_1}{\omega}, \quad \omega_1 := \frac{d\omega}{\omega}$$

Proposition 14.1 *Consider the situation of corollary (??). Then*

$$-\frac{h''(z)}{h'(z)} = \int_{\delta_{z,h(z)}} \omega_2 \omega_1 + h'(z) \left(\frac{\omega_1}{\omega} \Big|_{\Sigma_2} \right) (h(z)) - \frac{\omega_1}{\omega} \Big|_{\Sigma_1} (z) \quad (14.1)$$

Proof. By Corollary (??) we have

$$\begin{aligned} -\frac{h''(z)}{h'(z)} &= \frac{d}{dz} \left(\int_{\delta_{z,h(z)}} \omega_1 \right) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{(\int_{\delta_{z+\varepsilon,h(z+\varepsilon)}} - \int_{\delta_{z,h(z)}}) \omega_1}{\varepsilon} \\ &= \frac{\int_{\Delta_\varepsilon} d\omega_1 + \int_{\delta_{h(z),h(z+\varepsilon)}} \omega_1 - \int_{\delta_{z,z+\varepsilon}} \omega_1}{\varepsilon} \end{aligned}$$

Here $\delta_{z,z+h}$, resp. $\delta_{h(z+\varepsilon),h(z)}$ is a straight path in Σ_1 , resp. Σ_2 , which connects z to $z+h$, resp. $h(z+h)$ to $h(z)$, and Δ_ε is a two dimensional embedded disk with the boundary

$$\delta_{z+\varepsilon,h(z+\varepsilon)} - \delta_{h(z),h(z+\varepsilon)} - \delta_{z,h(z)} + \delta_{z,z+h}.$$

Now we write

$$d\omega_1 = \omega_2 I \wedge df$$

and we use

$$\frac{\partial}{\partial \varepsilon} \int_{\Delta_\varepsilon} \omega_2 I \wedge df = \int_{\delta_{z,h(z)}} \omega_2 I = \int_{\delta_{z,h(z)}} \omega_2 \omega_1$$

In the formula (14.1) if δ_p is a closed path and $h'(p) = 1$ we have

$$h''(p) = \int_{\delta_p} \omega_2 \omega_1 \quad (14.2)$$

Remark 14.1 Note that $\omega_1 = \frac{dI_\omega}{I_\omega}$ and so if $\int_{\delta_p} \omega_1 = 0$ then the analytic continuation of I_ω in the leaf L_p and along the cycle δ_p leads to the same value of I_ω in a neighborhood of p .

The formula (14.2) remains invariant if we substitute ω_1 with $\tilde{\omega}_1 = \omega_1 + f\omega$, where f is a holomorphic function in U :

$$\int_{\delta_p} \tilde{\omega}_2 \tilde{\omega}_1 = \int_{\delta_p} (\omega_2 + f\omega_1 - df)\omega_1 = \int_{\delta_p} \omega_2 \omega_1 + f(p) \int_{\delta_p} \omega_1 = \int_{\delta_p} \omega_2 \omega_1$$

Exercise 14.1 Consider the holonomy with $h'(0) = 1$, $h''(z) = \cdot = h^{(n)}(0) = 0$. Express the number $h^{(n+1)}$ in terms of iterated integrals.

Exercise 14.2 Another way to prove the Poincaré formula is as follows: We use Stokes formula for Δ_ε and ω and divide the equality by ε and let ε go to 0. The obtained formula is equivalent to the Poincaré formula after using some well-known properties of iterated integrals.

14.2 Unstable limit cycles

In my opinion an effective solution to the hilbert 16th problem would need a systematic study of the loci of unstable limit cycles. Bellow I explain this.

Let $\mathcal{F}_0 = \mathcal{F}(\omega)$ be an algebraic foliation with an unstable limit cycle δ_0 , i.e. $\int_{\delta} \frac{d\omega}{\omega} = 0$. We consider a perturbation \mathcal{F}_ε , $\varepsilon \in (\mathbb{C}^n, 0)$ of \mathcal{F}_0 .

