Some aspects of abelian integrals

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My research in mathematics turns around abelian integrals which I am going to explain with a simple example. Let us take the polynomial $f = y^2 - x^3 + 3x$ in two variables x and y and the family of elliptic curves $E_t : \{f - t = 0\},\ t \in \mathbb{C}$. Only for $t = -2, 2$ the curve E_t is singular. For t a real number between 2 and -2 the level surface $f^{-1}(t) \subset \mathbb{R}^2$ contains an oval δ_t. The polynomial f is a first integral of the differential equation

$$\mathcal{F}_\epsilon : \begin{cases} \dot{x} = 2y + \epsilon \frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + \epsilon s y \end{cases}.$$

with $\epsilon = 0$. If the abelian integral $\int_{\delta_t} (\frac{x^2}{2} dy - sy dx) = 0$ is zero for $t = 0$ or equivalently if $s \sim 0.9025$ then for ϵ near to 0, $\mathcal{F}_\epsilon$ has a limit cycle near δ_0. In fact for $\epsilon = 1$ such a limit cycle still exists and it is depicted in Figure (1). The origin of the above discussion comes form the second part of the Hilbert sixteen problem on limit cycles ([1]).

Every abelian integral satisfies a linear differential equation which is called the Picard-Fuchs equation (comming from geometry). For instance $\int_{\delta_t} \frac{dx}{y}$ satisfies

$$\frac{5}{36} I + 2tI' + (t^2 - 4)I'' = 0 \tag{1}$$

I general one has the linear system

$$Y' = \frac{1}{t^2 - 4} \begin{pmatrix} \frac{-1}{6} t & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{6} t \end{pmatrix} Y$$

which is called the Gauss-Manin connection of the family $E_t,\ t \in \mathbb{C}$. The main point behind the calculation of Picard-Fuchs equations and Gauss-Manin connections is the techniques of derivation of an integral with respect to a parameter and simplifying the result (see [4] for the implementation of algorithms for tame polynomials in Singular).

We may transfer the singularities $-2, 2$ of (1) to 0 and 1 and obtain a recursive formula for the coefficients of the Taylor series around 0 of its solutions. Since the integral $\int_{\delta_t} \frac{dx}{y}$ is holomorphic around $t = -2$, we get

$$\int_{\delta_t} \frac{dx}{y} = \frac{-2\pi}{\sqrt{3}} F(\frac{1}{6}, \frac{5}{6}, 1 | \frac{t+2}{4}),$$

where

$$F(a, b, c|z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

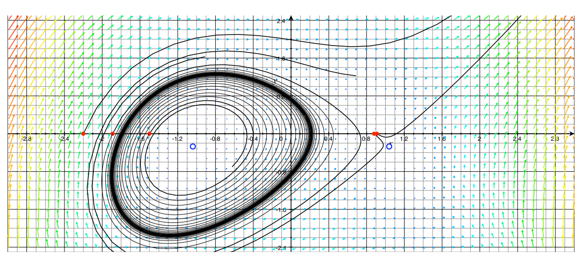

Figure 1: A limit cycle crossing $(x, y) \sim (-1.79, 0)$

is the Gauss hypergeometric function and $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$. An elegant way to prove the statement

$$F(\frac{5}{6},\frac{1}{6},1|\pm\frac{\sqrt{\frac{54001}{15}}}{120}+\frac{1}{2})\frac{\pi^2}{\Gamma(\frac{1}{3})^3}\in\bar{\mathbb{Q}}, \tag{2}$$

is as follows: The elliptic curve L_t has the j invariant $\frac{4}{t^2-4}$. For the values of t such that $j = 2^4\cdot 3^3\cdot 5^3$, L_t admits a complex multiplication by the field $\mathbb{Q}(\sqrt{-3})$. Now one uses the Chowla-Selberg Theorem on the periods of differential forms of the first kind on elliptic curves with complex multiplication. In the next paragraph we give another interpretation of (2) in terms of a Hodge cycle of a four dimensional cubic hypersurface.

Let us consider the affine hypersurface

$$U_c:\ x_1^3+x_2^3+\cdots+x_5^3-x_1-x_2-c=0,\ c\in\mathbb{C}-\{\pm\frac{4}{3\sqrt{3}},0\}$$

in $\mathbb{C}^5$ and its compactification M_c in the projective space of dimension 5. The Hodge decomposition of the 4-th primitive cohomology of M_c has the Hodge numbers $0,1,20,1,0$ and a generator of $H^{3,1}$ piece restricted to U_c is represented by the differential 4-form

$$\alpha := \big((972c^2-192)x_1x_2+(-405c^3-48c)x_2+(-405c^3-48c)x_1+(243c^4-36c^2+64)\big)$$
$$\cdot\sum_{i=1}^{5}(-1)^{i-1}x_i dx_1\wedge\cdots dx_{i-1}\wedge dx_{i+1}\wedge\cdots\wedge dx_5$$

(see [3,5]). Therefore, a cycle $\delta\in H_4(M_c,\mathbb{Z})$ with support in U_c is a Hodge cycle if and only if $\int_\delta\alpha = 0$. It turns out that the $\mathbb{Q}$-vector space of the periods of α is spanned over $\mathbb{Q}$ by $\Gamma(\frac{1}{3})^3\mathbb{Q}(\zeta_3)$ times the periods of $\frac{dx}{y}$ over the elliptic curve $L_t: y^2-x^3+3x-t,\ t=2-\frac{27}{4}c^2$. For $j=\frac{1}{t^2-4}=2^4\cdot3^4\cdot5^3$, L_t has a complex multiplication by $\mathbb{Q}(\zeta_3)$ and this gives us a Hodge cycle δ in $H_4(M_c,\mathbb{Q})$. One of the consequences of the Hodge conjecture is that for $c\in\bar{\mathbb{Q}}$ the integration over δ of any 4-differential form in $\mathbb{C}^5$, which is defined over $\bar{\mathbb{Q}}$ and is without residue at infinity, belongs to $\pi^2\bar{\mathbb{Q}}$. Since the Hodge conjecture is proved for cubic hypersurfaces of dimension 4, we get another interpretation of (2). For more details see [2].

Finally one can take families of elliptic curves depending on many parameters and investigate certain differential equations in parameter spaces. For instance, for the family of elliptic curves

$$y^2-4(x-t_1)^3+t_2(x-t_1)+t_3,$$

the abelian integral $\int\frac{xdx}{y}$ is contant along the solutions of the Ramanujan ordinary differential equation

$$\begin{cases}\dot{t}_1 = t_1^2-\frac{1}{12}t_2\\ \dot{t}_2 = 4t_1t_2-6t_3\\ \dot{t}_3=6t_1t_3-\frac{1}{3}t_2^2\end{cases}. \tag{3}$$

Using this, one can prove that every transcendental leaf of (3) intersects points with algebraic coordinates at most once. For more details see [4].

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