Proposition 14.2 *The loci of parameters ε with an unstable limit cycle near δ_0 is a germ of an analytic variety in $(\mathbb{C}^n, 0)$.*

Proof. Let m be the multiplicity of δ_0 . The foliation \mathcal{F}_ε has m limit cycles $\delta_1, \dots, \delta_m$, counting with multiplicity, around δ_0 . The function

$$A_1 = \prod_{1 \leq i \leq n} \int_{\delta_i} \frac{d\omega_\varepsilon}{\omega_\varepsilon}$$

is holomorphic function in $(\mathbb{C}^n, 0)$ and its zero set is the loci of unstable limit cycles around δ . Note that the function

$$A_2 = \left(\prod_{1 \leq i \leq n} \int_{\delta_i} \frac{d\omega_\varepsilon}{\omega_\varepsilon} \right) \left(\sum_{i=1}^n \int_{\delta_i} \frac{d\omega_\varepsilon}{\omega_\varepsilon} \right)^{-1}$$

is zero restricted to the zero set of A_1 .

Now consider $X(d)$ the topological closure of the set of foliations $\mathcal{F} \in \overline{\mathcal{F}}(d)$ with an unstable limit cycles. The set $\overline{\mathcal{F}}(d)$ is a projective space and hence by GAGA principle if $X(d)$ were locally an analytic variety, it would be an algebraic subset. The main point is that $X(d)$ is not an analytic set around foliations with a first integral.

Consider a family of cycles $\delta_t, t \in U$ in the Hamiltonian foliation \mathcal{F}_0 . The closure of the loci X of parameters ε such that \mathcal{F}_ε has a limit cycle near the family δ_t is an analytic set given by the zeros of $h_\varepsilon(z) - z, (z, \varepsilon) \in U \times (\mathbb{C}^n, 0)$. Now the loci of unstable limit cycles is given by the projection of $\text{Sing}(X)$ on ε coordiante. It is not a germ of an analytic variety because under projection $U \times \{0\}$ maps to 0 and in fact we have sectors of analytic varieties which can be extended analytically according to the Picard-Lefschetz theory of \mathcal{F}_0 .

14.1. Is $X(d)$ tangent to some polynomial 1-form in $\overline{\mathcal{F}}(d)$? In order to investigate this question, one must have some techniques for taking differential of integrals with respect to a deformation parameter. For example if we have a family of limit cycles δ_ε arising from a zero of an abelian integral in $df + \varepsilon\omega$ then find a formula for

$$\frac{\partial}{\partial \varepsilon} \int_{\delta_\varepsilon} \alpha \Big|_{\varepsilon=0}$$

14.3 Generic conditions

Theorem 14.1 *The projective variety \mathbb{P}^n is complete, that is, for all algebraic varieties V , the projection map $\pi : \mathbb{P}^n \times V \rightarrow V$ is closed. This means that any closed subset W of $\mathbb{P}^n \times V$, $\pi(W)$ is a closed subset of V . In particular, any closed subvariety of \mathbb{P}^n is complete*

For a proof of the above theorem see for instance [Mil]. Since $V \setminus \pi(W)$ is a Zariski open set, if we prove that it is non-empty then a generic point of V is not in the image of π . Sometimes, it is hard to find points in $V \setminus \pi(W)$ despite the fact that we are sure that it is a non-empty Zariski open set. In these situations, we take an arbitrary point of $x \in \mathbb{P}^n \times V$ and compute the map induced in the tangent spaces $(T\mathbb{P}^n \times V)_x \rightarrow (TV)_{\pi(x)}$ and prove that it is not surjective. For instance, if $\dim(W) < \dim(V)$ this

is always the case. This also prove that $V \setminus \pi(W)$ is non-empty. In some situations $W = \cup_{i=1}^s W_i$ is a union of some closed varieties, and it is easier to find points in each $V \setminus \pi(W_i)$. This implies that $V \setminus \pi(W)$ is non-empty and so we do not need to give explicit examples of its elements.

In the space of fibration $\frac{F^p}{G^q}$ the following conditions are generic:

1. $\{F = 0\}$ and $\{G = 0\}$ are smooth varieties;
2. $\{F = 0\}$ and $\{G = 0\}$ intersect each other transversally;
3. The restriction of f to $\mathbb{P}^n \setminus (\{F = 0\} \cup \{G = 0\})$ has nondegenerate critical points, namely p_1, p_2, \dots, p_r .
4. The images $c_1 = f(p_1), c_2 = f(p_2), \dots, c_r = f(p_r)$ are distincts.

For the proof we first take V the parameter space of $\frac{F^p}{G^q}$. The set $W_i \subset \mathbb{P}^n \times V$, $i = 1, 2$ is the algebraic closure of the set of $(\frac{F^p}{G^q}, p)$ such that for $i = 1$ (1) fails, and for $i = 2$ either (2) or (3) fails. Let also $W_3 \subset \mathbb{P}^n \times \mathbb{P}^n \times V$ the closure of the set of $(\frac{F^p}{G^q}, p, q)$ such that either (4) fails, that is, $f(p) = f(q)$. In this case we use the fact that $\mathbb{P}^n \times \mathbb{P}^n$ is complete.

It remains to find an explicit example of $\frac{F^p}{G^q}$ which satisfies the above generic conditions. This might get hard. That is why we have introduced many W_i 's. Finding a point in $V \setminus \pi(W_1)$ is trivial. For instance, we take $F = 0$ and $G = 0$ Fermat varieties. An example of a point in both $V \setminus \pi(W_i)$ $i = 2, 3$ is as follows. We take F and G a product of lines in general position. This example satisfies the condition (2), (3) or (4), see Exercise (14.2).

14.2. Show that the rational function

$$\frac{F^p}{G^q} = \frac{(x(x-1)(x-2) \cdots (x-a))^p}{(y(y-1)(y-2) \cdots (y-b))^q}, \quad ap = bq$$

has $(a-1)(b-1)$ non-degenerated critical points with distinct images. Show also that

$$f = x^{d+1} + y^{d+1} - (d+1)x - (d+1)y : \mathbb{C}^2 \rightarrow \mathbb{C}.$$

has d^2 non-degenerated critical points with distinct images.

14.4 Tame polynomials

Our main example for Picard-Lefschetz theory is the fibration of tame polynomials which has been extensively studied in [Mov11], Chapter 6.

Definition 14.1 We say that a polynomial $f \in \mathbb{C}[x, y]$ of degree d is tame if there are positive integers α_1 and α_2 such that the last homogeneous part g of f in the weighted ring $\mathbb{C}[x, y]$, $\text{weight}(x) = \alpha_1$ and $\text{weight}(y) = \alpha_2$, has an isolated singularity at the origin.

Note that For $\alpha_1 = \alpha_2 = 1$, g has an isolate singularity at the origin if and only if $g = (x - a_1y)(x - a_2y) \cdots (x - a_dy)$ and a_i 's are distinct.

Proposition 14.3 *Let $C := \{c_1, c_2, \dots, c_r\}$ be the set of critical values of f . Then f is a \mathbb{C}^∞ fibration over $\mathbb{C} \setminus C$*

Proof. A proof can be found in [Mov11], Theorem 6.1. We have to verify that there is no atypical fibers due to their behaviour at the indeterminacy points. If f is a tame polynomial with $\text{weight}(x) = \alpha_1$ and $\text{weight}(y) = \alpha_2$ then the same is true replacing α_i with $\frac{\alpha_i}{(\alpha_1, \alpha_2)}$, and so, we can assume that $(\alpha_1, \alpha_2) = 1$.

For $\alpha_1 = \alpha_2 = 1$ we only need to do just one blow-up at each indeterminacy point.

□

Example 14.1 *Fibrations with multiple fibers* Let us consider the fibration in \mathbb{P}^2 given by the rational function $\frac{F^p}{G^q}$, where F and G are two relatively prime irreducible polynomials in an affine chart \mathbb{C}^2 of \mathbb{P}^2 , $\frac{\deg(F)}{\deg(G)} = \frac{q}{p}$ and $\text{g.c.d.}(p, q) = 1$.

14.5 Monodromy and vanishing cycles

For this section we use the notation of the book [Mov19].

Theorem 14.2 *Suppose that $H_1(X - X_\infty, \mathbb{Q}) = 0$. Then a distinguished set of vanishing 1-cycles related to the critical points in the set $C \setminus \{\infty\} = \{c_1, c_2, \dots, c_r\}$ generates $H_1(L_b, \mathbb{Q})$.*

Note that in the above theorem ∞ can be a critical value of f .

Definition 14.2 *The cycle δ in a regular fiber L_b is called simple if the action of $\pi_1(B, b)$ on δ generates $H_1(L_b, \mathbb{Q})$.*

Note that in the above definition we have considered the homology group with rational coefficients. Of course, not all cycles are simple. For instance if the meromorphic function in a local coordinate (x, y) around $q \in \mathcal{R}$ has the form $\frac{x}{y}$, then the cycle around q in each leaf has this property that it is fixed under the action of monodromy, therefore it cannot be simple.